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HERMITE-HADAMARD INEQUALITY FOR FRACTIONAL INTEGRALS VIA η-CONVEX FUNCTIONS

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Abstract. In this paper, we prove Hermite-Hadamard inequality for fractional integrals by using η-convex function. We give some inequalities for Hermite-Hadamard type fractional integrals.

1. Introduction and Preliminaries

If $f\colon I \to \mathbb{R}$ is a convex function on the interval I, then for any $a, b \in I$ with $a \neq b$, we have the following double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

This significant result was given in ([**13**], 1893) and is well known in the literature as the Hermite-Hadamard inequality. Since then, many researchers have given considerable attention to the inequalities in (1) and a number of extensions, generalizations and variants have appeared in the literature of convex analysis, for example, see [**1, 2, 4, 5, 6, 7, 8, 9, 10, 16, 17, 18, 19, 21, 22, 24, 26, 27, 28**] and the references cited therein.

In [**12**], M. E. Gordji et al. introduced the idea of η-convex functions as generalization of ordinary convex functions and gave the following definition for η-convexity of functions.

Definition 1.1. A function $f\colon [a,b] \to \mathbb{R}$ is said to be η-convex (or convex with respect to η) if the inequality

$$f(tx+(1-t)y) \leq f(y) + t\eta(f(x),\, f(y)) \tag{2}$$

holds for all $x, y \in [a,b]$, $t \in [0,1]$, and η is defined by $\eta\colon f([a,b]) \times f([a,b]) \to \mathbb{R}$.

In the above definition if we set $\eta(x,y) = x - y$, then we can directly obtain the classical definition of a convex function.

Also in [**12**], the authors proved some important results but here we give only one of them in the following theorem based on the above definition, which is also known as η-convex version of Hermite-Hadamard inequality.

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Theorem 1.2 ([**12**])**.** *Suppose that* $f\colon [a,b] \to \mathbb{R}$ *is a* η*-convex function such that* η *is bounded above on* $f([a,b]) \times f([a,b])$*. Then the following inequalities hold.*

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) - \frac{M_\eta}{2} &\le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \\
&\le \frac{1}{2}\left[f(a)+f(b)\right] + \frac{1}{4}\left[\eta(f(a),f(b)) + \eta(f(b),f(a))\right] \\
&\le \frac{f(a)+f(b)}{2} + \frac{M_\eta}{2},
\end{aligned}
\tag{3}
$$

where M_η *is the upper bound of* η*.*

In the following, we give the definition of fractional Riemann-Liouville integral, which will be used in the later part of the paper. For more details, one can consult [**11, 23**].

Definition 1.3. Let $f \in L[a,b]$. The left-sided and right-sided Riemann-Liouville fractional integrals $J^\alpha_{a+}f$ and $J^\alpha_{b-}f$ of order $\alpha > 0$ with $a \ge 0$ are defined by

$$J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\mathrm{d}t \qquad \text{with } \ x > a$$

and

$$J^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\mathrm{d}t \qquad \text{with } \ x < b,$$

respectively, where $\Gamma(\alpha)$ is the Gamma function and its definition is

$$\Gamma(\alpha) = \int_0^\infty \mathrm{e}^{-u}\, u^{\alpha-1}\mathrm{d}u. \tag{4}$$

It is to be noted that $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$. In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

In [**25**], M. Z. Sarikaya et al. presented the following Hermite-Hadamard's inequalities for fractional integrals.

Theorem 1.4 ([**25**])**.** *Let* $f\colon I \to \mathbb{R}$ *be a positive function with* $0 \le a < b$ *and* $f \in L[a,b]$*. If* f *is a convex function on* $[a,b]$*, then the following inequality for fractional integrals holds.*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)\right] \le \frac{f(a)+f(b)}{2}. \tag{5}$$

Also in the same paper, the authors established an important lemma and proved the following Hermite-Hadamard's type inequalities for fractional integrals.

