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HERMITE-HADAMARD TYPE INEQUALITIES VIA CONFORMABLE FRACTIONAL INTEGRALS

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ABSTRACT. In this study, a new identity involving conformable fractional integrals is given. Then, by using this identity, some new Hermite-Hadamard type inequalities for conformable fractional integrals have been developed.

1. Introduction and Preliminaries

Since begining of the 20. century, the following famous inequality has been well known in the literature as Hermite-Hadamard's inequality. Many of researchers have extended, generalized and established lots of results with it.

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

The function which holds the above inequality is convex and the definition of convexity is given as follows:

Definition 1.1. Let $f\colon I \subseteq \mathbb{R} \to \mathbb{R}$ be a function and $a, b \in I$ with $a < b$, the function $f\colon I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f\left(tx + (1-t)y\right) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Definition 1.2. A function $f\colon \mathbb{R}_+ \to \mathbb{R}$ is said to be s-convex in the second sense if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

for all $x, y \in \mathbb{R}_+$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

We denote this by K_s^2. It is obvious that the s-convexity means just the convexity when $s = 1$, for more detail see also ([**2, 3, 4, 5, 7, 9**]).

In [**6**], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for s-convex functions in the second sense.

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Theorem 1.1. *Suppose that $f\colon [0,\infty) \to [0,\infty)$ is an s-convex function in the second sense, where $s \in (0,1]$, and let $a, b \in [0,\infty)$, $a < b$. If $f \in L_1[a,b]$, then the following inequality holds*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{s+1} \tag{1.2}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.2). Recently authors have generalized some identities and some results via Riemann-Liouville fractional integrals. In this part of paper, we will give some necessary definitions and properties which we use in this study, for more useful studies see [**8, 10, 11, 13, 14, 15, 16, 23**].

Definition 1.3. Let $f \in L_1[a,b]$. The Riemann-Lioville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$, are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, \qquad x > a,$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\mathrm{d}t, \qquad x < b,$$

respectively. Here $\Gamma(t)$ is the Gamma function and its definition is $\Gamma(t) = \int_0^\infty \mathrm{e}^{-x} x^{t-1}\mathrm{d}x$. It is to be noted that $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$ in the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

The Beta function is defined as follows

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1}(1-t)^{b-1}\mathrm{d}t, \qquad a, b > 0,$$

where $\Gamma(\alpha)$ is Gamma function, and the incompleted beta function is defined as

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\mathrm{d}t.$$

For $x = 1$, the incomplete beta function coincides with the complete beta function. In [**14**], Sarıkaya et al. gave remarkable integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals as follows.

Theorem 1.2. *Let $f\colon [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in [a,b]$. If f is convex function on $[a,b]$, then the following inequality for fractional integrals holds*

$$\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[(J_{a+}^{\alpha} f)(b) + (J_{b-}^{\alpha} f)(a)] \leq \frac{f(a)+f(b)}{2}. \tag{1.3}$$

It is obviously seen that if we take $\alpha = 1$ in Theorem 1.2, then the inequality (1.3) reduces to well known Hermite-Hadamard's inequality as (1.1).

Set et al. gave the Hermite Hadamard type inequality for s-convex functions on Riemann-Liouville fractional integral as follows.

Theorem 1.3. [**15**] *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L_1[a,b]$. If f is an s-convex mapping in the second sense on $[a,b]$, then the following inequality for fractional integral with $\alpha > 0$ and $s \in (0,1]$ hold*

$$\begin{aligned} 2^{s-1} f\left(\frac{a+b}{2}\right) &\le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[(J^{\alpha}_{a^+}f)(b) + (J^{\alpha}_{b^-}f)(a)] \\ &\le \alpha\Big[\frac{1}{\alpha+s} + B(\alpha, s+1)\Big]\frac{f(a)+f(b)}{2}, \end{aligned} \tag{1.4}$$

where $B(a,b)$ is Euler Beta function.

In [**13**], Özdemir et.al proved a new identity and obtained some new results by using this identity, as follows.

