HERMITE-HADAMARD INEQUALITY FOR FRACTIONAL INTEGRALS VIA \( \eta \)-CONVEX FUNCTIONS

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Abstract. In this paper, we prove Hermite-Hadamard inequality for fractional integrals by using \( \eta \)-convex function. We give some inequalities for Hermite-Hadamard type fractional integrals.

1. Introduction and Preliminaries

If \( f : I \to \mathbb{R} \) is a convex function on the interval \( I \), then for any \( a, b \in I \) with \( a \neq b \), we have the following double inequality

\[
f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(t) dt \leq \frac{f(a) + f(b)}{2}.
\]

This significant result was given in ([13], 1893) and is well known in the literature as the Hermite-Hadamard inequality. Since then, many researchers have given considerable attention to the inequalities in (1) and a number of extensions, generalizations and variants have appeared in the literature of convex analysis, for example, see [1, 2, 4, 5, 6, 7, 8, 9, 10, 16, 17, 18, 19, 21, 22, 24, 26, 27, 28] and the references cited therein.

In [12], M. E. Gordji et al. introduced the idea of \( \eta \)-convex functions as generalization of ordinary convex functions and gave the following definition for \( \eta \)-convexity of functions.

**Definition 1.1.** A function \( f : [a, b] \to \mathbb{R} \) is said to be \( \eta \)-convex (or convex with respect to \( \eta \)) if the inequality

\[
f(tx + (1-t)y) \leq f(y) + t\eta(f(x), f(y))
\]

holds for all \( x, y \in [a, b], t \in [0, 1] \), and \( \eta \) is defined by \( \eta : f([a, b]) \times f([a, b]) \to \mathbb{R} \).

In the above definition if we set \( \eta(x,y) = x - y \), then we can directly obtain the classical definition of a convex function.

Also in [12], the authors proved some important results but here we give only one of them in the following theorem based on the above definition, which is also known as \( \eta \)-convex version of Hermite-Hadamard inequality.

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Theorem 1.2 ([12]). Suppose that \( f: [a, b] \to \mathbb{R} \) is a \( \eta \)-convex function such that \( \eta \) is bounded above on \( f([a, b]) \times f([a, b]) \). Then the following inequalities hold.

\[
\frac{f \left( \frac{a+b}{2} \right) - M \eta}{2} \leq \frac{1}{b-a} \int_{a}^{b} f(x)dx \leq \frac{1}{2} [f(a) + f(b)] + \frac{1}{4} [\eta(f(a), f(b)) + \eta(f(b), f(a))]
\]

\[
\leq \frac{f(a) + f(b)}{2} + \frac{M \eta}{2},
\]

where \( M \eta \) is the upper bound of \( \eta \).

In the following, we give the definition of fractional Riemann-Liouville integral, which will be used in the later part of the paper. For more details, one can consult [11, 23].

Definition 1.3. Let \( f \in L[a, b] \). The left-sided and right-sided Riemann-Liouville fractional integrals \( J_{a+}^{\alpha}f \) and \( J_{b-}^{\alpha}f \) of order \( \alpha > 0 \) with \( a \geq 0 \) are defined by

\[
J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} (x-t)^{\alpha-1}f(t)dt \quad \text{with} \quad x > a
\]

and

\[
J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{b} (t-x)^{\alpha-1}f(t)dt \quad \text{with} \quad x < b,
\]

respectively, where \( \Gamma(\alpha) \) is the Gamma function and its definition is

\[
\Gamma(\alpha) = \int_{0}^{\infty} e^{-u}u^{\alpha-1}du.
\]

It is to be noted that \( J_{a+}^{0}f(x) = J_{b-}^{0}f(x) = f(x) \). In the case of \( \alpha = 1 \), the fractional integral reduces to the classical integral.

In [25], M. Z. Sarikaya et al. presented the following Hermite-Hadamard’s inequalities for fractional integrals.