Theorem 1.5 ([**25**]). *Let $f\colon [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a < b$. If $|f'|$ is a convex function on $[a,b]$, then the following inequality for fractional integrals holds*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a_+} f(b) + J^\alpha_{b_-} f(a)\right]\right| \\ \leq \frac{b-a}{2(\alpha+1)}\Big(1-\frac{1}{2^\alpha}\Big)\big(|f'(b)|+|f'(a)|\big).\end{aligned} \tag{6}$$

The following Hermite-Hadamard's type inequalities for fractional integrals given by M. Iqbal et al. based on [**15**, Lemma 1].

Theorem 1.6 ([**15**]). *Let $f\colon [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality for Riemann-Liouville fractional integrals holds for $0 < \alpha \leq 1$*

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right) + \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+} f(b) + J^\alpha_{b^-} f(a)]\right| \\ \leq \frac{b-a}{2^{\alpha+1}(\alpha+1)}\big(|f'(a)|+|f'(b)|\big).\end{aligned} \tag{7}$$

Theorem 1.7 ([**15**]). *Let $f\colon [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a < b$. If $|f|^q$ $(q = \frac{p}{p-1})$ is η-convex on $[a,b]$ for some fixed $p > 1$, then the following inequality for fractional integrals holds for $0 < \alpha \leq 1$*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+} f(b) + J^\alpha_{b^-} f(a)]\right| \\ &\leq \frac{b-a}{2^{\alpha+1}(\alpha p+1)^{\frac{1}{p}}}\left[\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right].\end{aligned} \tag{8}$$

Theorem 1.8 ([**15**]). *Let $f\colon [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a < b$. If $|f|^q$ where $q = \frac{p}{p-1}$, is convex on $[a,b]$ for some fixed $p > 1$, then the following inequality for fractional integrals holds for $0 < \alpha \leq 1$*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+} f(b) + J^\alpha_{b^-} f(a)]\right| \\ &\quad\leq \frac{b-a}{2^{\alpha+1}(\alpha+1)}\left[\left(\frac{(\alpha+1)|f'(b)|^q+(\alpha+3)|f'(a)|^q)}{2(\alpha+2)}\right)^{\frac{1}{q}}\right. \\ &\quad\quad \left. + \left(\frac{(\alpha+1)|f'(a)|^q+(\alpha+3)|f'(b)|^q)}{2(\alpha+2)}\right)^{\frac{1}{q}}\right].\end{aligned} \tag{9}$$

The main purpose of this paper is to establish a variant of Hermite-Hadamard inequalities for Riemann-Liouville fractional integral using η-convex function (Theorem 2.1). Then we give some interesting results (Theorems 3.2–3.9) connected with the left hand side of Hermite-Hadamard inequalities for Riemann-Liouville fractional integrals using the identities obtained for fractional integrals given in [**15, 25**]. Also we discuss the importance of our results (Remarks 2.2–3.10).

2. Hermite-Hadamard's inequalities for fractional integrals

η-convex version of Hermite-Hadamard's inequalities can be represented in the fractional integral form as follows.

Theorem 2.1. *Suppose that $f\colon [a,b] \to \mathbb{R}$ is an η-convex function such that η is bounded above by M_η, then for $\alpha > 0$, the following inequalities for fractional integrals hold:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) - M_\eta &\le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] \\ &\le \frac{f(a)+f(b)}{2} + \frac{\alpha(\eta(f(a),f(b)) + \eta(f(b),f(a)))}{2(\alpha+1)} \\ &\le \frac{f(a)+f(b)}{2} + \frac{\alpha M_\eta}{\alpha+1}. \end{aligned} \tag{10}$$

