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SELF ADJOINT OPERATOR KOROVKIN TYPE QUANTITATIVE APPROXIMATIONS

G. A. ANASTASSIOU

Abstract. Here we present self adjoint operator Korovkin type theorems via self adjoint operator Shisha-Mond type inequalities. This is a quantitative treatment to determine the degree of self adjoint operator uniform approximation with rates, of sequences of self adjoint operator positive linear operators. We give several applications involving the self adjoint operator Bernstein polynomials.

1. Background

Let A be a selfadjoint linear operator on a complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$. The Gelfand map establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all continuous functions defind on the spectrum of A, denoted $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see, e.g., [**6**, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$, we have

(i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$,

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ (the operation composition is on the right) and $\Phi(\overline{f}) = (\Phi(f))^*$,

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t\in Sp(A)} |f(t)|$,

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f) \qquad \text{for all } f \in C(Sp(A)),$$

and we call it the continuous functional calculus for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e., $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued continuous functions on $Sp(A)$, then the following important property holds

(P) $f(t) \geq g(t)$ for any $t \in Sp(A)$, implies that $f(A) \geq g(A)$ in the operator order of $B(H)$.

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Equivalently, we use (see [**4**, pp. 7–8])the following statement.

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$ with the spectrum $Sp\,(U)$ included in the interval $[m, M]$ for some real numbers $m < M$, and $\{E_\lambda\}_\lambda$ be its spectral family.

Then for any continuous function $f\colon [a,b] \to \mathbb{C}$, where $[m, M] \subset (a, b)$, it is well known that we have the following spectral representation in terms of the Riemann-Stieljes integral

$$\langle f(U)x, y\rangle = \int_{m-0}^{M} f(\lambda)\mathrm{d}(\langle E_\lambda x, y\rangle) \tag{1}$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y\rangle$ is of bounded variation on the interval $[m, M]$, and

$$g_{x,y}\,(m-0) = 0 \qquad \text{and} \qquad g_{x,y}(M) = \langle x, y\rangle$$

for any $x, y \in H$. Furthermore, it is known that $g_x(\lambda) := \langle E_\lambda x, x\rangle$ is increasing and right continuous on $[m, M]$.

In this article, we will often use the formula

$$\langle f(U)x, x\rangle = \int_{m-0}^{M} f(\lambda)\mathrm{d}(\langle E_\lambda x, x\rangle) \qquad \text{for all } x \in H. \tag{2}$$

As a symbol we can write

$$f(U) = \int_{m-0}^{M} f(\lambda)\mathrm{d}E_\lambda. \tag{3}$$

Above, $m = \min\{\lambda \mid \lambda \in Sp(U)\} := \min Sp(U)$, $M = \max\{\lambda \mid \lambda \in Sp(U)\} := \max Sp(U)$. The projections $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ are called the spectral family of A with the properties:

(a) $E_\lambda \le E_{\lambda'}$ for $\lambda \le \lambda'$,

(b) $E_{m-0} = 0_H$ (zero operator), $E_M = 1_H$ (identity operator) and $E_{\lambda+0} = E_\lambda$ for all $\lambda \in \mathbb{R}$.

Furthermore,

$$E_\lambda := \varphi_\lambda(U) \qquad \text{for all } \lambda \in \mathbb{R}, \tag{4}$$

is a projection which reduces U, with

$$\varphi_\lambda(s) := \begin{cases} 1 & \text{for } -\infty < s \le \lambda, \\ 0 & \text{for } \lambda < s < +\infty. \end{cases}$$

The spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ determines the self-adjoint operator U uniquely and vice versa.

For more on the topic, see [**5**, pp. 256–266], and for more details, see there pp. 157–266.

Some more basics are given (we follow [**4**, pp. 1–5]):

Let $(H; \langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbb{C}$. A bounded linear operator A defined on H is selfjoint, i.e., $A = A^*$, iff $\langle Ax, x\rangle \in \mathbb{R}$ for all $x \in H$, and if A is selfadjoint, then

$$\|A\| = \sup_{x \in H : \|x\|=1} |\langle Ax, x\rangle| . \tag{5}$$

Let A, B be selfadjoint operators on H. Then $A \leq B$ iff $\langle Ax, x\rangle \leq \langle Bx, x\rangle$ for all $x \in H$.

In particular, A is called positive if $A \geq 0$.

Denote by

$$\mathcal{P} := \left\{ \varphi(s) := \sum_{k=0}^{n} \alpha_k s^k \mid n \geq 0,\ \alpha_k \in \mathbb{C},\ 0 \leq k \leq n \right\}. \tag{6}$$

If $A \in \mathcal{B}(H)$ (the Banach algebra of all bounded linear operators defined on H, i.e., from H into itself) is selfadjoint and $\varphi(s) \in \mathcal{P}$ has real coefficients, then $\varphi(A)$ is selfadjoint, and

$$\|\varphi(A)\| = \max\{|\varphi(\lambda)|,\ \lambda \in Sp(A)\}. \tag{7}$$

If φ is any function defined on $\mathbb{R}$, we define

$$\|\varphi\|_A := \sup\{|\varphi(\lambda)|, \lambda \in Sp(A)\}. \tag{8}$$

If A is selfadjoint operator on Hilbert space H and φ is continuous and given that $\varphi(A)$ is selfadjoint, then $\|\varphi(A)\| = \|\varphi\|_A$. And if φ is a continuous real valued function so it is $|\varphi|$, then $\varphi(A)$ and $|\varphi|(A) = |\varphi(A)|$ are selfadjoint operators (by [**4**, p. 4, Theorem 7]).

Hence it holds

$$\begin{aligned} \||\varphi(A)|\| &= \||\varphi|\|_A = \sup\{||\varphi(\lambda)||,\ \lambda \in Sp(A)\} \\ &= \sup\{|\varphi(\lambda)|,\ \lambda \in Sp(A)\} = \|\varphi\|_A = \|\varphi(A)\|, \end{aligned}$$

that is,

$$\||\varphi(A)|\| = \|\varphi(A)\| . \tag{9}$$

For a selfadjoint operator $A \in \mathcal{B}(H)$ which is positive, there exists a unique positive selfadjoint operator $B := \sqrt{A} \in \mathcal{B}(H)$ such that $B^2 = A$, that is, $\left(\sqrt{A}\right)^2 = A$. We call B the square root of A.

Let $A \in \mathcal{B}(H)$, then A^*A is selfadjoint and positive. Define the "operator absolute value" $|A| := \sqrt{A^*A}$. If $A = A^*$, then $|A| = \sqrt{A^2}$.

For a continuous real valued function φ, we observe

$$\begin{aligned} &|\varphi(A)| \text{ (the functional absolute value)} \\ &= \int_{m-0}^{M} |\varphi(\lambda)| \mathrm{d}E_\lambda = \int_{m-0}^{M} \sqrt{(\varphi(\lambda))^2} \mathrm{d}E_\lambda = \sqrt{(\varphi(A))^2} \\ &= |\varphi(A)| \text{ (operator absolute value)}, \end{aligned}$$

where A is a selfadjoint operator.

That is, we have

$$|\varphi(A)| \text{ (functional absolute value)} = |\varphi(A)| \text{ (operator absolute value)}. \tag{10}$$

The next comes from [**3**, p. 3].

We say that a sequence $\{A_n\}_{n=1}^{\infty} \subset \mathcal{B}(H)$ converges uniformly to A (convergence in norm) iff

$$\lim_{n\to\infty} \|A_n - A\| = 0, \tag{11}$$

and we denote it as $\lim_{n\to\infty} A_n = A$.

We will be using Hölder's-McCarthy's inequality, 1967 ([**7**]): Let A be a selfadjoint positive operator on a Hilbert space H. Then

$$\langle A^r x, x\rangle \le \langle Ax, x\rangle^r \tag{12}$$

for all $0 < r < 1$ and $x \in H\colon \|x\| = 1$.

Let $A, B \in \mathcal{B}(H)$, then

$$\|AB\| \le \|A\| \, \|B\| \tag{13}$$

by Banach algebra property.

2. Auxiliary Results

All functions here are real valued.

Let $L : C([a,b]) \to C([a,b])$, $a < b$, be a linear operator. If $f, g \in C([a,b])$ such that $f \ge g$ implies $L(f) \ge L(g)$, we call L a positive linear operator. It is well-known that a positive linear operator is a bounded linear operator.

We need the following lemma.

Lemma 1. *Let $L\colon C([a,b]) \to C([a,b])$ be a positive linear operator, $0<\alpha\le 1$. Then the function*

$$g(x) := (L(|\cdot - x|^{\alpha}))(x) \tag{14}$$

is continuous in $x \in [a,b]$.