Theorem 1.4 ([25]). Let \( f: I \to \mathbb{R} \) be a positive function with \( 0 \leq a < b \) and \( f \in L[a, b] \). If \( f \) is a convex function on \([a, b] \), then the following inequality for fractional integrals holds.

\[
f \left( \frac{a+b}{2} \right) \leq \frac{\Gamma(\alpha + 1)}{2(b-a)^{\alpha}} \left[ J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a) \right] \leq \frac{f(a) + f(b)}{2}.
\]

Also in the same paper, the authors established an important lemma and proved the following Hermite-Hadamard’s type inequalities for fractional integrals.
Theorem 1.5 ([25]). Let \( f: [a, b] \to \mathbb{R} \) be a differentiable function on \((a, b)\) with \( a < b \). If \(|f'|\) is a convex function on \([a, b]\), then the following inequality for fractional integrals holds
\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)\alpha} \left[ J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] \right| \\
\leq \frac{b-a}{2(\alpha + 1)} \left( 1 - \frac{1}{2^\alpha} \right)(|f'(b)| + |f'(a)|).
\]

The following Hermite-Hadamard's type inequalities for fractional integrals holds for \( 0 < \alpha \leq 1 \)
\[
\left| f \left( \frac{a+b}{2} \right) - \frac{\Gamma(\alpha + 1)}{2(b-a)\alpha} \left[ J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] \right| \\
\leq \frac{b-a}{2^{\alpha+1}(\alpha + 1)} \left( |f'(a)|^q + |f'(b)|^q \right)^\frac{1}{q}.
\]

Theorem 1.7 ([15]). Let \( f: [a, b] \to \mathbb{R} \) be a differentiable function on \((a, b)\) with \( a < b \). If \(|f|^q\) is \(\eta\)-convex on \([a, b]\) for some fixed \(p > 1\), then the following inequality for fractional integrals holds for \( 0 < \alpha \leq 1 \)
\[
\left| f \left( \frac{a+b}{2} \right) - \frac{\Gamma(\alpha + 1)}{2(b-a)\alpha} \left[ J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] \right| \\
\leq \frac{b-a}{2^{\alpha+1}(\alpha + 1)^\frac{1}{p}} \left[ \left( \frac{3|f'(a)|^q + |f'(b)|^q}{4} \right)^\frac{1}{q} + \left( \frac{|f'(a)|^q + 3|f'(b)|^q}{4} \right)^\frac{1}{q} \right].
\]

Theorem 1.8 ([15]). Let \( f: [a, b] \to \mathbb{R} \) be a differentiable function on \((a, b)\) with \( a < b \). If \(|f|^q\) where \( q = \frac{p}{p-1} \) is \(\eta\)-convex on \([a, b]\) for some fixed \(p > 1\), then the following inequality for fractional integrals holds for \( 0 < \alpha \leq 1 \)
\[
\left| f \left( \frac{a+b}{2} \right) - \frac{\Gamma(\alpha + 1)}{2(b-a)\alpha} \left[ J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] \right| \\
\leq \frac{b-a}{2^{\alpha+1}(\alpha + 1)} \left[ \left( \frac{(\alpha + 1)|f'(b)|^q + (\alpha + 3)|f'(a)|^q}{2(\alpha + 2)} \right)^\frac{1}{q} \right.
\]
\[
+ \left. \left( \frac{(\alpha + 1)|f'(a)|^q + (\alpha + 3)|f'(b)|^q}{2(\alpha + 2)} \right)^\frac{1}{q} \right].
\]

The main purpose of this paper is to establish a variant of Hermite-Hadamard inequalities for Riemann-Liouville fractional integral using \(\eta\)-convex function (Theorem 2.1). Then we give some interesting results (Theorems 3.2–3.9) connected with the left hand side of Hermite-Hadamard inequalities for Riemann-Liouville fractional integrals using the identities obtained for fractional integrals given in [15, 25]. Also we discuss the importance of our results (Remarks 2.2–3.10).
2. Hermite-Hadamard’s Inequalities for Fractional Integrals

Theorem 2.1. Suppose that \( f : [a, b] \to \mathbb{R} \) is an \( \eta \)-convex function such that \( \eta \) is bounded above by \( M_\eta \), then for \( \alpha > 0 \), the following inequalities for fractional integrals hold:

\[
\frac{1}{\alpha} \left[ J^\alpha_{a+} f(b) + J^\alpha_{b-} f(a) \right] \leq \frac{f(a) + f(b)}{2} \leq \frac{\alpha f(a) - \alpha f(b)}{2(\alpha + 1)} + \frac{\alpha M_\eta}{\alpha + 1}.
\]