Lemma 1.1. *Let $f\colon I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, the interior of I, where $a, b \in I$ with $a < b$. If $f' \in L[a,b]$, then for all $x \in [a,b]$ and $\alpha > 0$, we have*

$$\begin{aligned} &\frac{(x-a)^{\alpha} f(a) + (b-x)^{\alpha} f(b)}{b-a} - \frac{\Gamma(\alpha+1)}{b-a}[J^{\alpha}_{x^-}f(a) + J^{\alpha}_{x^+}f(b)] \\ &= \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 (t^{\alpha}-1) f'(tx+(1-t)a)\mathrm{d}t \\ &\quad + \frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 (1-t^{\alpha}) f'(tx+(1-t)b)\mathrm{d}t, \end{aligned}$$

where $\Gamma(\alpha) = \int_0^{\infty} \mathrm{e}^{-t}\, u^{\alpha-1}\mathrm{d}u$.

Theorem 1.4. *Let $f\colon I \subset [0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is s-convex on $[a,b]$ for some fixed $s \in (0,1]$ and $x \in [a,b]$, then the following inequality for fractional integrals with $\alpha > 0$ holds*

$$\begin{aligned} &\left|\frac{(x-a)^{\alpha} f(a) + (b-x)^{\alpha} f(b)}{b-a} - \frac{\Gamma(\alpha+1)}{b-a}\left[J^{\alpha}_{x^-}f(a) + J^{\alpha}_{x^-}f(b)\right]\right| \\ &\le \frac{\alpha}{(s+1)(\alpha+s+1)}\left[\frac{(x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{b-a}\right]|f'(x)| \\ &\quad + \left[\frac{1}{s+1} - \frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}\right]\left[\frac{(x-a)^{\alpha+1}|f'(a)| + (b-x)^{\alpha+1}|f'(b)|}{b-a}\right], \end{aligned} \tag{1.5}$$

where Γ is Euler Gamma function.

Theorem 1.5. *Let $f\colon I \subset [0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is s-convex on $[a,b]$ for some fixed $s \in (0,1]$ and $x \in [a,b]$, then the following inequality for fractional integrals* *holds.*

$$\left|\frac{(x-a)^{\alpha}f(a)+(b-x)^{\alpha}f(b)}{b-a}-\frac{\Gamma(\alpha+1)}{b-a}\left[J^{\alpha}_{x^-}f(a)+J^{\alpha}_{x^-}f(b)\right]\right|$$
$$\leq\left(\frac{\Gamma(1+p)\Gamma(1+\frac{1}{\alpha})}{\Gamma(1+p+\frac{1}{\alpha})}\right)^{\frac{1}{p}}\left\{\frac{(x-a)^{\alpha+1}}{b-a}\left(\frac{|f'(x)|^q+|f'(a)|^q}{s+1}\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{(b-x)^{\alpha+1}}{b-a}\left(\frac{|f'(x)|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\right\}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $\alpha>0$ *and* Γ *is Euler Gamma function.*

Theorem 1.6. *Let* $f\colon I\subset[0,\infty)\to\mathbb{R}$ *be a differentiable function on* I° *such that* $f'\in L[a,b]$, *where* $a,b\in I$ *with* $a<b$. *If* $|f'|^q$ *is s-convex on* $[a,b]$ *for some fixed* $s\in(0,1]$, $q\geq 1$ *and* $x\in[a,b]$, *then the following inequality for fractional integrals holds*

$$\left|\frac{(x-a)^{\alpha}f(a)+(b-x)^{\alpha}f(b)}{b-a}-\frac{\Gamma(\alpha+1)}{b-a}\left[J^{\alpha}_{x^-}f(a)+J^{\alpha}_{x^-}f(b)\right]\right|$$
$$\leq\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}$$
$$\times\left\{\frac{(x-a)^{\alpha+1}}{b-a}\left(\frac{\alpha}{(s+1)(\alpha+s+1)}|f'(x)|^q+\left[\frac{1}{s+1}-\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}\right]|f'(a)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{(b-x)^{\alpha+1}}{b-a}\left(\frac{\alpha}{(s+1)(\alpha+s+1)}|f'(x)|^q+\left[\frac{1}{s+1}-\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}\right]|f'(b)|^q\right)^{\frac{1}{q}}\right\},$$