Proof. Let $x_n \to x$, $x_n, x \in [a,b]$. We notice that

$$\begin{aligned} &(L(|\cdot - x_n|^{\alpha}))(x_n) - (L(|\cdot - x|^{\alpha}))(x) \\ &= (L(|\cdot - x_n|^{\alpha}))(x_n) - (L(|\cdot - x|^{\alpha}))(x_n) + (L(|\cdot - x|^{\alpha}))(x_n) - (L(|\cdot - x|^{\alpha}))(x) \\ &= (L(|\cdot - x_n|^{\alpha} - |\cdot - x|^{\alpha}))(x_n) + [(L(|\cdot - x|^{\alpha}))(x_n) - (L(|\cdot - x|^{\alpha}))(x)]. \end{aligned} \tag{15}$$

Therefore, it holds

$$|(L(|\cdot - x|^{\alpha}))(x_n) - (L(|\cdot - x|^{\alpha}))(x)|$$

$$\le \|L(|\cdot - x|^{\alpha}) - |\cdot - x|^{\alpha})\|_{\infty} + |(L(|\cdot - x|^{\alpha}))(x_n) - (L(|\cdot - x|^{\alpha}))(x)| \tag{16}$$

$$\le \|L\| \, \||\cdot - x_n|^{\alpha} - |\cdot - x|^{\alpha}\|_{\infty} + |(L(|\cdot - x|^{\alpha}))(x_n) - (L(|\cdot - x|^{\alpha}))(x)| =: (\xi_1). \tag{17}$$

Notice that

$$|t - x_n| = |t - x + x - x_n| \le |t - x| + |t - x_n|,$$

hence

$$|t - x_n|^{\alpha} \le (|t - x| + |x - x_n|)^{\alpha} \le |t - x|^{\alpha} + |x - x_n|^{\alpha}.$$

That is,

$$|t - x_n|^\alpha - |t - x|^\alpha \leq |x - x_n|^\alpha. \tag{18}$$

Similarly,

$$|t - x| = |t - x_n + x_n - x| \leq |t - x_n| + |x_n - x|,$$

hence

$$|t - x|^\alpha \leq |t - x_n|^\alpha + |x_n - x|^\alpha$$

and

$$|t - x|^\alpha - |t - x_n|^\alpha \leq |x_n - x|^\alpha. \tag{19}$$

Consequently, it holds

$$||t - x_n|^\alpha - |t - x|^\alpha| \leq |x_n - x|^\alpha \tag{20}$$

and

$$\left\| |\cdot - x_n|^\alpha - |\cdot - x|^\alpha \right\|_\infty \leq |x_n - x|^\alpha. \tag{21}$$

Therefore, we get

$$(\xi_1) \leq \|L\| \, |x_n - x|^\alpha + |(L(| \cdot - x|^\alpha))(x_n) - (L(| \cdot - x|^\alpha))(x)| \to 0 \tag{22}$$

as $x_n \to x$, and by continuity of $(L(|\cdot - x|^\alpha))$, as $n \to \infty$, proving the claim. □

We make the following remark.

Remark 2. Let L be a positive linear operator from $C([a,b])$ into itself. Then

$$(t - x)^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} t^k x^{n-k}, \qquad t, x \in [a,b]. \tag{23}$$

Hence we get

$$(L((t - x)^n)) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} x^{n-k} L\left(t^k\right) \tag{24}$$

and

$$(L((t - x)^n))(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} x^{n-k} \left(L\left(t^k\right)\right)(x) \tag{25}$$

for all $x \in [a,b]$. Clearly we have that $(L((\cdot - x)^n))(x)$ is continuous in x for all $n \in \mathbb{N}$. So that $|(L((\cdot - x)^n))(x)|$ is continuous in $x \in [a,b]$.

It follows

Lemma 3. *Let L be a positive linear operator from $C([a,b])$ into itself. The function $(L(|\cdot - x|^m))(x)$ is continuous in $x \in [a,b]$ for any $m \in \mathbb{N}$.*

Proof. Let $x_n \to x$, $x_n, x \in [a,b]$, as $n \to \infty$. We observe that

$$\|L(|\cdot - x_n|^m - |\cdot - x|^m)\|_\infty \le \|L\| \, \||\cdot - x_n|^m - |\cdot - x|^m\|_\infty . \tag{26}$$

We notice that $(t, x_n, x \in [a,b])$

$$\begin{aligned} &||t-x_n|^m - |t-x|^m| \\ &= ||t-x_n| - |t-x|| \left\{ |t-x_n|^{m-1} + |t-x_n|^{m-2}|t-x| \right. \\ &\qquad \left. + |t-x_n|^{m-3}|t-x|^2 + \cdots + |t-x_n| \, |t-x|^{m-2} + |t-x|^{m-1} \right\} \\ &\le ||t-x_n| - |t-x|| \, m (b-a)^{m-1} \le |x_n - x| \, m (b-a)^{m-1} . \end{aligned} \tag{27}$$

Hence, we get

$$\||\cdot - x_n|^m - |\cdot - x|^m\|_\infty \le |x_n - x| \, m (b-a)^{m-1} . \tag{28}$$

Similarly, as in the proof of Lemma 1 (instead of α we set m), we obtain

$$|(L(|\cdot - x_n|^m))(x_n) - (L(|\cdot - x|^m))(x)|$$

(29)

$$\le \|L\| \, \||\cdot - x_n|^m - |\cdot - x|^m\|_\infty + |(L(|\cdot - x|^m))(x_n) - (L(|\cdot - x|^m))(x)|$$

(30)

$$\le \|L\| \, |x_n - x| \, m (b-a)^{m-1} + |(L(|\cdot - x|^m))(x_n) - (L(|\cdot - x|^m))(x)| \to 0,$$

proving the claim. □

We also need next lemma.

Lemma 4. *Let L be a positive linear operator from $C([a,b])$ into itself. The function $\Big(L\big(|\cdot - x|^{n+\alpha}\big)\Big)(x)$ is continuous in $x \in [a,b]$, $n \in \mathbb{N}$, $0 < \alpha \le 1$.*

Proof. Let $0 \le A, B \le b-a$, and $\gamma(z) := z^r$, $r > 1$, with $\gamma\colon [0, b-a] \to \mathbb{R}$, i.e., $\gamma(A) = A^r$, $\gamma(B) = B^r$. Then $\gamma'(z) = r z^{r-1}$ and $\|\gamma'\|_\infty = r(b-a)^{r-1}$.

Hence it holds

$$|A^r - B^r| \le r (b-a)^{r-1} |A - B| . \tag{31}$$

Let $t, x_m, x \in [a,b]$ with $x_m \to x$, as $m \to \infty$.

Therefore (for $r = n + \alpha > 1$, $A = |t - x_m|$, $B = |t - x|$) we get that

$$\begin{aligned} \left| |t - x_m|^{n+\alpha} - |t-x|^{n+\alpha} \right| &\le (n+\alpha)(b-a)^{n+\alpha-1} \, ||t - x_m| - |t-x|| \\ &\le (n+\alpha)(b-a)^{n+\alpha-1} \, |x_m - x| . \end{aligned} \tag{32}$$

So that it holds

$$\big\| |t - x_m|^{n+\alpha} - |t-x|^{n+\alpha} \big\|_\infty \le (n+\alpha)(b-a)^{n+\alpha-1} |x_m - x| \to 0. \tag{33}$$

We have that

$$\begin{aligned} &\Big| \big(L\big(|\cdot - x_m|^{n+\alpha}\big)\big)(x_m) - \big(L\big(|\cdot - x|^{n+\alpha}\big)\big)(x) \Big| \\ &\le \|L\| \big\| |\cdot - x_m|^{n+\alpha} - |\cdot - x|^{n+\alpha} \big\|_\infty \\ &\quad + \Big| \big(L\big(|\cdot - x|^{n+\alpha}\big)\big)(x_m) - \big(L\big(|\cdot - x|^{n+\alpha}\big)\big)(x) \Big| \end{aligned}$$

$$\leq \|L\| \, |x_m - x| \, (n+\alpha) \, (b-a)^{n+\alpha-1}$$
$$+ \left| \left(L\left(|\cdot - x|^{n+\alpha} \right)\right)(x_m) - \left(L\left(|\cdot - x|^{n+\alpha} \right)\right)(x) \right| \to 0, \tag{34}$$

proving the claim. □

We make the following remark.

Remark 5. Let L be a positive linear operator from $C([a,b])$ into itself, $a < b$. By Riesz representation theorem, for each $s \in [a,b]$, there exists a positive finite measure μ_s on $[a,b]$ such that

$$(L(f))(s) = \int_{[a,b]} f(t) \mathrm{d}\mu_s(t) \qquad \text{for all } f \in C([a,b]). \tag{35}$$

Therefore, $(k = 1, \ldots, n,\ 0 < \alpha \leq 1)$

$$\left| \left(L(\cdot - s)^k \right)(s) \right| = \left| \int_{[a,b]} (\lambda - s)^k \, \mathrm{d}\mu_s(\lambda) \right|$$
$$\leq \int_{[a,b]} |\lambda - s|^k \, \mathrm{d}\mu_s(\lambda) \text{ (by Hölder's inequality)}$$
$$\leq \left(\int_{[a,b]} 1 \mathrm{d}\mu_s(\lambda) \right)^{\frac{((n+\alpha)-k)}{n+\alpha}} \left(\int_{[a,b]} \left| \lambda - s \right|^{(n+\alpha)} \mathrm{d}\mu_s(\lambda) \right)^{\frac{k}{n+\alpha}} \tag{36}$$
$$= ((L(1))(s))^{\left(\frac{n+\alpha-k}{n+\alpha}\right)} \left(\left(L\left(|\cdot - s|^{n+\alpha} \right)\right)(s) \right)^{\frac{k}{n+\alpha}}.$$

We have proved that $(k = 1, \ldots, n;\ 0 < \alpha \leq 1)$

$$\left| \left(L(\cdot - s)^k \right)(s) \right| \leq ((L(1))(s))^{\left(\frac{n+\alpha-k}{n+\alpha}\right)} \left(\left(L\left(|\cdot - s|^{n+\alpha} \right)\right)(s) \right)^{\frac{k}{n+\alpha}} \tag{37}$$

for all $s \in [a,b]$.

