**k-GENERALIZED FIBONACCI NUMBERS CLOSE TO THE FORM**\(2^a + 3^b + 5^c\)

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Abstract. The \(k\)-generalized Fibonacci sequence \(\{F^{(k)}_n\}_{n \geq 0}\) is defined as the sum of the \(k\) proceeding terms and initial conditions are \(0, \ldots, 0, 1\) (\(k\) terms). In this paper, we solve the diophantine equation \(F^{(k)}_n = 2^a + 3^b + 5^c + \delta\), where \(a, b, c\) and \(\delta\) are nonnegative integers with \(\max\{a, b\} \leq c\) and \(0 \leq \delta \leq 5\). This work generalizes a recent Marques [9] and the first author, Szalay [6] results.

1. Introduction

Let \(k \geq 2\) be an integer. The \(k\)-generalized Fibonacci sequence \(\{F^{(k)}_n\}_{n \geq 0}\) is defined by the following recurrence relation

\[F^{(k)}_n = F^{(k)}_{n-1} + F^{(k)}_{n-2} + \cdots + F^{(k)}_{n-k}, \quad n \geq -(k-2)\]

with the initial conditions \(F^{(k)}_{-(k-2)} = F^{(k)}_{-(k-3)} = \cdots = F^{(k)}_0 = 0\) and \(F^{(k)}_{-(k-1)} = 1\).

Naturally, the case \(k = 2\) turns to well-known Fibonacci sequence \(\{F_n\}\). If \(k = 3\), then it gives the Tribonacci sequence \(\{T_n\}\). The problem of finding different type of the numbers among the terms of linear recurrence has a long history. One of the results by Bugeaud, Mignotte and Siksek [1] is that only 0, 1, 8, 144 in Fibonacci numbers and 1, 4 in Lucas numbers can be written in the form \(y^t\), where \(t > 1\). Szalay and Luca showed that there are only finitely quadruples \((n, a, b, p)\) such that \(F_n = p^a \pm p^b + 1\), where \(p\) is a prime number in [7]. Recently, Bravo and Luca [2] solved the diophantine equation \(F^{(k)}_n = 2^m\) for positive integers \(n, k, m\) with \(k \geq 2\). The paper of Marques and Togbe [8] determines the Fibonacci numbers and the Lucas numbers of the form \(2^n + 3^b + 5^c\) where Lucas sequence is defined by relation \(L_n = L_{n-1} + L_{n-2}\) for \(n \geq 2\), together with \(L_0 = 2\) and \(L_1 = 1\). Recently, Marques [9] solved the diophantine equation \(F^{(k)}_n = 2^n + 3^b + 5^c\) with \(\max\{a, b\} \leq c\). The first author and Szalay showed that there are 22 solutions to the diophantine equation \(0 \leq T_n - 2^n - 3^b - 5^c \leq 10\), where \(T_n\) is the Tribonacci sequence.

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In this paper, we solve the equation

\[ F_n^{(k)} = 2^a + 3^b + 5^c + \delta, \]

where \(a\), \(b\), \(c\) and \(\delta\) are nonnegative integers with \(\max\{a, b\} \leq c\) and \(0 \leq \delta \leq 5\).

There is a difference from the paper [9]. We add a diameter to the equation. If we take \(k = 3\) and \(0 \leq \delta \leq 5\) in the equation (1), then we obtain the result of the paper [6]. Taking \(\delta = 0\) in the equation (1) yields the paper [9].

Our result is the following theorem.

**Theorem 1.1.** For \(n \geq k + 2\), the solutions of the equation

\[ F_n^{(k)} = 2^a + 3^b + 5^c + \delta, \]

are given in the following tables

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where \(a\), \(b\) and \(c\) are positive integers with \(\max\{a, b\} \leq c\) and the sequence \(\{F_n^{(k)}\}\) is the \(k\)-generalized Fibonacci sequence.

When we take \(\delta = 0\), it coincides with the Marques [9] results. Moreover, we note that the solutions \((n, k, a, b, c, \delta) = (7, 4, 0, 1, 2, 0)\) and \((9, 7, 0, 3, 0)\) are not observed in [9].