Proof. Since \( f : [a, b] \to \mathbb{R} \) is an \( \eta \)-convex function such that \( \eta \) is bounded above by \( M_\eta \), so from (3), we have

\[
f \left( \frac{x + y}{2} \right) - \frac{M_\eta}{2} \leq \frac{f(x) + f(y)}{2} + \frac{M_\eta}{2},
\]

where \( x, y \in [a, b] \). Let \( x = ta + (1 - t)b \) and \( y = tb + (1 - t)a \), then from the above we have

\[
f \left( \frac{a + b}{2} \right) - \frac{M_\eta}{2} \leq \frac{f(ta + (1 - t)b) + f(tb + (1 - t)a)}{2} + \frac{M_\eta}{2},
\]

(11)

Multiplying both sides of (11) by \( t^{\alpha - 1} \) and then integrating the resulting inequality with respect to \( t \) over \([0, 1]\), we obtain

\[
\frac{2}{\alpha} f \left( \frac{a + b}{2} \right) - \frac{M_\eta}{\alpha} \leq \int_0^1 t^{\alpha - 1} f(ta + (1 - t)b) dt + \frac{M_\eta}{\alpha}.
\]

(12)

Let \( ta + (1 - t)b = u \) and \((1 - t)a + tb = v\), then

\[
\int_0^1 t^{\alpha - 1} f(ta + (1 - t)b) dt + \int_0^1 t^{\alpha - 1} f(tb + (1 - t)a) dt
\]

\[
= \int_b^a \left( \frac{b - u}{b - a} \right)^{\alpha - 1} f(u) \frac{du}{a - b} + \int_a^b \left( \frac{v - a}{b - a} \right)^{\alpha - 1} f(v) \frac{dv}{b - a}
\]

\[
= \frac{\Gamma(\alpha)}{(b - a)^\alpha} \left[ J^\alpha_{a+} f(b) + J^\alpha_{b-} f(a) \right].
\]

Therefore, the inequality (12) takes the following shape

\[
\frac{2}{\alpha} f \left( \frac{a + b}{2} \right) - \frac{M_\eta}{\alpha} \leq \frac{\Gamma(\alpha)}{(b - a)^\alpha} \left[ J^\alpha_{a+} f(b) + J^\alpha_{b-} f(a) \right] + \frac{M_\eta}{\alpha}
\]
and the rearrangement of terms provides

\begin{equation}
\frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} \left[ J^\alpha_{a+} f(b) + J^\alpha_{b-} f(a) \right] \geq f\left(\frac{a + b}{2}\right) - M_\eta,
\end{equation}

which proves the first inequality in (10). Now we proceed to prove the second inequality

\begin{align}
\begin{alignat}{2}
&f(ta + (1 - t)b) \leq f(b) + t\eta(f(a), f(b)), \quad (14) \\
&f(tb + (1 - t)a) \leq f(a) + t\eta(f(b), f(a)). \\
\end{alignat}
\end{align}

Adding (14), (15) and multiplying both sides by \(t^{\alpha - 1}\), and then integrating the resulting inequality with respect to \(t\) over \([0, 1]\), yield the following

\begin{equation}
\int_0^1 t^{\alpha - 1} \left( f(ta + (1 - t)b) + f(tb + (1 - t)a) \right) dt \\
\leq [f(a) + f(b)] \int_0^1 t^{\alpha - 1} dt \left( \eta(f(a), f(b)) + \eta(f(b), f(a)) \right) \int_0^1 t^{\alpha} dt
\end{equation}

(by definition of \(\eta\)-convex function). By simplifying inequality (16), we have

\begin{equation}
\frac{\Gamma(\alpha)}{(b - a)^\alpha} \left[ J^\alpha_{a+} f(b) + J^\alpha_{b-} f(a) \right] \\
\leq \frac{f(a) + f(b)}{\alpha} + \frac{\eta(f(a), f(b)) + \eta(f(b), f(a))}{\alpha + 1}.
\end{equation}

From inequalities (13) and (17), we have

\begin{equation}
f\left(\frac{a + b}{2}\right) - M_\eta \leq \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} \left[ J^\alpha_{a+} f(b) + J^\alpha_{b-} f(a) \right] \\
\leq \frac{f(a) + f(b)}{2} + \frac{\alpha(\eta(f(a), f(b)) + \eta(f(b), f(a)))}{2(\alpha + 1)}.
\end{equation}

Furthermore, since \(\eta\) is bounded above by \(M_\eta\), so from the above we can easily obtain the desired result for (10).