where $\alpha>0$ *and* Γ *is Euler Gamma function.*

In the following, we give some definitions and properties of conformable fractional integrals which help to obtain main identity and results. Recently, some authors, started to study on conformable fractional integrals. In [**12**], Khalil et al. defined the fractional integral of order $0<\alpha\leq 1$ only. In [**1**], Abdeljawad gave the definition of left and right conformable fractional integrals of any order $\alpha>0$.

Definition 1.4. Let $\alpha\in(n,n+1]$ and set $\beta=\alpha-n$, then the left conformable fractional integral starting at a is defined by

$$(I^a_{\alpha}f)(t)=\frac{1}{n!}\int_a^t(t-x)^n(x-a)^{\beta-1}f(x)\mathrm{d}x,$$

Analogously, the right conformable fractional integral is defined by

$$({}^bI_{\alpha}f)(t)=\frac{1}{n!}\int_t^b(x-t)^n(b-x)^{\beta-1}f(x)\mathrm{d}x.$$

Notice that if $\alpha=n+1$, then $\beta=\alpha-n=n+1-n=1$, where $n=0,1,2,\ldots$, and hence $(I^a_{\alpha}f)(t)=(J^a_{n+1}f)(t)$. For easy understanding the computation in our theorems, let us give some properties of beta and incompleted beta function

$$B(a,b)=B_t(a,b)+B_{1-t}(b,a),\quad\text{i.e.,}\quad B(a,b)=B_{\frac{1}{2}}(a,b)+B_{\frac{1}{2}}(b,a), \tag{1.6}$$

$$B_x(a+1,b) = \frac{aB_x(a,b) - x^a(1-x)^b}{a+b},$$

$$B_x(a,b+1) = \frac{bB_x(a,b) + x^a(1-x)^b}{a+b}.$$

In [**17**], Set et.al. gave Hermite Hadamard's inequality for conformable fractional integrals as follows.

Theorem 1.7. *Let $f\colon [a,b] \to \mathbb{R}$ be a function with $0 \le a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for conformable fractional integrals holds*

$$f\Big(\frac{a+b}{2}\Big) \le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha\Gamma(\alpha-n)}[(I_\alpha^a f)(b) + ({}^bI_\alpha f)(a)] \le \frac{f(a)+f(b)}{2} \tag{1.7}$$

with $\alpha \in (n, n+1]$, where Γ is Euler Gamma function.

In [**20**], Set et al. established a generalization of Hermite-Hadamard type inequality for s-convex functions and gave some remarks to show the relationships with the classical and Riemann Liouville fractional integrals inequality by using the given properties of conformable fractional integrals.

Theorem 1.8. *Let $f\colon [a,b] \to \mathbb{R}$ be a function with $0 \le a < b$, $s \in (0,1]$, and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for conformable fractional integrals hold*

(1.8)

$$\frac{\Gamma(\alpha-n)}{\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right) \le \frac{1}{(b-a)^\alpha 2^s}[(I_\alpha^a f)(b) + ({}^bI_\alpha f)(a)]$$
$$\le \left[\frac{B(n+s+1,\alpha-n) + B(n+1,\alpha-n+s)}{n!}\right]\frac{f(a)+f(b)}{2^s}$$

with $\alpha \in (n, n+1]$, $n \in \mathbb{N}$, $n = 0, 1, 2\ldots,$ where Γ is Euler Gamma function and $B(a,b)$ is a beta function.

Remark 1.1. If we choose $s = 1$ in Theorem 1.8, using relation between Γ and B functions, the inequality (1.8) reduces to inequality (1.7).