We mention the following theorem.

Theorem 6 (Shisha and Mond [**8**], 1968)**.** *Let $[a,b] \subset \mathbb{R}$ be a compact interval. Let $\{L_n\}_{n \in \mathbb{N}}$ be a sequence of positive linear operators acting from $C([a,b])$ into itself. For $n = 1, 2, \ldots,$ suppose $L_n(1)$ is bounded. Let $f \in C([a,b])$. Then for $n = 1, 2, \ldots,$ we have*

$$\|L_n f - f\|_\infty \leq \|f\|_\infty \|L_n 1 - 1\|_\infty + \|L_n(1) + 1\|_\infty \omega_1(f, \mu_n), \tag{38}$$

where

$$\mu_n := \left\| \left(L_n \left((t-x)^2 \right) \right)(x) \right\|_\infty^{\frac{1}{2}} \tag{39}$$

with

$$\omega_1(f, \delta) := \sup_{\substack{x,y \in [a,b]: \\ |x-y| \leq \delta}} |f(x) - f(y)|, \qquad \delta > 0, \tag{40}$$

and $\|\cdot\|_\infty$ stands for the sup-norm over $[a,b]$. In particular, if $L_n(1) = 1$, then (38) *becomes*

$$\|L_n(f) - f\|_\infty \leq 2\omega_1(f, \mu_n). \tag{41}$$

Note. (i) In forming μ_n^2, x is kept fixed, however, t forms the functions t, t^2 on which L_n acts.

(ii) One can easily find for $n = 1, 2, \dots,$

$$\mu_n^2 \le \left\| \left(L_n\left(t^2\right)\right)(x) - x^2 \right\|_\infty + 2c\|(L_n(t))(x) - x\|_\infty + c^2\|(L_n(1))(x) - 1\|_\infty, \tag{42}$$

where $c := \max(|a|, |b|)$.

So, if the Korovkin's assumptions are fulfilled, i.e., if $L_n\left(\mathrm{id}^2\right) \overset{u}{\to} \mathrm{id}^2$, $L_n(\mathrm{id}) \overset{u}{\to} \mathrm{id}$ and $L_n(1) \overset{u}{\to} 1$ as $n \to \infty$, where id is the identity map and u is the uniform convergence, then $\mu_n \to 0$, and then $\omega_1(f, \mu_n) \to 0$ as $n \to +\infty$, and, we obtain from (38) that $\|L_n f - f\|_\infty \to \infty$, i.e., $L_n f \overset{u}{\to} f$, as $n \to \infty$ for all $f \in C([a, b])$.

Clearly the assumption $\|L_n(1) - 1\|_\infty \to 0$ as $n \to \infty$, implies $\|L_n(1)\|_\infty \le \rho$ for all $n \in \mathbb{N}$ and for some $\rho > 0$.

Indeed we can write $L_n(1) = L_n(1) - 1 + 1$, hence

$$\|L_n(1)\|_\infty \le \|L_n(1) - 1\|_\infty + 1 \le \rho,$$

proving the boundedness of $L_n(1)$.

3. Main Results

Here we derive self adjoint operator-Korovkin type theorems via operator-Shisha-Mond type inequalities. This is a quantitative approach studying the degree of operator-uniform approximation with rates of sequences of operator-positive linear operators in the operator order of $\mathcal{B}(H)$.

In all of our results here we give direct self contained proofs by the use of spectral representation theorem. We are inspired by [**1**].

Our setting here follows:

Let A be a selfadjoint operator on the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, $I = [a, b]$, $a < b$, a, b real numbers, with $[m, M] \subset \overset{\circ}{I} = (a, b)$ (the interior of I). Let $f \in C(I)$, where $C(I)$ denotes all the continuous functions from I into $\mathbb{R}$. Let $n \in \mathbb{N}$ and $\{L_n\}_{n \in \mathbb{N}}$ be a sequence of positive linear operators from $C(I)$ into itself.

We give the following theorems

Theorem 7. *It holds*

$$\|(L_n(f))(A) - f(A)\| \le \|L_n f - f\|_{\infty,[a,b]} \qquad \text{for all } n \in \mathbb{N}. \tag{43}$$

If $L_n 1 \overset{u}{\to} 1$, $L_n(\mathrm{id}) \overset{u}{\to} \mathrm{id}$, $L_n\left(\mathrm{id}^2\right) \overset{u}{\to} \mathrm{id}^2$, *then* $\|L_n f - f\|_{\infty,[a,b]} \underset{[a,b]}{\to} 0$ *(see Theorem 6 and Note).*

By (43), *we get* $\|(L_n(f))(A) - f(A)\| \to 0$ *as* $n \to \infty$, *i.e.,*

$$\lim_{n\to\infty} (L_n(f))(A) = f(A) \text{ uniformly} \qquad \text{for all } f \in C(I).$$

Proof. Here we use the spectral representation theorem.
For any $x \in H : \|x\| = 1$, we have that $\int_{m-0}^{M} d\langle E_\lambda x, x\rangle = 1$, see (1) and (2).
We observe that

$$\begin{aligned}
\|(L_n(f))(A) - f(A)\| &= \sup_{\substack{x \in H \\ \|x\|=1}} |\langle((L_n(f))(A) - f(A))x, x\rangle| \qquad (44)\\
&= \sup_{\|x\|=1} \left| \int_{m-0}^{M} ((L_n(f))(\lambda) - f(\lambda)) \mathrm{d}\langle E_\lambda x, x\rangle \right| \\
&\leq \sup_{\|x\|=1} \int_{m-0}^{M} |(L_n(f))(\lambda) - f(\lambda)| \mathrm{d}\langle E_\lambda x, x\rangle \\
&\leq \|L_n(f) - f\|_{\infty,[a,b]} \left(\sup_{\|x\|=1} \int_{m-0}^{M} \mathrm{d}\langle E_\lambda x, x\rangle \right) \\
&= \|L_n(f) - f\|_{\infty,[a,b]} \cdot 1 = \|L_n(f) - f\|_{\infty,[a,b]}, \qquad (45)
\end{aligned}$$

proving the claim. □

Next we give special Korovkin type quantitative convergence results for a self adjoint operator A.

Theorem 8. *Let $f\colon [a, b] \to \mathbb{R}$. Assume that*

$$|f(t) - f(s)| \leq K|t - s|^\alpha \qquad \text{for all } t, s \in [a, b], \tag{46}$$

where $0 < \alpha \leq 1$, $K > 0$.
Assume that

$$\|L_n(1)\|_{\infty,[a,b]} \leq \mu, \quad \mu > 0 \qquad \text{for all } n \in \mathbb{N}, \tag{47}$$

and set

$$\rho := \mu^{\frac{2-\alpha}{2}}. \tag{48}$$

Set also $c := \max(|a|, |b|)$.
Then it holds

$$\begin{aligned}
\|(L_n(f))(A) - f(A)\| &\leq \|f(A)\| \|(L_n(1))(A) - 1_H\| \\
&\quad + K\rho \big[\|(L_n(\mathrm{id}^2))(A) - A^2\| + 2c\|(L_n(\mathrm{id}))(A) - A\| \\
&\quad + c^2 \|(L_n(1))(A) - 1_H\| \big]^{\frac{\alpha}{2}},
\end{aligned} \tag{49}$$

for all $n \in \mathbb{N}$.
If we assume that $(L_n(\mathrm{id}^2))(A) \to A^2$, $(L_n(\mathrm{id}))(A) \to A$, $(L_n(1))(A) \to 1_H$, uniformly as $n \to \infty$, we get $(L_n(f))(A) \to f(A)$ uniformly, as $n \to \infty$ for all $f \in C([a, b])$ fulfilling (46).

Proof. Here we consider the sequence of positive linear operators $\{L_n\}_{n\in\mathbb{N}}$ from $C([a, b])$ into itself. By Riesz representation theorem, we have that

$$(L_n(f))(s) = \int_{[a,b]} f(t)\mu_{ns}(\mathrm{d}t) \tag{50}$$

for all $f \in C([a,b])$; where μ_{ns} is a non-negative finite measure for all $s \in [a,b]$ and all $n \in \mathbb{N}$.