Proof. Since $f\colon [a,b] \to \mathbb{R}$ is an η-convex function such that η is bounded above by M_η, so from (3), we have

$$f\left(\frac{x+y}{2}\right) - \frac{M_\eta}{2} \le \frac{f(x)+f(y)}{2} + \frac{M_\eta}{2},$$

where $x, y \in [a,b]$. Let $x = ta + (1-t)b$ and $y = tb + (1-t)a$, then from the above we have

$$f\left(\frac{a+b}{2}\right) - \frac{M_\eta}{2} \le \frac{f(ta+(1-t)b) + f(tb+(1-t)a)}{2} + \frac{M_\eta}{2},$$

$$2f\left(\frac{a+b}{2}\right) - M_\eta \le f(ta+(1-t)b) + f(tb+(1-t)a) + M_\eta. \tag{11}$$

Multiplying both sides of (11) by $t^{\alpha-1}$ and then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned} \frac{2}{\alpha} f\left(\frac{a+b}{2}\right) - \frac{M_\eta}{\alpha} \le &\int_0^1 t^{\alpha-1} f(ta+(1-t)b)\mathrm{d}t \\ &+ \int_0^1 t^{\alpha-1} f(tb+(1-t)a)\mathrm{d}t + \frac{M_\eta}{\alpha}. \end{aligned} \tag{12}$$

Let $ta + (1-t)b = u$ and $(1-t)a + tb = v$, then

$$\begin{aligned} &\int_0^1 t^{\alpha-1} f(ta+(1-t)b)\mathrm{d}t + \int_0^1 t^{\alpha-1} f(tb+(1-t)a)\mathrm{d}t \\ &= \int_b^a \left(\frac{b-u}{b-a}\right)^{\alpha-1} f(u)\frac{\mathrm{d}u}{a-b} + \int_a^b \left(\frac{v-a}{b-a}\right)^{\alpha-1} f(v)\frac{\mathrm{d}v}{b-a} \\ &= \frac{\Gamma(\alpha)}{(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right]. \end{aligned}$$

Therefore, the inequality (12) takes the following shape

$$\frac{2}{\alpha} f\left(\frac{a+b}{2}\right) - \frac{M_\eta}{\alpha} \le \frac{\Gamma(\alpha)}{(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] + \frac{M_\eta}{\alpha}$$

and the rearrangement of terms provides

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\Big[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)\Big]\geq f\Big(\frac{a+b}{2}\Big)-M_\eta, \tag{13}$$

which proves the first inequality in (10). Now we proceed to prove the second inequality

$$f(ta+(1-t)b)\leq f(b)+t\eta(f(a),f(b)), \tag{14}$$
$$f(tb+(1-t)a)\leq f(a)+t\eta(f(b),f(a)). \tag{15}$$

Adding (14), (15) and multiplying both sides by $t^{\alpha-1}$, and then integrating the resulting inequality with respect to t over $[0,1]$, yield the following

$$\begin{aligned}&\int_0^1 t^{\alpha-1}\Big(f(ta+(1-t)b)+f(tb+(1-t)a)\Big)\mathrm{d}t\\ &\leq [f(a)+f(b)]\int_0^1 t^{\alpha-1}\mathrm{d}t\Big(\eta(f(a),f(b))+\eta(f(b),f(a))\Big)\int_0^1 t^\alpha\mathrm{d}t\end{aligned} \tag{16}$$

(by definition of η-convex function). By simplifying inequality (16), we have

$$\begin{aligned}&\frac{\Gamma(\alpha)}{(b-a)^\alpha}\Big[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)\Big]\\ &\qquad\leq\frac{f(a)+f(b)}{\alpha}+\frac{\eta(f(a),f(b))+\eta(f(b),f(a))}{\alpha+1}.\end{aligned} \tag{17}$$

From inequalities (13) and (17), we have

$$\begin{aligned}f\Big(\frac{a+b}{2}\Big)-M_\eta&\leq\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\Big[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)\Big]\\ &\leq\frac{f(a)+f(b)}{2}+\frac{\alpha(\eta(f(a),f(b))+\eta(f(b),f(a)))}{2(\alpha+1)}.\end{aligned}$$

Furthermore, since η is bounded above by M_η, so from the above we can easily obtain the desired result for (10). ☐

Remark 2.2. If f is η-convex with respect to η defined by $\eta(x,y)=x-y$, then (10) reduces to the inequality of Theorem 1.4.