Remark 1.2. If we choose $\alpha = n+1$ in Theorem 1.8, the inequality (1.8) reduces to inequality (1.4). And also if we choose $\alpha = s = 1$ in the inequality (1.8), then we get the Hermite Hadamard's inequality as (1.2).

Also Set et al. established some results for some kind of inequalities via conformable fractional integrals in [**18, 19, 21, 22**].

2. Main Results

Lemma 2.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, the interior of I, where $a, b \in I$ with $a < b$. If $f' \in L[a,b]$, then for all $x \in [a,b]$ and $\alpha > 0$,* *we have*

$$\frac{B(n+1,\alpha-n)}{b-a}\left[(x-a)^{\alpha}f(a)+(b-x)^{\alpha}f(b)\right]-\frac{n!}{b-a}\left[{}^{b}I_{\alpha}f(a)+I_{\alpha}^{a}f(b)\right]$$

$$=\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1\left[B_t(n+1,\alpha-n)-B(n+1,\alpha-n)\right]f'(tx+(1-t)a)\mathrm{d}t$$

$$+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1\left[B(n+1,\alpha-n)-B_t(n+1,\alpha-n)\right]f'(tx+(1-t)b)\mathrm{d}t.$$

Proof. Using the Definiton 1.4, integrating by parts and changing variables with $u=tx+(1-t)a$ and $u=tx+(1-t)b$ in

$$I_1=\int_0^1\left[B_t(n+1,\alpha-n)-B(n+1,\alpha-n)\right]f'(tx+(1-t)a)\mathrm{d}t$$

and

$$I_2=\int_0^1\left[B(n+1,\alpha-n)-B_t(n+1,\alpha-n)\right]f'(tx+(1-t)b)\mathrm{d}t.$$

$$\begin{aligned}I_1&=\int_0^1\left[B_t(n+1,\alpha-n)-B(n+1,\alpha-n)\right]f'(tx+(1-t)a)\mathrm{d}t\\&=\left(B_t(n+1,\alpha-n)-B(n+1,\alpha-n)\right)\frac{f(tx+(1-t)a)}{x-a}\bigg|_0^1\\&\quad-\int_0^1 t^n(1-t)^{\alpha-n-1}\frac{f(tx+(1-t)a)}{x-a}\mathrm{d}t\\&=B(n+1,\alpha-n)\frac{f(a)}{x-a}-\frac{1}{x-a}\int_a^x\left(\frac{u-a}{x-a}\right)^n\left(\frac{x-u}{x-a}\right)^{\alpha-n-1}\frac{f(u)}{x-a}\mathrm{d}u\\&=B(n+1,\alpha-n)\frac{f(a)}{x-a}-\frac{n!}{(x-a)^{\alpha+1}}I_{x^-}f(a)\end{aligned}$$

and

$$\begin{aligned}I_2&=\int_0^1\left[B(n+1,\alpha-n)-B_t(n+1,\alpha-n)\right]f'(tx+(1-t)b)\mathrm{d}t\\&=\left(B(n+1,\alpha-n)-B_t(n+1,\alpha-n)\right)\frac{f(tx+(1-t)b)}{x-b}\bigg|_0^1\\&\quad+\int_0^1 t^n(1-t)^{\alpha-n-1}\frac{f(tx+(1-t)b)}{x-b}\mathrm{d}t\\&=B(n+1,\alpha-n)\frac{f(b)}{b-x}+\frac{1}{x-b}\int_b^x\left(\frac{u-b}{x-b}\right)^n\left(\frac{x-u}{x-b}\right)^{\alpha-n-1}\frac{f(u)}{x-b}\mathrm{d}u\\&=B(n+1,\alpha-n)\frac{f(b)}{b-x}-\frac{n!}{(b-x)^{\alpha+1}}I_{x^+}f(b).\end{aligned}$$

By multiplying I_1 by $\frac{(x-a)^{\alpha+1}}{b-a}$ and I_2 by $\frac{(b-x)^{\alpha+1}}{b-a}$, we get the desired result. □

Remark 2.1. If we choose $\alpha=n+1$ in Lemma 2.1, the above Lemma reduces to Lemma 1.1.