**Remark 2.2.** If \(f\) is \(\eta\)-convex with respect to \(\eta\) defined by \(\eta(x, y) = x - y\), then (10) reduces to the inequality of Theorem 1.4.

3. **Hermite-Hadamard type inequalities for fractional integrals**

In order to prove our next result, we need the following Lemma.
Lemma 3.1 ([25]). Let \( f: [a, b] \to \mathbb{R} \) be a differentiable function on \( (a, b) \) with \( a < b \). If \( f' \in L[a, b] \), then the following equality holds

\[
\frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^{\alpha}} [J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)]
\]

(18)

\[
= \frac{b - a}{2} \int_0^1 [(1 - t)^\alpha - t^\alpha] f'(at + (1 - t)b) dt.
\]

Theorem 3.2. Let \( f: [a, b] \to \mathbb{R} \) be a differentiable function on \( (a, b) \) with \( a < b \). If \( |f'| \) is an \( \eta \)-convex function on \( [a, b] \), then the following inequality for fractional integrals holds

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^{\alpha}} [J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)] \right|
\]

(19)

\[
\leq \frac{b - a}{2(\alpha + 1)} \left( 1 - \frac{1}{2^\alpha} \right) [2|f'(b)| + \eta(|f'(a)|, |f'(b)|)].
\]

Proof. By using Lemma 3.1 together with the fundamental property of absolute value of real numbers, we have

(20)

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^{\alpha}} [J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)] \right|
\]

\[
\leq \frac{b - a}{2} \int_0^1 ||(1 - t)^\alpha - t^\alpha|| |f'(at + (1 - t)b)| dt
\]

\[
\leq \frac{b - a}{2} \int_0^1 ||(1 - t)^\alpha - t^\alpha|| (|f'(b)| + t\eta(|f'(a)|, |f'(b)|)) dt \quad \text{(by } \eta\text{-convexity of } |f'|)\]

\[
= \frac{b - a}{2} \left[ \int_0^1 [(1 - t)^\alpha - t^\alpha]|f'(b)| + t\eta(|f'(a)|, |f'(b)|) dt 
\right.
\]

\[
\left. + \int_{\frac{1}{2}}^1 [t^\alpha - (1 - t)^\alpha]|f'(b)| + \eta(|f'(a)|, |f'(b)|) dt \right]
\]

\[
= \frac{b - a}{2} \left[ |f'(b)| \left\{ \int_0^1 [(1 - t)^\alpha - t^\alpha] dt \right\} + \eta(|f'(a)|, |f'(b)|) \left\{ \int_0^1 t[(1 - t)^\alpha - t^\alpha] dt \right\} 
\right.
\]

\[
\left. + |f'(b)| \left\{ \int_{\frac{1}{2}}^1 [t^\alpha - (1 - t)^\alpha] dt \right\} + \eta(|f'(a)|, |f'(b)|) \left\{ \int_{\frac{1}{2}}^1 t[t^\alpha - (1 - t)^\alpha] dt \right\} \right].
\]
Hence the R. H. S of (20) is equivalent to
\[ |f'(b)| \left[ \frac{1}{\alpha + 1} - \frac{1}{2^\alpha(\alpha + 1)} \right] + \eta(|f'(a)|, |f'(b)|) \left[ \frac{1}{(\alpha + 1)(\alpha + 2)} - \frac{1}{2^{\alpha+1}(\alpha + 1)} \right] \]
\[ + |f'(b)| \left[ \frac{1}{\alpha + 1} - \frac{1}{2^\alpha(\alpha + 1)} \right] + \eta(|f'(a)|, |f'(b)|) \left[ \frac{1}{(\alpha + 2)} - \frac{1}{2^{\alpha+1}(\alpha + 1)} \right]. \]
and furthermore, the simplification of the above terms provides the following
\[ \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[ J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a) \right] \right| \]
\[ \leq \frac{b - a}{2(\alpha + 1)} \left( 1 - \frac{1}{2^\alpha} \right) \left( 2 |f'(b)| + \eta(|f'(a)|, |f'(b)|) \right). \]
This completes the desired proof of the result.