2. Auxiliary Results

Before proceeding further, we recall some facts and tools which will be used next. Dresden ([4, Theorem 1]) gave the Binet-type formula of the terms of the sequence \(\{F_n^{(k)}\}\) as follows

\[ F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_i - 1}{2 + (k + 1)(\alpha_i - 2)} \alpha_i^{n-1}, \]
for \( \alpha_1, \ldots, \alpha_k \), the roots \( x^k - x^{k-1} - \cdots - 1 = 0 \). Also, it was proven in the same paper that
\[
\left| F_n^{(k)} - g(\alpha, k)\alpha^{n-1} \right| < \frac{1}{2}
\]
where \( \alpha \) is the dominant root of the characteristic equation \( x^k - x^{k-1} - \cdots - 1 = 0 \) and the notation \( g(\alpha, k) := (\alpha - 1)/(2 + (k + 1)(\alpha - 2)) \). Also, Bravo and Luca [2] proved that
\[
\alpha^n - 2 \leq F_n^{(k)} \leq \alpha^n - 1
\]
for all \( n \geq 1 \).

Another tool to prove our theorem is a lower bound for linear forms in logarithms of algebraic numbers, given by Matveev [10]. The first one is the following lemma.

**Lemma 2.1.** Let \( K \) be a number field of degree \( D \) over \( \mathbb{Q} \), \( \gamma_1, \gamma_2, \ldots, \gamma_t \) be positive real numbers of \( K \), and \( b_1, b_2, \ldots, b_t \) be rational integers. Put
\[
B \geq \max \{|b_1|, |b_2|, \ldots, |b_t|\}
\]
and
\[
\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1.
\]
Let \( A_1, \ldots, A_t \) be real numbers such that
\[
A_i \geq \max \{Dh(\gamma_i), |\log \gamma_i|, 0.16\}, \quad i = 1, \ldots, t.
\]
Then, assuming that \( \Lambda \neq 0 \), we have
\[
|\Lambda| > \exp \left(-1.4 \times 30^{t+3} \times t^{1.5} \times D^2 \times (1 + \log D) \times (1 + \log B) \times A_1 \cdots A_t \right).
\]
As usual, in the above lemma, the logarithmic height of algebraic number \( \eta \) is defined as
\[
h(\eta) = \frac{1}{d} \left( \log a_0 + \sum_{i=1}^{d} \left( \max \{|\eta^{(i)}|, 1|\} \right) \right)
\]
where \( d \) is the degree of \( \eta \) over \( \mathbb{Q} \), \( \{\eta^{(i)}\}_{1 \leq i \leq d} \) are the conjugates of \( \eta \) over \( \mathbb{Q} \), and \( a_0 \) is the positive leading coefficient of the minimal polynomial of \( \eta \) over the integers.

The application of Matveev theorem gives a large upper bound. In order to reduce this bound, we use the following lemma ([5, Lemma 2.2]).

**Lemma 2.2.** Suppose that \( M \) is a positive integer. Let \( p/q \) be a convergent of the continued fraction expansion of the irrational number \( \gamma \) such that \( q > 6M \) and \( \varepsilon = \|\mu q - M\|q\| \), where \( \mu \) is a real number and \( \| \cdot \| \) denotes the distance from the nearest integer. If \( \varepsilon > 0 \), then there is no solution to the inequality
\[
0 < m\gamma - n + \mu < AB^{-m}
\]
in positive integers \( m \) and \( n \) with
\[
\frac{\log(Aq/\varepsilon)}{\log B} \leq m < M.
\]
We use also the following lemma from the paper [2].
Lemma 2.3. For every positive integer \( n \geq 2 \), we have
\[
F_n^{(k)} \leq 2^{n-2}.
\]