We can write the following

$$(51)\quad \begin{aligned}(L_n(f))(s) - f(s) &= (L_n(f))(s) - f(s) + f(s)(L_n(1))(s) - f(s)(L_n(1))(s) \\ &= \int_{[a,b]} (f(t) - f(s))\, \mu_{ns}(\mathrm{d}t) + f(s)((L_n(1))(s) - 1).\end{aligned}$$

By the assumption (46), we obtain

$$(52)\quad \begin{aligned}|(L_n(f))(s) - f(s)| &\le \int_{[a,b]} |f(t) - f(s)|\, \mu_{ns}(\mathrm{d}t) + |f(s)|\, |(L_n(1))(s) - 1| \\ &\le K \int_{[a,b]} |t-s|^{\alpha}\, \mu_{ns}(\mathrm{d}t) + |f(s)|\, |(L_n(1))(s) - 1| \\ &= K\left(L_n\left(|\cdot - s|^{\alpha}\right)(s)\right) + |f(s)|\, |(L_n(1))(s) - 1|.\end{aligned}$$

That is, we get

$$(53)\quad |(L_n(f))(s) - f(s)| \le |f(s)|\, |(L_n(1))(s) - 1| + K\left(L_n\left(|\cdot - s|^{\alpha}\right)(s)\right).$$

Notice that by Hölder's inequality

$$(54)\quad \begin{aligned}\left(L_n\left(|\cdot - s|^{\alpha}\right)(s)\right) &= \int_{[a,b]} |t-s|^{\alpha}\, \mu_{ns}(\mathrm{d}t) \\ &\le ((L_n(1))(s))^{\frac{2-\alpha}{2}} \left(\int_{[a,b]} (t-s)^2 \mu_{ns}(\mathrm{d}t)\right)^{\frac{\alpha}{2}}.\end{aligned}$$

Hence it holds

$$(55)\quad \begin{aligned}|(L_n(f))(s) - f(s)| \le\ & |f(s)|\, |(L_n(1))(s) - 1| \\ &+ K\, ((L_n(1))(s))^{\frac{2-\alpha}{2}} \left(\left(L_n\left((\cdot - s)^2\right)\right)(s)\right)^{\frac{\alpha}{2}}.\end{aligned}$$

By the assumption (47) and (48), we get

$$(56)\quad \begin{aligned}|(L_n(f))(s) - f(s)| \le\ & |f(s)|\, |(L_n(1))\, s) - 1| \\ &+ K\rho\left(\left(L_n\left((\cdot - s)^2\right)\right)(s)\right)^{\frac{\alpha}{2}}\end{aligned}$$

for all $s \in [a,b]$.

We see that

$$(57)\quad \begin{aligned}&\left(L_n\left((t-s)\right)^2\right)(s) = \left(L_n\left(t^2 - 2ts + s^2\right)\right)(s) \\ &= \left(L_n\left(t^2\right)\right)(s) - 2s\left(L_n(t)\right)(s) + s^2\left(L_n(1)\right)(s) \\ &= \left(\left(L_n\left(t^2\right)\right)(s) - s^2\right) - 2s\left(\left(L_n(t)\right)(s) - s\right) + s^2\left((L_n(1))(s) - 1\right).\end{aligned}$$

Calling $c := \max(|a|, |b|)$, we obtain

$$(58)\quad \begin{aligned}\left(L_n\left((t-s)^2\right)\right)(s) \le\ & \left|\left(L_n\left(t^2\right)\right)(s) - s^2\right| \\ &+ 2c\left|\left(L_n(t)\right)(s) - s\right| + c^2\left|(L_n(1))(s) - 1\right|.\end{aligned}$$

Therefore,

$$|(L_n(f))(s) - f(s)| \leq |f(s)||(L_n(1))(s) - 1| + K\rho\Big[|\left(L_n\left(t^2\right)\right)(s) - s^2| + 2c\,|(L_n(t))(s) - s| + c^2|(L_n(1))(s) - 1|\Big]^{\frac{\alpha}{2}} \tag{59}$$

for all $s \in [a, b]$ and all $n \in \mathbb{N}$.

Here we take $x \in H : \|x\| = 1$. We find that

$$|\langle((L_n(f))(A) - f(A))\,x, x\rangle| = \left|\int_{m-0}^{M}((L_n(f))(s) - f(s))\mathrm{d}\langle E_s x, x\rangle\right| \tag{60}$$

$$\leq \int_{m-0}^{M}|(L_n(f))(s) - f(s)|\,\mathrm{d}\langle E_s x, x\rangle$$

$$\leq \int_{m-0}^{M}|f(s)|\,|(L_n(1))(s) - 1|\,\mathrm{d}\langle E_s x, x\rangle + K\rho\int_{m-0}^{M}\Big[\,\big|\left(L_n\left(t^2\right)\right)(s) - s^2\big| + 2c\,|(L_n(t))(s) - s| + c^2|(L_n(1))(s) - 1|\Big]^{\frac{\alpha}{2}}\cdot\mathrm{d}\langle E_s x, x\rangle \tag{61}$$

$$\overset{(10)}{=} \Big\langle(|f(A)|\,|(L_n(1))(A) - 1_H|)x, x\Big\rangle + K\rho\Big\langle\big[\,|(L_n(t^2))(A) - A^2| + 2c|(L_n(t))(A) - A| + c^2|(L_n(1))(A) - 1_H|\big]^{\frac{\alpha}{2}}x, x\Big\rangle \tag{62}$$

(by Hölder-McCarthy inequality (12) and (5), (9), and (13))

$$\leq \|f(A)\|\,\|(L_n(1))(A) - 1_H\| + K\rho\cdot$$

$$\big(\big\langle\big[|(L_n(\mathrm{id}^2))(A) - A^2| + 2c\,|(L_n(\mathrm{id}))(A) - A| + c^2\,|(L_n(1))(A) - 1_H|\,\big]x, x\big\rangle\big)^{\frac{\alpha}{2}} \tag{63}$$

$$= \|f(A)\|\,\|(L_n(1))(A) - 1_H\| + K\rho\big(\big\langle|(L_n(\mathrm{id}^2))(A) - A^2|x, x\big\rangle + 2c\,\langle|(L_n(\mathrm{id}))(A) - A|\,x, x\rangle + c^2\,\langle|(L_n(1))(A) - 1_H|\,x, x\rangle\,\big)^{\frac{\alpha}{2}} \tag{64}$$

$$\leq \|f(A)\|\,\|(L_n(1))(A) - 1_H\| + K\rho\Big(\,\big\|(L_n(\mathrm{id}^2))(A) - A^2\big\| + 2c\,\|(L_n(\mathrm{id}))(A) - A\| + c^2\,\|(L_n(1))(A) - 1_H\|\,\Big)^{\frac{\alpha}{2}}, \tag{65}$$

proving (49). □

It follows a related result.

Theorem 9. *Let $f\colon [a, b] \to \mathbb{R}$. Assume that*

$$|f(t) - f(s)| \leq K|t - s|^{\alpha} \qquad \text{for all } t, s \in [a, b], \tag{66}$$

where $0 < \alpha \leq 1$, $K > 0$.

Then

$$\|(L_n(f))(A) - f(A)\| \leq \|f(A)\| \|(L_n(1))(A) - 1_H\| + K \|(L_n(|\cdot - A|^{\alpha}))(A)\| \qquad \text{for all } n \in \mathbb{N}. \tag{67}$$

Clearly, if $(L_n(1))(A) \to 1_H$ *and* $(L_n(|\cdot - A|^{\alpha}))(A) \to 0_H$ *uniformly as* $n \to \infty$, *then by* (67), *we get that* $(L_n(f))(A) \to f(A)$ *uniformly as* $n \to \infty$.

Proof. We have established (53) from which follows:

$$|(L_n(f))(s) - f(s)| \leq |f(s)||(L_n(1))(s) - 1| + K((L_n(|\cdot - s|^{\alpha}))(s)). \tag{68}$$

Consider $x \in H\colon \|x\| = 1$. Then

$$|\langle ((L_n(f))(A) - f(A))x, x\rangle| = \left| \int_{m-0}^{M} ((L_n(f))(s) - f(s))\, \mathrm{d}\langle E_s x, x\rangle \right| \tag{69}$$

$$\leq \int_{m-0}^{M} |(L_n(f))(s) - f(s)| \mathrm{d}\langle E_s x, x\rangle$$

$$\overset{(68)}{\leq} \int_{m-0}^{M} |f(s)||(L_n(1))(s) - 1| \mathrm{d}\langle E_s x, x\rangle + K \int_{m-0}^{M} ((L_n(|\cdot - s|^{\alpha}))(s))\, \mathrm{d}\langle E_s x, x\rangle$$

$$= \langle (|f(A)||(L_n(1))(A) - 1_H|)x, x\rangle + K \langle ((L_n(|\cdot - A|^{\alpha}))(A))x, x\rangle$$

$$\leq \|f(A)\| \|(L_n 1))(A) - 1_H\| + K \|(L_n(|\cdot - A|^{\alpha}))(A)\|, \tag{70}$$

proving the claim. □

We continue with next statement.