3. Hermite-Hadamard type inequalities for fractional integrals

In order to prove our next result, we need the following Lemma.

Lemma 3.1 ([**25**])**.** *Let $f\colon [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a<b$. If $f' \in L[a,b]$, then the following equality holds*

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\big[J^{\alpha}_{a+}f(b)+J^{\alpha}_{b-}f(a)\big] = \frac{b-a}{2}\int_0^1 [(1-t)^{\alpha}-t^{\alpha}]f'(at+(1-t)b)\mathrm{d}t. \tag{18}$$

Theorem 3.2. *Let $f\colon [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a<b$. If $|f'|$ is an η-convex function on $[a,b]$, then the following inequality for fractional integrals holds*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\big[J^{\alpha}_{a+}f(b)+J^{\alpha}_{b-}f(a)\big]\right| \leq \frac{b-a}{2(\alpha+1)}\big(1-\frac{1}{2^{\alpha}}\big)\big(2|f'(b)|+\eta(|f'(a)|,|f'(b)|)\big). \tag{19}$$

Proof. By using Lemma 3.1 together with the fundamental property of absolute value of real numbers, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\big[J^{\alpha}_{a+}f(b)+J^{\alpha}_{b-}f(a)\big]\right| \\
&\leq \frac{b-a}{2}\int_0^1 |[(1-t)^{\alpha}-t^{\alpha}]||f'(at+(1-t)b)|\mathrm{d}t \\
&\leq \frac{b-a}{2}\int_0^1 |[(1-t)^{\alpha}-t^{\alpha}]|(|f'(b)|+t\eta(|f'(a)|,|f'(b)|))\mathrm{d}t \quad (\text{by } \eta\text{-convexity of } |f'|) \\
&= \frac{b-a}{2}\Bigg[\int_0^{\frac{1}{2}} [(1-t)^{\alpha}-t^{\alpha}](|f'(b)|+t\eta(|f'(a)|,|f'(b)|))\mathrm{d}t \\
&\quad + \int_{\frac{1}{2}}^1 [t^{\alpha}-(1-t)^{\alpha}](|f'(b)|+t\eta(|f'(a)|,|f'(b)|))\Bigg]\mathrm{d}t \\
&= \frac{b-a}{2}\Bigg[|f'(b)|\Big\{\int_0^{\frac{1}{2}}[(1-t)^{\alpha}-t^{\alpha}]\mathrm{d}t\Big\} + \eta(|f'(a)|,|f'(b)|)\Big\{\int_0^{\frac{1}{2}} t[(1-t)^{\alpha}-t^{\alpha}]\mathrm{d}t\Big\} \\
&\quad + |f'(b)|\Big\{\int_{\frac{1}{2}}^1 [t^{\alpha}-(1-t)^{\alpha}]\mathrm{d}t\Big\} + \eta(|f'(a)|,|f'(b)|)\Big\{\int_{\frac{1}{2}}^1 t[t^{\alpha}-(1-t)^{\alpha}]\mathrm{d}t\Big\}\Bigg].
\end{aligned} \tag{20}$$

Hence the R. H. S of (20) is equivalent to

$$|f'(b)|\Big[\frac{1}{\alpha+1}-\frac{1}{2^{\alpha}(\alpha+1)}\Big]+\eta(|f'(a)|,|f'(b)|)\Big[\frac{1}{(\alpha+1)(\alpha+2)}-\frac{1}{2^{\alpha+1}(\alpha+1)}\Big]$$
$$+|f'(b)|\Big[\frac{1}{\alpha+1}-\frac{1}{2^{\alpha}(\alpha+1)}\Big]+\eta(|f'(a)|,|f'(b)|)\Big[\frac{1}{(\alpha+2)}-\frac{1}{2^{\alpha+1}(\alpha+1)}\Big].$$

and furthermore, the simplification of the above terms provides the following

$$\Big|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\big[J^{\alpha}_{a+}f(b)+J^{\alpha}_{b-}f(a)\big]\Big|$$
$$\leq\frac{b-a}{2(\alpha+1)}\big(1-\frac{1}{2^{\alpha}}\big)\big(2|f'(b)|+\eta(|f'(a)|,|f'(b)|)\big).$$