3. Proof of Theorem 1.1

The formula (2) together with the diophantine equation yields that
\[
\sum_{i=1}^{k} \frac{\alpha_i - 1}{2 + (k + 2)(\alpha_i - 2)} \alpha_i^{n-1} = 2^a + 3^b + 5^c + \delta.
\]
Then we have
\[
g(\alpha, k) \frac{\alpha^{n-1}}{5} - 1 = \frac{2^a + 3^b - \xi + \delta}{5^c}
\]
where \( 1 < \alpha = \alpha_1 \in \mathbb{R} \) is the dominant root of the characteristic equation
\[
x^k - x^{k-1} - \cdots - 1 = 0
\]
and \( \xi = \sum_{i=2}^{k} g(\alpha_i, k)\alpha_i^{n-1} \) is the real number whose absolute value is less than 1. Therefore,
\[
g(\alpha, k) \frac{\alpha^{n-1}}{5^c} - 1 > 0.
\]
Consequently, we get
\[
g(\alpha, k) \frac{\alpha^{n-1}}{5^c} - 1 < \frac{2^a + 3^b + \delta}{5^c} < 3^{5^c}.
\]
since \( 2 < \sqrt{5} \), \( 3 < 5^{0.7} \), and \( c \geq \max\{a, b\} \).

In order to apply the Lemma 2.1, we take \( t := 3 \) and
\[
\gamma_1 := g(\alpha, k), \quad \gamma_2 := \alpha, \quad \gamma_3 := 5,
\]
together with the exponents \( b_1 := 1, \ b_2 := n - 1 \) and \( b_3 := -c \). For this choice, we have \( D = k, A_2 = k \log 5, A_3 = 0.7, \) and by the paper [2, page 73], \( A_1 = 4k \log k \).

By the inequality (3), \( 5^c < r_n^{(k)} < \alpha^{n-1} \) gives that \( \frac{n - 1}{c} > \frac{\log 5}{\log \alpha} > 1 \). Hence,
\[
B = \max\{1, n - 1, c\} = n - 1.
\]
When we compare the upper and lower bounds for \( \Lambda \), we get
\[
e^{-T(1+\log k)(1+\log(n-1))(4k \log k)(k \log 5)0.7} < \frac{3}{1.6^c},
\]
where \( T = 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot k^2 \). After taking logarithm of both sides and some simplifications together with \( 1 + \log (n - 1) < 2 \log (n - 1) \) and \( 1 + \log k < 2 \log k \), we obtain
\[
c \log 1.6 - \log 2 < 3.9 \cdot 10^{12} k^4 \log k \log (n - 1).
\]
Since \( \alpha^{n-2} < F_n^{(k)} < 3 \cdot 5^c < 5^{c+1} \), then \( (n - 2) \frac{\log \alpha}{\log 5} - 1 < c \). So, the inequality
\[
\left( (n - 2) \frac{\log \alpha}{\log 5} - 1 \right) \log 1.6 < 3.9 \cdot 10^{12} k^4 \log k \log (n - 1)
\]
yields that
\[
\frac{n - 1}{\log (n - 1)} < 2.8 \cdot 10^{13} k^4 \log k^2.
\]
Since the function \( x \to x/\log x \) is increasing for all \( x > e \), it is easy to check that the inequality
\[
\frac{x}{\log x} < A \quad \text{yields} \quad x < 2A \log A.
\]
Thus, taking \( A := 5.1 \cdot 10^{12} k^4 (\log k)^2 \), we have
\[
n - 1 < 2 \left( 2.8 \cdot 10^{13} k^4 (\log k)^2 \right)
\times \log \left( 2.8 \cdot 10^{13} k^4 (\log k)^2 \right)
< 5.6 \cdot 10^{13} k^4 (\log k)^2
\times (\log 2.8 + 13 \log 10 + 4 \log k + 2 \log (\log k))
\times (31 + 4 \log k + 2 \log (\log k))
\]
since \( (31 + 4 \log k + 2 \log (\log k)) < 48 \log k \) for all \( k \geq 2 \), then
\[
n < 2.7 \cdot 10^{13} k^4 (\log k)^3.
\]
Since \( 5^c < F_n^{(k)} < \alpha^{n-1} \) and \( \alpha^{n-2} < F_n^{(k)} < 5^{c+1} \), then the inequality
\[
2.3 \cdot c + 1 < n < 3.4 (c + 1) + 2
\]
holds.