**Remark 3.3.** If $|f'|$ is $\eta$-convex with respect to $\eta$ defined by $\eta(x, y) = x - y$, then (19) reduces to the inequality of Theorem 1.5.

The following lemma is needed in the proof of our next result, which given in [15].

**Lemma 3.4.** Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable function on $(a, b)$. If $f' \in L^1[a, b]$, then the following identity for Riemann-Liouville fractional integrals holds
\[ f \left( \frac{a + b}{2} \right) - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[ J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a) \right] = \frac{b - a}{2} \sum_{k=1}^{4} I_k, \]
where
\[ I_1 = \int_0^{\frac{1}{2}} t^\alpha f'(tb + (1-t)a) dt, \quad I_2 = \int_0^{\frac{1}{2}} (-t^\alpha) f'(ta + (1-t)b) dt, \]
\[ I_3 = \int_{\frac{1}{2}}^{1} (t^\alpha - 1) f'(tb + (1-t)a) dt, \quad I_4 = \int_{\frac{1}{2}}^{1} (1-t^\alpha) f'(ta + (1-t)b) dt. \]

**Theorem 3.5.** Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable function on $(a, b)$ with $a < b$. If $|f'|$ is $\eta$-convex on $[a, b]$ and $0 < \alpha \leq 1$, then the following inequality for Riemann-Liouville fractional integrals holds:
\[ \left| f \left( \frac{a + b}{2} \right) - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[ J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a) \right] \right| \]
\[ \leq \frac{b - a}{2^{\alpha+1}(\alpha + 1)} \left( |f'(a)| + |f'(b)| + \eta(|f'(a)|, |f'(b)|) + \eta(|f'(b)|, |f'(a)|) \right). \]

**Proof.** By using the well-known triangular inequality on Lemma 3.4, we have
\[ \left| f \left( \frac{a + b}{2} \right) - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[ J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a) \right] \right| \leq \frac{b - a}{2} \sum_{k=1}^{4} |I_k| \]
and then by applying the \(\eta\)-convexity of \(|f'|\), we get
\[
|I_1| \leq \frac{1}{\alpha+1} \int_0^1 tf'(tb + (1-t)a)dt \\
\leq \frac{1}{\alpha+1} \int_0^1 tf'(a)dt + \frac{1}{\alpha+1} \int_0^1 t^{\alpha+1}\eta(|f'(b)|, |f'(a)|)dt \\
= \frac{1}{2^\alpha+1(\alpha+1)} |f'(a)| + \frac{1}{2^\alpha+2(\alpha+2)} \eta(|f'(b)|, |f'(a)|).
\]
Similarly,
\[
|I_2| \leq \frac{1}{2^\alpha+1(\alpha+1)} |f'(b)| + \frac{1}{2^\alpha+2(\alpha+2)} \eta(|f'(a)|, |f'(b)|).
\]
Again using the \(\eta\)-convexity of \(|f'|\) and the fact \(|t_1^\alpha - t_2^\alpha| \leq |t_1 - t_2|^\alpha\) for all \(\alpha \in (0,1]\) and \(t_1, t_2 \in [0,1]\), leads to the following
\[
|I_3| \leq \frac{1}{2^\alpha+1(\alpha+1)} |f'(a)| + \frac{\alpha+3}{2^\alpha+2(\alpha+2)(\alpha+1)} \eta(|f'(b)|, |f'(a)|)
\]
and similarly
\[
|I_4| \leq \frac{1}{2^\alpha+1(\alpha+1)} |f'(b)| + \frac{\alpha+3}{2^\alpha+2(\alpha+2)(\alpha+1)} \eta(|f'(a)|, |f'(b)|).
\]
The addition of the above inequalities take us to the required conclusion. \(\square\)

**Remark 3.6.** If \(|f'|\) is \(\eta\)-convex with respect to \(\eta\) defined by \(\eta(x,y) = x - y\), then (22) reduces to the inequality of Theorem 1.6.