This completes the desired proof of the result. □

Remark 3.3. If $|f'|$ is η-convex with respect to η defined by $\eta(x,y)=x-y$, then (19) reduces to the inequality of Theorem 1.5.

The following lemma is needed in the proof of our next result, which given in [**15**].

Lemma 3.4. *Let $f\colon[a,b]\to\mathbb{R}$ be a differentiable function on (a,b). If $f'\in L^1[a,b]$, then the following identity for Riemann-Liouville fractional integrals holds*

$$f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)]=\frac{b-a}{2}\sum_{k=1}^{4}I_k, \tag{21}$$

where

$$I_1=\int_0^{\frac12}t^{\alpha}f'(tb+(1-t)a)\mathrm{d}t,\qquad I_2=\int_0^{\frac12}(-t^{\alpha})f'(ta+(1-t)b)\mathrm{d}t,$$
$$I_3=\int_{\frac12}^{1}(t^{\alpha}-1)f'(tb+(1-t)a)\mathrm{d}t,\qquad I_4=\int_{\frac12}^{1}(1-t^{\alpha})f'(ta+(1-t)b)\mathrm{d}t.$$

Theorem 3.5. *Let $f\colon[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) with $a<b$. If $|f'|$ is η-convex on $[a,b]$ and $0<\alpha\leq1$, then the following inequality for Riemann-Liouville fractional integrals holds:*

(22)
$$\Big|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)]\Big|$$
$$\leq\frac{b-a}{2^{\alpha+1}(\alpha+1)}\big(|f'(a)|+|f'(b)|+\eta(|f'(a)|,|f'(b)|)+\eta(|f'(b)|,|f'(a)|)\big).$$

Proof. By using the well-known triangular inequality on Lemma 3.4, we have

$$\Big|f\big(\frac{a+b}{2}\big)-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)]\Big|\leq\frac{b-a}{2}\sum_{k=1}^{4}|I_k|$$

and then by applying the η-convexity of $|f'|$, we get

$$\begin{aligned}|I_1| &\leq \int_0^{\frac{1}{2}} t^\alpha |f'(tb+(1-t)a)|\mathrm{d}t \\ &\leq \int_0^{\frac{1}{2}} t^\alpha |f'(a)|\mathrm{d}t + \int_0^{\frac{1}{2}} t^{\alpha+1}\eta(|f'(b)|,|f'(a)|)\mathrm{d}t \\ &= \frac{1}{2^{\alpha+1}(\alpha+1)}|f'(a)| + \frac{1}{2^{\alpha+2}(\alpha+2)}\eta(|f'(b)|,|f'(a)|).\end{aligned}$$

Similarly,

$$|I_2| \leq \frac{1}{2^{\alpha+1}(\alpha+1)}|f'(b)| + \frac{1}{2^{\alpha+2}(\alpha+2)}\eta(|f'(a)|,|f'(b)|).$$

Again using the η-convexity of $|f'|$ and the fact $|t_1^\alpha - t_2^\alpha| \leq |t_1 - t_2|^\alpha$ for all $\alpha \in (0,1]$ and $t_1, t_2 \in [0,1]$, leads to the following

$$|I_3| \leq \frac{1}{2^{\alpha+1}(\alpha+1)}|f'(a)| + \frac{\alpha+3}{2^{\alpha+2}(\alpha+2)(\alpha+1)}\eta(|f'(b)|,|f'(a)|)$$

and similarly

$$|I_4| \leq \frac{1}{2^{\alpha+1}(\alpha+1)}|f'(b)| + \frac{\alpha+3}{2^{\alpha+2}(\alpha+2)(\alpha+1)}\eta(|f'(a)|,|f'(b)|).$$

The addition of the above inequalities take us to the required conclusion. □

Remark 3.6. If $|f'|$ is η-convex with respect to η defined by $\eta(x,y) = x - y$, then (22) reduces to the inequality of Theorem 1.6.