In the sequel, assume that \( k \in [2,372] \). In order to apply Lemma 2.2, let
\[
t := (n - 1) \log \alpha + \log g(\alpha,k) - c \log 5.
\]
By the equation (4), we can write that
\[
et - 1 < \frac{3}{5^{0.3c}}.
\]
Since \( e^t - 1 > 0 \), then \( t > 0 \). Together with the equation (5),
\[
(n - 1) \log \alpha + \log g(\alpha,k) - c \log 5 < \frac{3}{5^{0.3c}} < \frac{3}{(1.6)^n}
< \frac{3}{(1.6)^{0.29n-1.6}}
< 6.37 \cdot (1.14)^{-n}
\]
holds. Dividing both sides of the above inequality by \( \log 5 \), we get
\[
n \left( \frac{\log \alpha}{\log 5} \right) + \frac{\log g(\alpha,k) - \log \alpha}{\log 5} - c < \frac{6.37}{\log 5} \cdot (1.14)^{-n}
\]
holds. With
\[
\gamma := \frac{\log \alpha}{\log 5}, \quad \mu := \frac{\log g(\alpha,k) - \log \alpha}{\log 5},
\]
\(A := 3.96, \quad B := 1.14,\)
the inequality (6) yields
\[ 0 < n\gamma - c + \mu < A \cdot B^{-n}. \]

It is obvious that \( \gamma \) is a irrational number. Take \( M := 3.96 \cdot 10^{15}k^4(\log k)^3 \). We use the Lemma 2.2 for each case \( k \in [2,372] \). Mathematica programme reveals that maximum value of \( \log(Aq/\varepsilon)/\log B \) is 1941, 25. . . . Hence, we deduce that possible solutions of the diophantine equation (1) are in the range \( k \in [2,372] \) and \( n \in [4,1941] \).

In order to decrease the upper bound for \( n \), we use the inequality
\[ \log_3 \left( F_n^{(k)} - 5^{[\log_5 F_n^{(k)} - \delta]} \right) \leq \log_5 \left( F_n^{(k)} - \delta \right) \]
for \( 0 \leq \delta \leq 5 \). Since we assume that \( \max\{a,b\} \leq c \), this inequality must hold. For \( k \in [2,372] \) and \( n \in [4,1941] \), the inequality \( \log_3 \left( F_n^{(k)} - 5^{[\log_5 F_n^{(k)} - \delta]} \right) \leq \log_5 \left( F_n^{(k)} - \delta \right) \) yields that \( n \leq 30 \). We go through the solutions of the diophantine equation. We find them as in Theorem 1.1.

From now on, assume that \( k \geq 373 \). Under this condition, the inequality (7)
\[ n < 2.7 \cdot 10^{15}k^4(\log k)^3 \leq 2^\frac{3}{2} \]
holds. Using the same arguments in [2], with Lemma 2.3, we have
\[ g(\alpha,k)\alpha^{n-1} = 2^{n-2} - \frac{\delta}{2} + 2^{n-1}\eta + \eta \delta, \]
where \( |\delta| < \frac{2^n}{2k^{1/2}} \) and \( |\eta| < \frac{2k}{\pi} \). Then,
\[ |2^a + 3^b + 5^c + \delta - 2^{n-2}| = \left| 2^a + 3^b + 5^c - g(\alpha,k)\alpha^{n-1} - \frac{\delta}{2} + 2^{n-1}\eta + \eta \delta \right| < \frac{5 \cdot 2^{n-2}}{2k^{1/2}}, \]
where we used the facts \( 1/2^{n-1} < 1/2^{k/2} \), \( 4k/2^k < 1/2^{k/2} \) and \( 8k/2^{3k/2} < 1/2^{k/2} \) for \( k > 363 \). If we divide both sides by \( 2^{n-2} \), then we get
\[ \left| \frac{2^a + 3^b + 5^c}{2^{n-2}} - 1 \right| < \frac{5}{2k^{1/2}}. \]
This inequality gives that
\[ \left| 1 - \frac{5^c}{2^{n-2}} \right| < \frac{5}{2k^{1/2}} + \frac{2^n + 3^b}{2^{n-2}} + \delta \cdot \frac{2}{2^{n-2}}. \]
The facts \( 2^{2a} < 5^c < F_n^{(k)} < 2^{n-2} \) and \( 3^{1.456} < 5^c < F_n^{(k)} < 2^{n-2} \) yield that
\[ \frac{5}{2k^{1/2}} + \frac{2^n + 3^b}{2^{n-2}} + \delta \cdot \frac{2}{2^{n-2}} < \frac{12}{20.3k}. \]
So,
\[ \left| 1 - \frac{5^c}{2^{n-2}} \right| < \frac{12}{20.3k}. \]
Now we apply the Lemma 2.1 again. We take \( t := 2, D := 1, \gamma_1 := 5, \gamma_2 := 2, b_1 := c, b_2 := -(n - 2), B := n, A_1 := \log 5, \) and \( A_2 := \log 2. \) Then the Lemma 2.2 yields that