**Theorem 3.7.** Let \(f : [a, b] \to \mathbb{R}\) be a differentiable function on \((a, b)\) with \(a < b\). If \(|f|^q (q = \frac{p}{p+1})\) is \(\eta\)-convex on \([a, b]\) for some fixed \(p > 1\) and \(0 < \alpha \leq 1\), then the following inequality for fractional integrals holds
\[
\left| f \left( \frac{a+b}{2} \right) - \frac{\Gamma(\alpha+1)}{2(b-a)\alpha} \left[ J_a^\alpha f(b) + J_b^\alpha f(a) \right] \right| \\
\leq \frac{b-a}{2^{\alpha+1}(\alpha p + 1)^{\frac{1}{2}}} \left[ \left( 4|f'(a)|^q + \eta(|f'(b)|^q, |f'(a)|^q) \right)^\frac{1}{2} + \left( 4|f'(b)|^q + \eta(|f'(b)|^q, |f'(a)|^q) \right)^\frac{1}{2} \right].
\]

**Proof.** By using the well-known triangular and Holder inequalities on Lemma 3.4 in turn, we have
\[
\left| f \left( \frac{a+b}{2} \right) - \frac{\Gamma(\alpha+1)}{2(b-a)\alpha} \left[ J_a^\alpha f(b) + J_b^\alpha f(a) \right] \right| \leq \frac{b-a}{2} \sum_{k=1}^4 |I_k|,
\]
\[ |I_1| \leq \left( \int_0^{t_1} \frac{dt}{p^{\alpha} \, dt} \right)^{\frac{1}{p}} \left( \int_0^{t_1} |f'(tb + (1-t)a)|^q \, dt \right)^{\frac{1}{q}} \]

\[ \leq \left( \frac{1}{2p^{\alpha+1}(p\alpha + 1)} \right)^{\frac{1}{p}} \left( \int_0^{t_1} |f'(a)|^q \, dt + \int_0^{t_1} \eta(|f'(b)|^q, |f'(a)|^q) \, dt \right)^{\frac{1}{q}} \]

\[ = \left( \frac{1}{2p^{\alpha+1}(p\alpha + 1)} \right)^{\frac{1}{p}} \left[ \frac{|f'(a)|}{2} + \frac{\eta(|f'(b)|^q, |f'(a)|^q)}{8} \right] \]

(by \( \eta \)-convexity of \( f \)). Similarly,

\[ |I_2| \leq \left( \frac{1}{2p^{\alpha+1}(p\alpha + 1)} \right)^{\frac{1}{p}} \left[ \frac{|f'(b)|}{2} + \frac{\eta(|f'(b)|^q, |f'(a)|^q)}{8} \right]^{\frac{1}{q}} \]

and

\[ |I_3| \leq \left( \int_0^{1} (1-t^\alpha)^p \, dt \right)^{\frac{1}{p}} \left( \int_0^{1} (1-t^\alpha)^q \, dt \right)^{\frac{1}{q}} \]

Let \( \alpha \in (0, 1] \) and for all \( t_1, t_2 \in [0, 1] \), \( |t_1^\alpha - t_2^\alpha| \leq |t_1 - t_2|^\alpha \), therefore,

\[ \int_0^{1} (1-t^\alpha)^p \, dt \leq \int_0^{1} (1-t)^{p\alpha} \, dt = \frac{1}{2p^{\alpha+1}(p\alpha + 1)} \]

Hence

\[ |I_3| \leq \left( \frac{1}{2p^{\alpha+1}(p\alpha + 1)} \right)^{\frac{1}{p}} \left[ \frac{|f'(a)|}{2} + \frac{\eta(|f'(b)|^q, |f'(a)|^q)}{8} \right]^{\frac{1}{q}} \]

analogously,

\[ |I_4| \leq \left( \frac{1}{2p^{\alpha+1}(p\alpha + 1)} \right)^{\frac{1}{p}} \left[ \frac{|f'(b)|}{2} + \frac{\eta(|f'(b)|^q, |f'(a)|^q)}{8} \right]^{\frac{1}{q}} \]

By adding the above four inequalities, we get the required result. This completes the proof. \( \Box \)

**Remark 3.8.** If \( |f|^\frac{p}{p+1} \) is \( \eta \)-convex with respect to \( \eta \) defined by \( \eta(x, y) = x - y \), then inequality (23) becomes the inequality obtained in Theorem 1.7.