Theorem 10. *Let* $\{L_N\}_{N\in\mathbb{N}}$ *be a sequence of positive linear operators from* $C([a,b])$ *into itself. Let* $f\colon [a,b] \to \mathbb{R}$ *be such that* $f^{(n)} \in C([a,b])$ *and*

$$|f^{(n)}(z) - f^{(n)}(s)| \leq K|z - s|^{\alpha}, \qquad K > 0, \tag{71}$$

$0 < \alpha \leq 1$ *and for all* $z, s \in [a,b]$.

Then it holds

$$\|(L_N(f))(A) - f(A)\| \leq \|f(A)\| \|(L_N(1))(A) - 1_H\| + \sum_{k=1}^{n} \frac{1}{k!} \left\|f^{(k)}(A)\right\| \left\|\left(L_N\left((\cdot - A)^k\right)\right)(A)\right\| + \frac{K}{\prod_{i=1}^{n}(i+\alpha)} \left\|\left(L_N\left(|\cdot - A|^{n+\alpha}\right)\right)(A)\right\| \tag{72}$$

for all $N \in \mathbb{N}$. *Assuming further that*

$$\|L_N(1)\|_{\infty} \leq \mu \qquad \text{for all } N \in \mathbb{N},\ \mu > 0, \tag{73}$$

and $(L_N(1))(A) \to 1_H$, $(L_N(|\cdot - A|^{n+\alpha}))(A) \to 0_H$ *uniformly as* $N \to \infty$, *we get that* $(L_N(f))(A) \to f(A)$ *uniformly as* $N \to \infty$ *for all* $f \in C^n([a,b])$, *fulfilling* (71).

Proof. Here $f : [a,b] \to \mathbb{R}$ is such that $f^{(n)} \in C([a,b])$. Let $s \in [a,b]$, $n \in \mathbb{N}$. Then

$$f(t) = \sum_{k=0}^{n} \frac{f^{(k)}s)}{k!}(t-s)^k + R_n(t,s), \tag{74}$$

where

$$R_n(t,s) = \frac{1}{(n-1)!}\int_s^t \left[f^{(n)}(z) - f^{(n)}(s)\right](t-z)^{n-1}\,\mathrm{d}z \tag{75}$$

for all $t, s \in [a,b]$.

Under the assumption (71), next we estimate $R_n(t,s)$. Let $t \geq s$, then

$$\begin{aligned}
&\Big|\int_s^t \left[f^{(n)}(z) - f^{(n)}(s)\right](t-z)^{n-1}\,\mathrm{d}z\Big| \\
&\leq \int_s^t |f^{(n)}(z) - f^{(n)}(s)|\,(t-z)^{n-1}\,\mathrm{d}z \\
&\leq K\int_s^t |z-s|^{\alpha}(t-z)^{n-1}\,\mathrm{d}z \\
&= K\int_s^t (t-z)^{n-1}(z-s)^{(\alpha+1)-1}\mathrm{d}z = K\frac{\Gamma(n)\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)}(t-s)^{n+\alpha}.
\end{aligned} \tag{76}$$

So, when $t \geq s$, we get

$$\Big|\int_s^t \left[f^{(n)}(z) - f^{(n)}(s)\right](t-z)^{n-1}\mathrm{d}z\Big| \leq K\frac{\Gamma(n)\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)}(t-s)^{n+\alpha}. \tag{77}$$

Let $t \leq s$, then

$$\begin{aligned}
&\Big|\int_s^t \left[f^{(n)}(z) - f^{(n)}(s)\right](t-z)^{n-1}\mathrm{d}z\Big| \\
&= \Big|\int_t^s \left[f^{(n)}(z) - f^{(n)}(s)\right](z-t)^{n-1}\mathrm{d}z\Big| \\
&\leq \int_t^s \left|f^{(n)}(z) - f^{(n)}(s)\right|(z-t)^{n-1}\mathrm{d}z \leq K\int_t^s |z-s|^{\alpha}(z-t)^{n-1}\,\mathrm{d}z \\
&= K\int_t^s (s-z)^{(\alpha+1)-1}(z-t)^{n-1}\,\mathrm{d}z = K\frac{\Gamma(\alpha+1)\Gamma(n)}{\Gamma(n+1+\alpha)}(s-t)^{n+\alpha}.
\end{aligned} \tag{78}$$

We have proved that

$$\begin{aligned}
&\Big|\int_s^t \left[f^{(n)}(z) - f^{(n)}(s)\right](t-z)^{n-1}\mathrm{d}z\Big| \leq K\frac{\Gamma(n)\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)}|t-s|^{n+\alpha} \\
&= K\frac{(n-1)!}{(1+\alpha)(2+\alpha)\ldots(n+\alpha)}|t-s|^{n+\alpha} \qquad \text{for all } t,s \in [a,b].
\end{aligned} \tag{79}$$

Hence it holds

$$
\begin{aligned}
|R_n(t,s)| &= \frac{1}{(n-1)!}\Big|\int_s^t \left[f^{(n)}(z)-f^{(n)}(s)\right](t-z)^{n-1}\mathrm{d}z\Big| \\
&\le \frac{K}{(1+\alpha)(2+\alpha)\dots(n+\alpha)}|t-s|^{n+\alpha} \qquad \text{for all } t,s\in[a,b].
\end{aligned} \tag{80}
$$

Let now L_N, $N\in\mathbb{N}$, be a sequence of positive linear operators from $C([a,b])$ into itself. Then we get

$$
\begin{aligned}
|\left(L_N\left(R_n\left(\cdot,s\right)\right)\right)(s)| &\le \left(L_N\left(\|\left(R_n\left(\cdot,s\right)\right)|\right)\right)(s) \\
&\le \frac{K}{\prod\limits_{i=1}^{n}(i+\alpha)}\left(L_N\left(|\cdot-s|^{n+\alpha}\right)\right)(s) \quad \text{for all } s\in[a,b].
\end{aligned} \tag{81}
$$

Above, $\left(L_N\left(|\cdot-s|^{n+\alpha}\right)\right)(s)$ is continuous in $s\in[a,b]$ (by Lemma 4).

We can rewrite (74) as follows

$$
f(\cdot)-f(s)=\sum_{k=1}^{n}\frac{f^{(k)}(s)}{k!}(\cdot-s)^k+R_n(\cdot,s), \tag{82}
$$

and we notice that $R_n(\cdot,s)\in C([a,b])$, here we keep s fixed.
Hence we find

$$
\begin{aligned}
&(L_N(f))(s)-f(s)\,(L_N(1))(s) \\
&=\sum_{k=1}^{n}\frac{f^{(k)}(s)}{k!}\left(L_N(\cdot-s)^k\right)(s)+\left(L_N\left(R_n\left(\cdot,s\right)\right)\right)(s) \qquad \text{for all } s\in[a,b].
\end{aligned} \tag{83}
$$

Therefore, we have

$$
\begin{aligned}
&(L_N(f))(s)-f(s)=(L_N(f))(s)-f(s)-f(s)\,(L_N(1))(s)+f(s)\,(L_N(1))(s) \\
&=(L_N(f))(s)-f(s)\,(L_N(1))(s)+f(s)\,((L_N(1))(s)-1) \\
&=f(s)\,((L_N(1))(s)-1)+\sum_{k=1}^{n}\frac{f^{(k)}(s)}{k!}\left(L_N(\cdot-s)^k\right)(s)+\left(L_N\left(R_n\left(\cdot,s\right)\right)\right)(s)
\end{aligned} \tag{84}
$$

for all $s\in[a,b]$. Thus, it holds

$$
\begin{aligned}
(L_N(f))(s)-f(s) &= f(s)\,((L_N(1))(s)-1)+\sum_{k=1}^{n}\frac{f^{(k)}(s)}{k!}\left(L_N(\cdot-s)^k\right)(s) \\
&\quad +\left(L_N\left(R_n\left(\cdot,s\right)\right)\right)(s) \qquad \text{for all } s\in[a,b].
\end{aligned} \tag{85}
$$

Above, $\left(L_N(\cdot-s)^k\right)(s)$ is continuous in $s\in[a,b]$ for all $k=1,\dots,n$.
Furthermore, it holds
(86)

$$
\begin{aligned}
|(L_N(f))(s)-f(s)| &\overset{(81)}{\le} |f(s)|\,|(L_N(1))(s)-1|+\sum_{k=1}^{n}\frac{\left|f^{(k)}(s)\right|}{k!}\left|\left(L_N\left((\cdot-s)^k\right)\right)(s)\right| \\
&\quad +\frac{K}{\prod_{i=1}^{n}(i+\alpha)}\left(L_N\left(|\cdot-s|^{n+\alpha}\right)\right)(s) \quad \text{for all } s\in[a,b].
\end{aligned}
$$