\[
e^{-1.4 \cdot 30^5 \cdot 2^{4.5} (1 + \log n) \log 5 \log 2} < \frac{12}{20 \cdot 3k}.
\]

Taking both sides with logarithm function, we have

\[
\frac{3k}{10} \log 2 - \log 12 < 1.4 \cdot 30^5 \cdot 2^{4.5} \log 5 \cdot \log 2 \cdot 1.5 \cdot \log n,
\]

where we use the fact \( 1 + \log n < \frac{3}{2} \log n. \) Since

\[
\log n < \log \left( 2.7 \cdot 10^{15} \cdot k^4 \cdot (\log k)^3 \right) < 11 \log k
\]

for \( k \geq 373, \) then

\[
\frac{3k}{10} \log 2 - \log 7 < 1.42 \cdot 10^{10} \cdot \log k.
\]

When we solve the above inequality, we get

\[
2 \leq k < 2 \cdot 10^{12} \quad \text{and} \quad 4 \leq n < 9.9 \cdot 10^{68}.
\]

In order to reduce the upper bound of \( n, \) we use Lemma 2.2 again. Let \( \tau := c \log 5 - (n - 2) \log 2. \) Since \( 5^e < 2^{4.5}, \) then \( c \log 5 - (n - 2) \log 2 < 0 \) yields that \( \tau < 0. \) Thus, we obtain

\[
|\tau| < e^{|\tau|} - 1 = e^{|\tau|} (e^{|\tau|} - 1) < \frac{24}{20.3k},
\]

where we use that \( e^{|\tau|} < 2 \) since \( |e^{|\tau|} - 1| < \frac{1}{2}. \) Since \( n - 2 \geq k, \) then

\[
k \log 2 - c \log 5 \leq (n - 2) \log 2 - c \log 5 < 24 \cdot (2^{0.3})^{-k}.
\]

After dividing both sides by \( \log 5, \) we get

\[
k \log \frac{2}{\log 5} - c < 15 (2^{0.3})^{-k}.
\]

Let \( \gamma := \log \frac{2}{\log 5}, [a_0, a_1, a_2, \ldots] = [0, 2, 3, 9, \ldots] \) be the continued fraction of \( \gamma \) and \( \frac{p_k}{q_k} \) denote the \( k \)th convergent. Then we have \( q_{142} > 9.9 \cdot 10^{68}. \) Furthermore, \( a_M := \max \{a_i; i = 0, 1, \ldots, 142\}. \) Then we find \( a_M = a_{137} = 5394. \) Using the properties of the continued fraction, we get that

\[
|n - 2) \gamma - c| > \frac{1}{(a_M + 2) (n - 2)}.
\]

It yields that

\[
\frac{1}{5395 (n - 2)} < |(n - 2) \gamma - c| < 9 \cdot (2^{0.3})^{-k}.
\]

Then, by the inequality (7),

\[
2^{0.3k} < 9 \cdot 5395 \cdot (n - 2) < 1.32 \cdot 10^{30} k^4 \cdot (\log k)^3
\]

holds. This inequality implies that \( k \leq 361. \) This case is already treated.
Therefore, Theorem 1.1 is completed.

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