**Theorem 3.9.** Let \( f : [a, b] \to \mathbb{R} \) be a differentiable function on \((a, b)\) with \( a < b \). If \( |f|^q \ (q = \frac{p}{p+1}) \) is \( \eta \)-convex on \([a, b]\) for some fixed \( p > 1 \) and \( 0 < \alpha \leq 1 \),
then the following inequality for fractional integrals holds

\[(24)\]

\[
\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_a^\alpha f(b) + J_b^\alpha f(a)] \right|
\]

\[\leq \frac{b-a}{2^{\alpha+1}(\alpha + 1)} \left[ \left( \frac{2(\alpha + 2)\|f'(a)\|_q + (\alpha + 1)\eta(\|f'(a)\|_q, \|f'(a)\|_q) }{2(\alpha + 2)} \right)^\frac{1}{\eta} 
+ \left( \frac{2(\alpha + 2)\|f'(b)\|_q + (\alpha + 1)\eta(\|f'(b)\|_q, \|f'(a)\|_q) }{2(\alpha + 2)} \right)^\frac{1}{\eta} 
+ \left( \frac{2(\alpha + 2)\|f'(b)\|_q + (\alpha + 3)\eta(\|f'(b)\|_q, \|f'(a)\|_q) }{2(\alpha + 2)} \right)^\frac{1}{\eta} \right].
\]

Proof. By using the triangular and power mean integral inequalities on Lemma 3.4 in turn, we have

\[
\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_a^\alpha f(b) + J_b^\alpha f(a)] \right|
\]

\[\leq \frac{b-a}{2} \sum_{k=1}^{4} |I_k|,
\]

\[
|I_1| \leq \left( \int_0^\frac{b-a}{2} t^\alpha dt \right)^{1-\frac{1}{\eta}} \left( \int_0^\frac{b-a}{2} t^\alpha |f(t^b + (1-t)a)|^q dt \right)^\frac{1}{\eta}
\]

\[\leq \left( \frac{1}{2^{\alpha+1}(\alpha + 1)} \right)^{1-\frac{1}{\eta}} \left( \int_0^\frac{b-a}{2} t^\alpha |f'(a)|^q dt + \int_0^\frac{b-a}{2} t^{\alpha+1} \eta(\|f'(b)\|_q, \|f'(a)\|_q) dt \right)^\frac{1}{\eta}
\]

\[= \left( \frac{1}{2^{\alpha+1}(\alpha + 1)} \right) \left( \frac{2(\alpha + 2)\|f'(a)\|_q + (\alpha + 1)\eta(\|f'(b)\|_q, \|f'(a)\|_q) }{2(\alpha + 2)} \right)^\frac{1}{\eta},
\]

(by \(\eta\)-convexity of \(f\)). Similarly,

\[
|I_2| \leq \left( \frac{1}{2^{\alpha+1}(\alpha + 1)} \right) \left( \frac{2(\alpha + 2)\|f'(b)\|_q + (\alpha + 1)\eta(\|f'(b)\|_q, \|f'(a)\|_q) }{2(\alpha + 2)} \right)^\frac{1}{\eta}
\]

and

\[
|I_3| \leq \left( \frac{2(\alpha + 2)\|f'(a)\|_q + (\alpha + 3)\eta(\|f'(b)\|_q, \|f'(a)\|_q) }{2(\alpha + 2)} \right)^\frac{1}{\eta}.
\]

Analogously,

\[
|I_4| \leq \left( \frac{2(\alpha + 2)\|f'(b)\|_q + (\alpha + 3)\eta(\|f'(b)\|_q, \|f'(a)\|_q) }{2(\alpha + 2)} \right)^\frac{1}{\eta}.
\]

By adding all the above inequalities, we can reach the conclusion. \(\square\)
Remark 3.10. If $|f|^\frac{p}{p-1}$ is $\eta$-convex with respect to $\eta$ defined by $\eta(x,y) = x-y$, then (24) reduces to the inequality of Theorem 1.8.

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