Theorem 2.1. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|$ is s-convex on $[a,b]$ for some fixed $s \in [0,1]$ and $x \in [a,b]$, then the following inequality for conformable fractional integrals with $\alpha > 0$ holds*

$$\begin{aligned}&\left|\frac{B(n+1,\alpha-n)}{b-a}(x-a)^\alpha f(a)+(b-x)^\alpha f(b)-\frac{n!}{b-a}\left[{}^bI_\alpha f(a)+I_\alpha^a f(b)\right]\right|\\&\le\frac{B(n+s+2,\alpha-n)}{s+1}\left(\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\right)|f'(x)|\\&\quad+\frac{B(n+1,\alpha-n)-B(n+1,\alpha-n+s+1)}{s+1}\left(\frac{(x-a)^{\alpha+1}|f(a)|+(b-x)^{\alpha+1}|f(b)|}{b-a}\right),\end{aligned}$$

where $\alpha \in (n,n+1]$, $n = 0,1,2\ldots$, $B(a,b)$ is Euler Beta function.

Proof. Taking modulus in Lemma 2.1 and using the s-convexity of $|f'|$, we get

$$\begin{aligned}&\left|\frac{B(n+1,\alpha-n)}{b-a}\left((x-a)^\alpha f(a)+(b-x)^\alpha f(b)\right)-\frac{n!}{b-a}\left[{}^bI_\alpha f(a)+I_\alpha^a f(b)\right]\right|\\&\le\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1|B_t(n+1,\alpha-n)-B(n+1,\alpha-n)|\left|f'(tx+(1-t)a)\right|\mathrm{d}t\\&\quad+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1|B(n+1,\alpha-n)-B_t(n+1,\alpha-n)|\left|f'(tx+(1-t)b)\right|\mathrm{d}t\\&\le\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1B(n+1,\alpha-n)-B_t(n+1,\alpha-n)\left(t^s|f'(x)|+(1-t)^s|f'(a)|\right)\mathrm{d}t\\&\quad+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1B(n+1,\alpha-n)-B_t(n+1,\alpha-n)\left(t^s|f'(x)|+(1-t)^s|f'(b)|\right)\mathrm{d}t.\end{aligned}\tag{2.1}$$

After some calculation we can write as follows:

$$\begin{aligned}&\int_0^1B(n+1,\alpha-n)-B_t(n+1,\alpha-n)\left(t^s|f'(x)|+(1-t)^s|f'(a)|\right)\mathrm{d}t\\&=|f'(x)|\int_0^1[B(n+1,\alpha-n)-B_t(n+1,\alpha-n)]t^s\mathrm{d}t\\&\quad+|f'(a)|\int_0^1[B(n+1,\alpha-n)-B_t(n+1,\alpha-n)](1-t)^s\mathrm{d}t\\&=|f'(x)|\left[(B(n+1,\alpha-n)-B_t(n+1,\alpha-n))\frac{t^{s+1}}{s+1}\Big|_0^1+\int_0^1t^n(1-t)^{\alpha-n-1}\frac{t^{s+1}}{s+1}\mathrm{d}t\right]\quad(2.2)\\&\quad+|f'(a)|\left[(B_t(n+1,\alpha-n)-B(n+1,\alpha-n))\frac{(1-t)^{s+1}}{s+1}\Big|_0^1\right.\\&\quad\left.-\int_0^1t^n(1-t)^{\alpha-n-1}\frac{(1-t)^{s+1}}{s+1}\mathrm{d}t\right]\\&=|f'(x)|\frac{B(n+s+2,\alpha-n)}{s+1}+|f'(a)|\left(\frac{B(n+1,\alpha-n)-B(n+1,\alpha-n+s+1)}{s+1}\right)\end{aligned}$$