Theorem 3.7. *Let $f\colon [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a < b$. If $|f|^q$ ($q = \frac{p}{p-1}$) is η-convex on $[a,b]$ for some fixed $p > 1$ and $0 < \alpha \leq 1$, then the following inequality for fractional integrals holds*

$$\begin{aligned}&\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \right| \\ &\quad \leq \frac{b-a}{2^{\alpha+1}(\alpha p+1)^{\frac{1}{p}}} \left[\left(\frac{4|f'(a)|^q + \eta(|f'(b)|^q, |f'(a)|^q)}{4} \right)^{\frac{1}{q}} \right. \\ &\quad\quad \left. + \left(\frac{4|f'(a)|^q + \eta(|f'(b)|^q, |f'(a)|^q)}{4} \right)^{\frac{1}{q}} \right]. \qquad (23)\end{aligned}$$

Proof. By using the well-known triangular and Holder inequalities on Lemma 3.4 in turn, we have

$$\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \right| \leq \frac{b-a}{2}\sum_{k=1}^{4}|I_k|,$$

$$
\begin{aligned}
|I_1| &\le \left(\int_0^{\frac{1}{2}} t^{p\alpha}\mathrm{d}t\right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} |f'(tb+(1-t)a)|^q \mathrm{d}t\right)^{\frac{1}{q}} \\
&\le \left(\frac{1}{2^{p\alpha+1}(p\alpha+1)}\right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} |f'(a)|^q \mathrm{d}t + \int_0^{\frac{1}{2}} t\eta(|f'(b)|^q, |f'(a)|^q)\mathrm{d}t\right)^{\frac{1}{q}} \\
&= \left(\frac{1}{2^{p\alpha+1}(p\alpha+1)}\right)^{\frac{1}{p}} \left[\frac{|f'(a)|}{2} + \frac{\eta(|f'(b)|^q, |f'(a)|^q)}{8}\right]^{\frac{1}{q}}
\end{aligned}
$$

(by η-convexity of f). Similarly,

$$
|I_2| \le \left(\frac{1}{2^{p\alpha+1}(p\alpha+1)}\right)^{\frac{1}{p}} \left[\frac{|f'(b)|}{2} + \frac{\eta(|f'(a)|^q, |f'(b)|^q)}{8}\right]^{\frac{1}{q}}
$$

and

$$
|I_3| \le \left(\int_{\frac{1}{2}}^{1} (1-t^\alpha)^p \mathrm{d}t\right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^{1} |f'(tb+(1-t)a)|^q \mathrm{d}t\right)^{\frac{1}{q}}.
$$

Let $\alpha \in (0,1]$ and for all $t_1, t_2 \in [0,1]$, $|t_1^\alpha - t_2^\alpha| \le |t_1 - t_2|^\alpha$, therefore,

$$
\int_{\frac{1}{2}}^{1} (1-t^\alpha)^p \mathrm{d}t \le \int_{\frac{1}{2}}^{1} (1-t)^{p\alpha} \mathrm{d}t = \frac{1}{2^{p\alpha+1}(p\alpha+1)}.
$$

Hence

$$
|I_3| \le \left(\frac{1}{2^{p\alpha+1}(p\alpha+1)}\right)^{\frac{1}{p}} \left[\frac{|f'(a)|}{2} + \frac{\eta(|f'(b)|^q, |f'(a)|^q)}{8}\right]^{\frac{1}{q}},
$$

analogously,

$$
|I_4| \le \left(\frac{1}{2^{p\alpha+1}(p\alpha+1)}\right)^{\frac{1}{p}} \left[\frac{|f'(b)|}{2} + \frac{\eta(|f'(a)|^q, |f'(b)|^q)}{8}\right]^{\frac{1}{q}}.
$$

By adding the above four inequalities, we get the required result. This completes the proof. □

Remark 3.8. If $|f|^{\frac{p}{p-1}}$ is η-convex with respect to η defined by $\eta(x,y) = x - y$, then inequality (23) becomes the inequality obtained in Theorem 1.7.