Next we observe that

$$\|(L_N(f))(A) - f(A)\| = \||(L_N(f))(A) - f(A)|\| \tag{87}$$
$$= \sup_{x\in H:\|x\|=1} \| \langle \| (L_N(f))(A) - f(A)|x,x\rangle |$$
$$= \sup_{x\in H:\|x\|=1} \Big| \int_{m-0}^{M} |(L_N(f))(s) - f(s)| \mathrm{d}\langle E_s x, x\rangle \Big| \tag{88}$$
$$= \sup_{x\in H:\|x\|=1} \int_{m-0}^{M} |(L_N(f))(s) - f(s)| \mathrm{d}\langle E_s x, x\rangle$$
$$\le \sup_{x\in H:\|x\|=1} \langle |f(A)||(L_n(1))A - 1_H|x,x\rangle$$
$$+ \sum_{k=1}^{n} \frac{1}{k!} \sup_{x\in H:\|x\|=1} \Big\langle |f^{(k)}(A)|\big|\big(L_n\big((\cdot - A)^k\big)\big)(A)|x,x\Big\rangle$$
$$+ \frac{K}{\prod_{i=1}^{n}(i+\alpha)} \sup_{x\in H:\|x\|=1} \left\langle \left(L_n\left(|\cdot - A|^{n+\alpha}\right)\right)(A)x,x\right\rangle$$
$$= \||f(A)||(L_n(1))(A) - 1_H|\| + \sum_{k=1}^{n} \frac{1}{k!} \Big\| |f^{(k)}(A)|\big|\big(L_N\big((\cdot - A)^k\big)\big)(A)| \Big\| \tag{89}$$
$$+ \frac{K}{\prod_{i=1}^{n}(i+\alpha)} \left\| \left(L_n\left(|\cdot - A|^{n+\alpha}\right)\right)(A)\right\|$$
$$\le \|f(A)\|\|(L_n(1))(A) - 1_H\| + \sum_{k=1}^{n} \frac{1}{k!} \Big\| f^{(k)}(A)\Big\| \Big\|\big(L_N\big((\cdot - A)^k\big)\big)(A)\Big\|$$
$$+ \frac{K}{\prod_{i=1}^{n}(i+\alpha)} \left\| \left(L_n\left(|\cdot - A|^{n+\alpha}\right)\right)(A)\right\|,$$

proving the inequality (72).

By (37), we have ($k = 1, \ldots, n$, $0 < \alpha \le 1$) that

$$\left|\left(L_N(\cdot - s)^k\right)(s)\right| \le ((L_N(1))(s))^{\left(\frac{n+\alpha-k}{n+\alpha}\right)} \left(\left(L_N\left(|\cdot - s|^{n+\alpha}\right)\right)(s)\right)^{\frac{k}{n+\alpha}}, \tag{90}$$

for all $s \in [a,b]$, $N \in \mathbb{N}$.

By assumption (73), we get

$$\left|\left(L_N(\cdot - s)^k\right)(s)\right| \le \mu^{\left(\frac{n+\alpha-k}{n+\alpha}\right)} \left(\left(L_N\left(|\cdot - s|^{n+\alpha}\right)\right)(s)\right)^{\frac{k}{n+\alpha}} \tag{91}$$

for all $s \in [a,b]$, $N \in \mathbb{N}$, $k = 1, \ldots, n$ and $0 < \alpha \le 1$.

Hence we derive

$$\left\|\big(L_N\big((\cdot - A)^k\big)\big)(A)\right\| = \left\|\left|L_N\big((\cdot - A)^k\big)(A)\right|\right\|$$
$$= \sup_{x\in H:\|x\|=1} \left|\left\langle \left|L_N\big((\cdot - A)^k\big)(A)\right|x,x\right\rangle\right|$$
$$= \sup_{x\in H:\|x\|=1} \left\langle \left|L_N\big((\cdot - A)^k\big)(A)\right|x,x\right\rangle \tag{92}$$

$$= \sup_{x\in H:\|x\|=1} \int_{m-0}^{M} \left|\left(L_N\left((\cdot - s)^k\right)\right)(s)\right| \mathrm{d}\langle E_s x, x\rangle$$

$$\leq \mu^{\left(\frac{n+\alpha-k}{n+\alpha}\right)} \sup_{x\in H:\|x\|=1} \int_{m-0}^{M} \left(\left(L_N\left(|\cdot - s|^{n+\alpha}\right)\right)(s)\right)^{\frac{k}{n+\alpha}} \mathrm{d}\langle E_s x, x\rangle$$

$$= \mu^{\left(\frac{n+\alpha-k}{n+\alpha}\right)} \sup_{x\in H:\|x\|=1} \left\langle \left(\left(L_N\left(|\cdot - A|^{n+\alpha}\right)\right)(A)\right)^{\frac{k}{n+\alpha}} x, x\right\rangle$$

(by Hölder-McCarthy's inequality (12))

$$\leq \mu^{\left(\frac{n+\alpha-k}{n+\alpha}\right)} \sup_{x\in H:\|x\|=1} \left(\left\langle \left(\left(L_N\left(|\cdot - A|^{n+\alpha}\right)\right)(A)\right)x, x\right\rangle\right)^{\frac{k}{n+\alpha}} \tag{93}$$

$$= \mu^{\left(\frac{n+\alpha-k}{n+\alpha}\right)} \left(\sup_{x\in H:\|x\|=1} \left|\left\langle \left(\left(L_N\left(|\cdot - A|^{n+\alpha}\right)\right)(A)\right)x, x\right\rangle\right|\right)^{\frac{k}{n+\alpha}}$$

$$= \mu^{\left(\frac{n+\alpha-k}{n+\alpha}\right)} \left\|\left(L_N\left(|\cdot - A|^{n+\alpha}\right)\right)(A)\right\|^{\frac{k}{n+\alpha}}.$$

We have proved that

$$\left\|\left(L_N\left((\cdot - A)^k\right)\right)(A)\right\| \leq \mu^{\left(\frac{n+\alpha-k}{n+\alpha}\right)} \left\|\left(L_N\left(|\cdot - A|^{n+\alpha}\right)\right)(A)\right\|^{\frac{k}{n+\alpha}} \tag{94}$$

for all $k = 1, \ldots, n$, $0 < \alpha \leq 1$ and $N \in \mathbb{N}$.

By (94) and assuming that $(L_N(1))(A) \to 1_H$ and $\left(L_N\left(|\cdot - A|^{n+\alpha}\right)\right)(A) \to 0_H$, uniformly as $N \to \infty$, we get that $\left(L_N\left((\cdot - A)^k\right)\right)(A) \to 0_H$ and $(L_N(f))(A) \to f(A)$ uniformly for all $f \in C([a,b])$ under assumptions (71), (73). □

We continue with next theorem.

Theorem 11. *Let $\{L_N\}_{N\in\mathbb{N}}$ be a sequence of positive linear operators from $C([a,b])$ into itself. Let $f\colon [a,b] \to \mathbb{R}$ be such that $f^{(n)} \in C([a,b])$ and*

$$\left|f^{(n)}(z) - f^{(n)}(s)\right| \leq K\,|z-s|^{\alpha}, \qquad K > 0, \tag{95}$$

for all $0 < \alpha \leq 1$ and $z, s \in [a,b]$.

Then it holds

$$\begin{aligned} &\left\|(L_N(f))(A) - f(A) - \sum_{k=1}^{n} \frac{f^{(k)}(A)}{k!}\left(L_N(\cdot - A)^k\right)(A)\right\| \\ &\leq \|f(A)\|\,\|(L_N(1))(A) - 1_H\| + \frac{K}{\prod_{i=1}^{n}(i+\alpha)} \left\|\left(L_N\left(|\cdot - A|^{n+\alpha}\right)\right)(A)\right\|. \end{aligned} \tag{96}$$

Conclusion. If $(L_N(1))(A) \to 1_H$ and $\left(L_N\left(|\cdot - A|^{n+\alpha}\right)\right)(A) \to 0_H$, uniformly as $N \to \infty$, then

$$\left[(L_N(f))(A) - \sum_{k=1}^{n} \frac{f^{(k)}(A)}{k!}\left(L_N(\cdot - A)^k\right)(A)\right] \to f(A)$$

uniformly as $N \to \infty$ and for all f as above.