and

$$
\begin{aligned}
&\int_0^1 B(n+1,\alpha-n)-B_t(n+1,\alpha-n)\left(t^s|f'(x)|+(1-t)^s|f'(b)|\right)\mathrm{d}t\\
&=|f'(x)|\left[(B(n+1,\alpha-n)-B_t(n+1,\alpha-n))\left.\frac{t^{s+1}}{s+1}\right|_0^1+\int_0^1 t^n(1-t)^{\alpha-n-1}\frac{t^{s+1}}{s+1}\mathrm{d}t\right]\\
&+|f'(b)|\left[(B_t(n+1,\alpha-n)-B(n+1,\alpha-n))\left.\frac{(1-t)^{s+1}}{s+1}\right|_0^1\right.\\
&\left.-\int_0^1 t^n(1-t)^{\alpha-n-1}\frac{(1-t)^{s+1}}{s+1}\mathrm{d}t\right]\\
&=|f'(x)|\frac{B(n+s+2,\alpha-n)}{s+1}+|f'(b)|\left(\frac{B(n+1,\alpha-n)-B(n+1,\alpha-n+s+1)}{s+1}\right).
\end{aligned}
\tag{2.3}
$$

Combining (2.3) and (2.2) with (2.1), we get desired result. □

Remark 2.2. If we choose $\alpha = n+1$, then the inequality (2.1) reduces to the inequality (1.5).

Theorem 2.2. *Let $f\colon I\subset[0,\infty)\to\mathbb{R}$ be a differentiable function on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|^q$ is s-convex on $[a,b]$ for some fixed $s\in(0,1]$ and $x\in[a,b]$ with $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality for conformable fractional integrals holds*

$$
\begin{aligned}
&\left|\frac{B(n+1,\alpha-n)}{b-a}\left((x-a)^\alpha f(a)+(b-x)^\alpha f(b)\right)-\frac{n!}{b-a}\left({}^bI_\alpha f(a)+I_\alpha^a f(b)\right)\right|\\
&\le\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\lambda^{\frac{1}{p}}\left[\frac{(x-a)^{\alpha+1}}{b-a}\left(|f'(x)|^q+|f'(a)|^q\right)^{\frac{1}{q}}+\left(\frac{b-x}{b-a}\right)^{\alpha+1}\left(|f'(x)|^q+|f'(b)|^q\right)^{\frac{1}{q}}\right],
\end{aligned}
$$

where $\lambda=\int_0^1[B(n+1,\alpha-n)-B_t(n+1,\alpha-n)]^p\mathrm{d}t$ and $\alpha\in(n,n+1]$, $n=0,1,2\ldots$ Also $B(a,b)$ is Euler beta function and $B_t(a,b)$ is incompleted Euler beta function.

Proof. By using Hölder inequality and taking modulus in Lemma 2.1, we have

$$
\begin{aligned}
&\left|\frac{B(n+1,\alpha-n)}{b-a}\left((x-a)^\alpha f(a)+(b-x)^\alpha f(b)\right)-\frac{n!}{b-a}\left({}^bI_\alpha f(a)+I_\alpha^a f(b)\right)\right|\\
&\le\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1|B_t(n+1,\alpha-n)-B(n+1,\alpha-n)|\left|f'(tx+(1-t)a)\right|\mathrm{d}t\\
&\quad+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1|B(n+1,\alpha-n)-B_t(n+1,\alpha-n)|\left|f'(tx+(1-t)b)\right|\mathrm{d}t\\
&\le\frac{(x-a)^{\alpha+1}}{b-a}\left\{\left[\int_0^1[B(n+1,\alpha-n)-B_t(n+1,\alpha-n)]^p\,\mathrm{d}t\right]^{\frac{1}{p}}\right.\\
&\qquad\left.\times\left[\int_0^1|f'(tx+(1-t)a)|^q\mathrm{d}t\right]^{\frac{1}{q}}\right\}\\
&\quad+\frac{(b-x)^{\alpha+1}}{b-a}\left\{\left[\int_0^1[(B(n+1,\alpha-n)-B_t(n+1,\alpha-n))]^p\,\mathrm{d}t\right]^{\frac{1}{p}}\right.\\
&\qquad\left.\times\left[\int_0^1|f'(tx+(1-t)b)|^q\mathrm{d}t\right]^{\frac{1}{q}}\right\}.
\end{aligned}
\tag{2.4}
$$