Theorem 3.9. *Let $f\colon [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a < b$. If $|f|^q$ ($q = \frac{p}{p-1}$) is η-convex on $[a,b]$ for some fixed $p > 1$ and $0 < \alpha \le 1$,* *then the following inequality for fractional integrals holds*

(24)

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+}f(b)+J^\alpha_{b^-}f(a)]\right|$$
$$\leq \frac{b-a}{2^{\alpha+2}(\alpha+1)}\left[\left(\frac{2(\alpha+2)|f'(a)|^q+(\alpha+1)\eta(|f'(b)|^q,|f'(a)|^q)}{2(\alpha+2)}\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{2(\alpha+2)|f'(b)|^q+(\alpha+1)\eta(|f'(a)|^q,|f'(b)|^q)}{2(\alpha+2)}\right)^{\frac{1}{q}}$$
$$+\left(\frac{2(\alpha+2)|f'(a)|^q+(\alpha+3)\eta(|f'(b)|^q,|f'(a)|^q)}{2(\alpha+2)}\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{2(\alpha+2)|f'(b)|^q+(\alpha+3)\eta(|f'(a)|^q,|f'(b)|^q)}{2(\alpha+2)}\right)^{\frac{1}{q}}\right].$$

Proof. By using the triangular and power mean integral inequalities on Lemma 3.4 in turn, we have

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+}f(b)+J^\alpha_{b^-}f(a)]\right|\leq\frac{b-a}{2}\sum_{k=1}^{4}|I_k|,$$

$$|I_1|\leq\left(\int_0^{\frac{1}{2}}t^\alpha\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}t^\alpha|f'(tb+(1-t)a)|^q\mathrm{d}t\right)^{\frac{1}{q}}$$
$$\leq\left(\frac{1}{2^{\alpha+1}(\alpha+1)}\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}t^\alpha|f'(a)|^q\mathrm{d}t+\int_0^{\frac{1}{2}}t^{\alpha+1}\eta(|f'(b)|^q,|f'(a)|^q)\mathrm{d}t\right)^{\frac{1}{q}}$$
$$=\left(\frac{1}{2^{\alpha+1}(\alpha+1)}\right)\left(\frac{2(\alpha+2)|f'(a)|^q+(\alpha+1)\eta(|f'(b)|^q,|f'(a)|^q)}{2(\alpha+2)}\right)^{\frac{1}{q}}$$

(by η-convexity of f). Similarly,

$$|I_2|\leq\left(\frac{1}{2^{\alpha+1}(\alpha+1)}\right)\left(\frac{2(\alpha+2)|f'(b)|^q+(\alpha+1)\eta(|f'(a)|^q,|f'(b)|^q)}{2(\alpha+2)}\right)^{\frac{1}{q}}$$

and

$$|I_3|\leq\left(\frac{2(\alpha+2)|f'(a)|^q+(\alpha+3)\eta(|f'(b)|^q,|f'(a)|^q)}{2(\alpha+2)}\right)^{\frac{1}{q}}.$$

Analogously,

$$|I_4|\leq\left(\frac{2(\alpha+2)|f'(b)|^q+(\alpha+3)\eta(|f'(a)|^q,|f'(b)|^q)}{2(\alpha+2)}\right)^{\frac{1}{q}}.$$

By adding all the above inequalities, we can reach the conclusion. □

Remark 3.10. If $|f|^{\frac{p}{p-1}}$ is η-convex with respect to η defined by $\eta(x,y) = x-y$, then (24) reduces to the inequality of Theorem 1.8.

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