Proof. The next is a continuous function in $s \in [a,b]$:

$$
\begin{aligned}
& (L_N(f))(s) - f(s) - \sum_{k=1}^{n} \frac{f^{(k)}(s)}{k!} \left(L_N(\cdot - s)^k\right)(s) \\
& \overset{(85)}{=} f(s)\left((L_N(1))(s) - 1\right) + \left(L_N\left(R_n(\cdot, s)\right)(s)\right) \qquad \text{for all } s \in [a,b].
\end{aligned} \tag{97}
$$

Hence it holds

$$
\begin{aligned}
& \left\| (L_N(f))(A) - f(A) - \sum_{k=1}^{n} \frac{f^{(k)}(A)}{k!} \big(L_N(\cdot - A)^k\big)(A) \right\| \\
& = \left\| \left| (L_N(f))(A) - f(A) - \sum_{k=1}^{n} \frac{f^{(k)}(A)}{k!} \big(L_N(\cdot - A)^k\big)(A) \right| \right\| \\
& = \sup_{x \in H : \|x\| = 1} \left| \left\langle \left| (L_N(f))(A) - f(A) - \sum_{k=1}^{n} \frac{f^{(k)}(A)}{k!} \big(L_N(\cdot - A)^k\big)(A) \right| x, x \right\rangle \right| \\
& = \sup_{x \in H : \|x\| = 1} \left\langle \left| (L_N(f))(A) - f(A) - \sum_{k=1}^{n} \frac{f^{(k)}(A)}{k!} \big(L_N(\cdot - A)^k\big)(A) \right| x, x \right\rangle \qquad (98) \\
& = \sup_{x \in H : \|x\| = 1} \int_{m-0}^{M} \left| f(s)\left((L_N(1))(s) - 1\right) + \left(L_N\left(R_n(\cdot, s)\right)\right)(s) \right| \mathrm{d}\langle E_s x, x\rangle \\
& \le \sup_{x \in H : \|x\| = 1} \int_{m-0}^{M} |f(s)| \left| (L_N(1))(s) - 1 \right| \mathrm{d}\langle E_s x, x\rangle \qquad (99) \\
& \quad + \sup_{x \in H : \|x\| = 1} \int_{m-0}^{M} \left(L_N\left(|R_n(\cdot, s)|\right)\right)(s) \mathrm{d}\langle E_s x, x\rangle \\
& \overset{(81)}{\le} \sup_{x \in H : \|x\| = 1} \left\langle |f(A)| \left| (L_N(1))(A) - 1_H \right| x, x \right\rangle \\
& \quad + \sup_{x \in H : \|x\| = 1} \frac{K}{\prod_{i=1}^{n}(i+\alpha)} \int_{m-0}^{M} \big(L_N\big(|\cdot - s|^{n+\alpha}\big)\big)(s) \mathrm{d}\langle E_s x, x\rangle \\
& \overset{(13)}{\le} \| |f(A)| \| \, \| |(L_N(1))(A) - 1_H| \| + \frac{K}{\prod_{i=1}^{n}(i+\alpha)} \left\| \big(L_N\big(|\cdot - A|^{n+\alpha}\big)\big)(A) \right\| \\
& = \|f(A)\| \, \|(L_N(1))(A) - 1_H\| + \frac{K}{\prod_{i=1}^{n}(i+\alpha)} \left\| \big(L_N\big(|\cdot - A|^{n+\alpha}\big)\big)(A) \right\|, \qquad (100)
\end{aligned}
$$

proving the claim. ∎

4. Applications

For the next, see [**2**, pp. 169-170].

Let $g \in C([0,1])$, $N \in \mathbb{N}$, the Nth basic Bernstein polynomial for g is defined by

$$(\beta_N(g))(z) = \sum_{k=0}^{N} \binom{N}{k} g\Big(\frac{k}{N}\Big) z^k (1-z)^{N-k} \qquad \text{for all } z \in [0,1]. \tag{101}$$

It has the properties:

$$\begin{aligned} &\beta_N(1) = 1, && (\beta_N(\mathrm{id}))(z) = z, \\ &(\beta_N((\cdot - z)))(z) = 0, && \big(\beta_N(\mathrm{id}^2)\big)(z) = \Big(1 - \frac{1}{N}\Big) z^2 + \frac{1}{N} z, \end{aligned} \tag{102}$$

and

$$\big(\beta_N\big((\cdot - z)^2\big)\big)(z) = \frac{z(1-z)}{N} \qquad \text{for all } z \in [0,1].$$

Here we consider $f \in C([a,b])$ and the general Bernstein positive linear polynomial operators from $C([a,b])$ into itself, defined by (see [**9**, p. 80])

$$(B_N f)(s) = \sum_{i=0}^{N} \binom{N}{i} f\left(a + i\frac{(b-a)}{N}\right) \left(\frac{s-a}{b-a}\right)^i \left(\frac{b-s}{b-a}\right)^{N-i} \tag{103}$$

for all $s \in [a,b]$. By [**9**, p. 81], we get that

$$\|B_N f - f\|_\infty \le \frac{5}{4} \omega_1\Big(f; \frac{b-a}{\sqrt{N}}\Big), \tag{104}$$

i.e., $B_N f \to f$, uniformly as $N \to \infty$ and for all $f \in C([a,b])$, the convergence is given with rates.

We obtain easily that

$$(B_N(1))(s) = 1 \qquad \text{for all } s \in [a,b] \text{ , i.e.,} \qquad B_N(1) = 1. \tag{105}$$

We notice that

$$\left(\frac{s-a}{b-a}\right) + \left(\frac{b-s}{b-a}\right) = 1 \qquad \text{for all } s \in [a,b], \tag{106}$$

calling $y := \frac{s-a}{b-a}$, we have $\frac{b-s}{b-a} = 1 - y$.

So we can write

$$(B_N f)(s) = \sum_{i=0}^{N} \binom{N}{i} f\left(a + \frac{i(b-a)}{N}\right) y^i (1-y)^{N-i}. \tag{107}$$

We observe that

$$\begin{aligned} (B_N(\mathrm{id}))(s) &= \sum_{i=0}^{N} \binom{N}{i} \left(a + \frac{i(b-a)}{N}\right) y^i (1-y)^{N-i} \\ &= a + (b-a) \sum_{i=0}^{N} \binom{N}{i} \left(\frac{i}{N}\right) y^i (1-y)^{N-i} \\ &\overset{(102)}{=} a + (b-a)y = a + s - a = s, \end{aligned}$$

proving $\left(B_n(\mathrm{id})\right)(s) = s$, i.e., it holds

$$B_N(\mathrm{id}) = \mathrm{id}\,. \tag{108}$$

We see that

$$\left(B_N\left(\left(\mathrm{id} - s\right)\right)\right)(s) = \left(B_N(\mathrm{id})\right)(s) - s\left(B_N\left(1\right)\right)(s) \overset{(105)}{=} s - s = 0,$$

i.e.,

$$\left(B_N\left(\left(\cdot - s\right)\right)\right)(s) = 0 \qquad \text{for all } s \in [a,b]. \tag{109}$$

Next we calculate

$$\begin{aligned}
(B_N(\mathrm{id}^2))(s) &= \sum_{i=0}^{N} \binom{N}{i} \Big(a + \frac{i(b-a)}{N}\Big)^2 y^i \left(1-y\right)^{N-i} \\
&= \sum_{i=0}^{N} \binom{N}{i} \Big(a^2 + 2a\left(b-a\right)\frac{i}{N} + \left(b-a\right)^2 \Big(\frac{i}{N}\Big)^2\Big) y^i \left(1-y\right)^{N-i} \\
&= a^2 + 2a\left(b-a\right) \sum_{i=0}^{N} \binom{N}{i} \Big(\frac{i}{N}\Big) y^i \left(1-y\right)^{N-i} \\
&\qquad + \left(b-a\right)^2 \sum_{i=0}^{N} \binom{N}{i} \Big(\frac{i}{N}\Big)^2 y^i \left(1-y\right)^{N-i} \\
&\overset{(102)}{=} a^2 + 2a\left(b-a\right)y + \left(b-a\right)^2 \Big[\Big(1 - \frac{1}{N}\Big) y^2 + \frac{1}{N} y\Big] \\
&= a^2 + 2a\left(s-a\right) + \left(b-a\right)^2 \Big[\Big(1 - \frac{1}{N}\Big) \frac{\left(s-a\right)^2}{\left(b-a\right)^2} + \frac{1}{N}\Big(\frac{s-a}{b-a}\Big)\Big] \\
&= a^2 + 2a\left(s-a\right) + \Big(1 - \frac{1}{N}\Big)\left(s-a\right)^2 + \frac{1}{N}\left(b-a\right)\left(s-a\right) \\
&= a^2 + \left(s-a\right)\Big(2a + \frac{b-a}{N}\Big) + \Big(1 - \frac{1}{N}\Big)\left(s-a\right)^2 .
\end{aligned} \tag{110}$$

We have proved that

$$\left(B_N(\mathrm{id}^2)\right)(s) = a^2 + \left(s-a\right)\Big(2a + \frac{b-a}{N}\Big) + \Big(1 - \frac{1}{N}\Big)\left(s-a\right)^2 \tag{111}$$

for all $s \in [a,b]$. Finally we calculate

$$\begin{aligned}
\left(B_N\left(\left(\mathrm{id} - s\right)^2\right)\right)(s) &= \left(B_N\left(\left(\mathrm{id}^2 - 2s\ \mathrm{id} + s^2\right)\right)\right)(s) \\
&= \left(B_N(\mathrm{id}^2)\right)(s) - 2s\left(B_N(\mathrm{id})\right)(s) + s^2\left(B_N\left(1\right)\right)(s) \\
&= a^2 + \left(s-a\right)\Big(2a + \frac{b-a}{N}\Big) + \Big(1 - \frac{1}{N}\Big)\left(s-a\right)^2 - 2s^2 + s^2 \\
&= a^2 + \left(s-a\right)\Big(2a + \frac{b-a}{N}\Big) + \Big(1 - \frac{1}{N}\Big)\left(s-a\right)^2 - s^2 \\
&= 2a^2 + \left(s-a\right)\Big(2a + \frac{b-a}{N}\Big) - 2as - \frac{1}{N}\left(s-a\right)^2 .
\end{aligned} \tag{112}$$

We have proved that

$$\left(B_N\left(\,(\mathrm{id}-s)^2\,\right)\right)(s) = 2a^2 + (s-a)\left(2a + \frac{b-a}{N}\right) - 2as - \frac{1}{N}(s-a)^2 \tag{113}$$

for all $s \in [a,b]$.