Since $|f'|^q$ is s-convex on $[a, b]$, we get

$$\int_0^1 |f'(tx + (1-t)a)|^q \mathrm{d}t \le \frac{|f'(x)|^q + |f'(a)|^q}{s+1} \tag{2.5}$$

and

$$\int_0^1 |f'(tx + (1-t)b)|^q \mathrm{d}t \le \frac{|f'(x)|^q + |f'(b)|^q}{s+1}. \tag{2.6}$$

Let

$$\lambda = \int_0^1 [B(n+1, \alpha - n) - B_t(n+1, \alpha - n)]^p \mathrm{d}t. \tag{2.7}$$

Hence, combining (2.5), (2.6) and (2.7) with (2.4), we get the desired result. □

Remark 2.3. If we choose $\alpha = n+1$, then Theorem 2.2 reduces the Theorem 1.5.

Theorem 2.3. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is s-convex on $[a, b]$ for some fixed $s \in [0, 1]$, $q \ge 1$ and $x \in [a, b]$, then the following inequality for conformable fractional integrals holds*

(2.8)

$$\begin{aligned}
&\left| \frac{B(n+1, \alpha - n)}{b-a} \left((x-a)^\alpha f(a) + (b-x)^\alpha f(b)\right) - \frac{n!}{b-a} \left({}^b I_\alpha f(a) + I_\alpha^a f(b)\right) \right| \\
&\le \left(B(n+2, \alpha - n)\right)^{1-\frac{1}{q}} \left(\frac{1}{s+1}\right)^{\frac{1}{q}} \times \Bigg\{ \left(\frac{(x-a)^{\alpha+1}}{b-a}\right) \\
&\quad \times \left[|f'(x)|^q B(n+s+2, \alpha-n) + |f'(a)|^q \left(B(n+1, \alpha-n) - B(n+1, \alpha-n+s+1)\right)\right]^{\frac{1}{q}} \\
&\quad + \left(\frac{(b-x)^{\alpha+1}}{b-a}\right) \\
&\quad \times \left[|f'(x)|^q B(n+s+2, \alpha-n) + |f'(b)|^q \left(B(n+1, \alpha-n) - B(n+1, \alpha-n+s+1)\right)\right]^{\frac{1}{q}} \Bigg\},
\end{aligned}$$

where $\alpha \in (n, n+1]$, $n = 0, 1, 2 \ldots$ Also $B(a, b)$ is Euler beta function and $B_t(a, b)$ is incompleted Euler beta function.

Proof. Taking modulus on Lemma 2.1 and using well-known power-mean inequality, we get

$$\begin{aligned}
&\left| \frac{B(n+1, \alpha - n)}{b-a} \left[(x-a)^\alpha f(a) + (b-x)^\alpha f(b)\right] - \frac{n!}{b-a} \left({}^b I_\alpha f(a) + I_\alpha^a f(b)\right) \right| \\
&\le \frac{(x-a)^{\alpha+1}}{b-a} \Bigg\{ \left[\int_0^1 (B(n+1, \alpha - n) - B_t(n+1, \alpha - n)) \mathrm{d}t\right]^{1-\frac{1}{q}} \\
&\quad \times \left[\int_0^1 \left(B(n+1, \alpha - n) - B_t(n+1, \alpha - n)\right) |f'(tx + (1-t)a)|^q \mathrm{d}t\right]^{\frac{1}{q}} \Bigg\} \qquad (2.9) \\
&\quad + \frac{(b-x)^{\alpha+1}}{b-a} \Bigg\{ \left[\int_0^1 \left(B(n+1, \alpha - n) - B_t(n+1, \alpha - n)\right) \mathrm{d}t\right]^{1-\frac{1}{q}} \\
&\quad \times \left[\int_0^1 \left(B(n+1, \alpha - n) - B_t(n+1, \alpha - n)\right) |f'(tx + (1-t)b)|^q \mathrm{d}t\right]^{\frac{1}{q}} \Bigg\}.
\end{aligned}$$