Notice that

$$\lim_{N\to\infty} \left(B_N\left(\,(\mathrm{id}-s)^2\,\right)\right)(s) = 0, \tag{114}$$

as well as

$$\lim_{N\to\infty} \left(B_N(\mathrm{id}^2)\right)(s) = s^2 \qquad \text{for all } s \in [a,b], \tag{115}$$

both uniformly.

Next we apply the results of section 3 for the case of B_N operators. Here again $Sp(A) \subseteq [m,M] \subset (a,b)$; A as a selfadjoint operator on the Hilbert space H. By (1) and (2) and $x \in H : \|x\| = 1$, we get $\int_{m-0}^{M} \mathrm{d}\langle E_\lambda x, x\rangle = 1$.

So Theorem 7 is going to read as follows.

Corollary 12. *Let $f \in C([a,b])$. Then*

$$\|(B_N(f))(A) - f(A)\| \le \|B_N f - f\|_{\infty,[a,b]} \qquad \textit{for all } N \in \mathbb{N}. \tag{116}$$

By earlier comments on B_N, see (104), we get that $\lim_{N\to\infty}(B_N(f))(A) = f(A)$ uniformly for all $f \in C([a,b])$.

Next we apply Theorem 8 to B_N operators.

Corollary 13. *Let $f\colon [a,b] \to \mathbb{R}$. Assume that*

$$|f(t) - f(s)| \le K\,|t-s|^{\alpha} \qquad \textit{for all } t,s \in [a,b], \tag{117}$$

where $0 < \alpha \le 1$, $K > 0$.

Then

$$\|(B_N(f))(A) - f(A)\| \le K\left\|\left(B_N(\mathrm{id}^2)\right)(A) - A^2\right\|^{\frac{\alpha}{2}} \qquad \textit{for all } N \in \mathbb{N}. \tag{118}$$

Since (see Remark 14 next) $\left(B_N(\mathrm{id}^2)\right)(A) \to A^2$, uniformly as $N \to \infty$, we get that $(B_N(f))(A) \to f(A)$ uniformly as $N \to \infty$ and for all $f \in C([a,b])$, fulfilling (117).

Remark 14. Indeed it holds

(119)

$$\begin{aligned}
\left\|\left(B_N(\mathrm{id}^2)\right)(A) - A^2\right\| &= \sup_{\substack{x\in H:\\ \|x\|=1}} \left|\left\langle\left(\left(B_N(\mathrm{id}^2)\right)(A) - A^2\right)x, x\right\rangle\right| \\
&= \left|\int_{m-0}^{M} \left(\left(B_N(\mathrm{id}^2)\right)(s) - s^2\right)\mathrm{d}\langle E_s x, x\rangle\right| \\
&\le \int_{m-0}^{M} \left|\left(B_N(\mathrm{id}^2)\right)(s) - s^2\right|\mathrm{d}\langle E_s x, x\rangle \le \left\|\left(B_N(\mathrm{id}^2)\right)(s) - s^2\right\|_{\infty,[a,b]} \to 0,
\end{aligned}$$

as $N \to \infty$, by (115).

Proving that $\left(B_N(\mathrm{id}^2)\right)(A) \to A^2$, uniformly as $N \to \infty$.

Corollary 15 (to Theorem 9)**.** *Let* $f\colon[a,b]\to\mathbb{R}$ *such that*

$$|f(t)-f(s)|\le K\,|t-s|^{\alpha}\qquad \text{for all } t,s\in[a,b], \tag{120}$$

where $0<\alpha\le 1$, $K>0$.

Then

$$\|(B_N(f))(A)-f(A)\|\le K\,\|(B_N(|\cdot-A|^{\alpha}))(A)\|\qquad \text{for all } N\in\mathbb{N}. \tag{121}$$

Since $(B_N(|\cdot-A|^{\alpha}))(A)\to 0_H$ uniformly, (see Remarks 16 and 18) then $(B_N(f))(A)\to f(A)$, uniformly as $N\to\infty$ for every f as above, see (120).

Remark 16. We easily obtain (by Hölder's inequality and (105))

$$(B_N(|\cdot-s|^{\alpha}))(s)\le\left(\left(B_N\left((\cdot-s)^2\right)\right)(s)\right)^{\frac{\alpha}{2}}\qquad \text{for all } s\in[a,b]. \tag{122}$$

Hence it holds

$$\begin{aligned}
\|(B_N(|\cdot-A|^{\alpha}))(A)\| &= \sup_{\substack{x\in H:\\ \|x\|=1}}|\langle((B_N(|\cdot-A|^{\alpha}))(A))\,x,x\rangle|\\
&= \sup_{\substack{x\in H:\\ \|x\|=1}}\langle((B_N(|\cdot-A|^{\alpha}))(A))\,x,x\rangle\\
&= \sup_{\substack{x\in H:\\ \|x\|=1}}\int_{m-0}^{M}(B_N(|\cdot-s|^{\alpha}))(s)\mathrm{d}\,\langle E_s x,x\rangle \qquad (123)\\
&\overset{(122)}{\le} \sup_{\substack{x\in H:\\ \|x\|=1}}\int_{m-0}^{M}\left(\left(B_N\left(|\cdot-s|^2\right)\right)(s)\right)^{\frac{\alpha}{2}}\mathrm{d}\,\langle E_s x,x\rangle\\
&= \sup_{\substack{x\in H:\\ \|x\|=1}}\left\langle\left(\left(B_N\left(|\cdot-A|^2\right)\right)(A)\right)^{\frac{\alpha}{2}}x,x\right\rangle\\
&\overset{(12)}{\le} \sup_{\substack{x\in H:\\ \|x\|=1}}\left(\left\langle\left(\left(B_N\left(|\cdot-A|^2\right)\right)(A)\right)x,x\right\rangle\right)^{\frac{\alpha}{2}} \qquad (124)\\
&= \sup_{\substack{x\in H:\\ \|x\|=1}}\left|\left\langle\left(\left(B_N\left(|\cdot-A|^2\right)\right)(A)\right)x,x\right\rangle\right|^{\frac{\alpha}{2}}=\left\|\left(B_N\left((\cdot-A)^2\right)\right)(A)\right\|^{\frac{\alpha}{2}}.
\end{aligned}$$

That is, we have

$$\|(B_N(|\cdot-A|^{\alpha}))(A)\|\le\left\|\left(B_N\left((\cdot-A)^2\right)\right)(A)\right\|^{\frac{\alpha}{2}}. \tag{125}$$

Corollary 17 (to Theorem 10 for $n=1$, $\alpha=1$)**.** *Let* $f\colon[a,b]\to\mathbb{R}$ *such that*

$$|f'(z)-f'(s)|\le K\,|z-s|\,,\qquad K>0,\ \text{for all } z,s\in[a,b]. \tag{126}$$

Then

$$\|(B_N(f))(A)-f(A)\|\le\frac{K}{2}\left\|\left(B_N\left((\cdot-A)^2\right)\right)(A)\right\|\qquad \text{for all } N\in\mathbb{N}. \tag{127}$$

Proof. See also (105) and (109). ☐

We make

Remark 18. We observe

$$\left\|\left(B_N\left((\cdot - A)^2\right)\right)(A)\right\| = \sup_{\substack{x\in H:\\ \|x\|=1}} \left|\left\langle \left(\left(B_N\left((\cdot - A)^2\right)\right)(A)\right)x, x\right\rangle\right|$$

$$= \sup_{\substack{x\in H:\\ \|x\|=1}} \left\langle \left(\left(B_N\left((\cdot - A)^2\right)\right)(A)\right)x, x\right\rangle \tag{128}$$

$$= \sup_{\substack{x\in H:\\ \|x\|=1}} \int_{m-0}^{M} \left(B_N\left((\cdot - s)^2\right)\right)(s)\mathrm{d}\left\langle E_s x, x\right\rangle$$

$$\leq \left\|\left(B_N\left((\cdot - s)^2\right)\right)(s)\right\|_{\infty,[a,b]} \to 0$$

as $N \to \infty$, (by (114)). That proves

$$\left\|\left(B_N\left((\cdot - A)^2\right)\right)(A)\right\| \to 0,$$

and by (127), we derive that $(B_N(f))(A) \to f(A)$ uniformly as $N \to \infty$ for every f as in (126).

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G. A. Anastassiou, Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, U.S.A., *e-mail*: `ganastss@memphis.edu`