By using integrating by parts, we get

$$\begin{aligned}&\int_0^1 (B(n+1,\alpha-n)-B_t(n+1,\alpha-n))\,\mathrm{d}t\\ &=(B(n+1,\alpha-n)-B_t(n+1,\alpha-n))\,t\Big|_0^1+\int_0^1 t^n(1-t)^{\alpha-n-1}t\mathrm{d}t\\ &=B(n+2,\alpha-n).\end{aligned} \tag{2.10}$$

Since $|f'|^q$ is s-convex in the second sense, we can write

$$\begin{aligned}&\int_0^1 [B(n+1,\alpha-n)-B_t(n+1,\alpha-n)]|f'(tx+(1-t)a)|^q\mathrm{d}t\\ &\le \int_0^1 (B(n+1,\alpha-n)-B_t(n+1,\alpha-n))\,t^s|f'(x)|^q\mathrm{d}t\\ &\quad+\int_0^1 (B(n+1,\alpha-n)-B_t(n+1,\alpha-n))\,(1-t)^s|f'(a)|^q\mathrm{d}t\\ &=|f'(x)|^q\left[(B(n+1,\alpha-n)-B_t(n+1,\alpha-n))\,\frac{t^{s+1}}{s+1}\Big|_0^1+\int_0^1 t^n(1-t)^{\alpha-n-1}\frac{t^{s+1}}{s+1}\mathrm{d}t\right]\\ &\quad+|f'(a)|^q\Bigg[(-B(n+1,\alpha-n)+B_t(n+1,\alpha-n))\,\frac{(1-t)^{s+1}}{s+1}\Big|_0^1\\ &\quad-\int_0^1 t^n(1-t)^{\alpha-n-1}\frac{(1-t)^{s+1}}{s+1}\mathrm{d}t\Bigg]\\ &=\frac{|f'(x)|^q}{s+1}B(n+s+2,\alpha-n)+\frac{|f'(a)|^q}{s+1}\left[B(n+1,\alpha-n)-B_t(n+1,\alpha-n+s+1)\right]\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}&\int_0^1 [B(n+1,\alpha-n)-B_t(n+1,\alpha-n)]|f'(tx+(1-t)b)|^q\mathrm{d}t\\ &\le \int_0^1 [B(n+1,\alpha-n)-B_t(n+1,\alpha-n)]\left(t^s|f'(x)|^q+(1-t)^s|f'(b)|^q\right)\mathrm{d}t\\ &=|f'(x)|^q\left[[B(n+1,\alpha-n)-B_t(n+1,\alpha-n)]\,\frac{t^{s+1}}{s+1}\Big|_0^1+\int_0^1 t^n(1-t)^{\alpha-n-1}\frac{t^{s+1}}{s+1}\mathrm{d}t\right]\\ &\quad+|f'(b)|^q\Bigg[[-B(n+1,\alpha-n)+B_t(n+1,\alpha-n)]\,\frac{(1-t)^{s+1}}{s+1}\Big|_0^1\\ &\quad-\int_0^1 t^n(1-t)^{\alpha-n-1}\frac{(1-t)^{s+1}}{s+1}\mathrm{d}t\Bigg]\\ &=\frac{|f'(x)|^q}{s+1}B(n+s+2,\alpha-n)+\frac{|f'(b)|^q}{s+1}\left[B(n+1,\alpha-n)-B_t(n+1,\alpha-n+s+1)\right].\end{aligned} \tag{2.12}$$

Hence by combining (2.10), (2.11), and (2.12) with (2.8), we get the desired result. □

Remark 2.4. If we choose $\alpha=n+1$, then Theorem 2.3 reduces to the Theorem 1.6.

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