Acta Math. Univ. Comenianae
Vol. LXXXVII, 1 (2018), pp. 1–13

RELATING THE ANNIHILATION NUMBER AND THE ROMAN DOMINATION NUMBER

H. ARAM, R. KHOEILAR, S. M. SHEIKHOLESLAMI AND L. VOLKMANN

Abstract. A *Roman dominating function* (RDF) on a graph G is a labeling $f\colon V(G) \to \{0,1,2\}$ such that every vertex with label 0 has a neighbor with label 2. The *weight* of an RDF f is the value $\omega(f) = \sum_{v\in V} f(v)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, equals the minimum weight of an RDF on G. The annihilation number $a(G)$ is the largest integer k such that the sum of the first k terms of the non-decreasing degree sequence of G is at most the number of edges in G. In this paper, we prove that for any tree T of order at least two, $\gamma_R(T) \leq \frac{4a(T)+2}{3}$.

1. Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V(G)$, the *open neighborhood* $N_G(v) = N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = \deg(v) = |N(v)|$.

We write P_n for a path of order n. For a subset $S \subseteq V(G)$, we define

$$\sum(S,G) = \sum_{v\in S} \deg_G(v).$$

A *leaf* of a tree T is a vertex of degree 1, a *support vertex* is a vertex adjacent to a leaf and a *strong support vertex* is a vertex adjacent to at least two leaves. For $r, s \geq 1$, a double star $S(r, s)$ is a tree with exactly two vertices that are not leaves, with one adjacent to r leaves and the other one to s leaves. For a vertex v in a rooted tree T, let $D(v)$ denote the set of descendants of v and $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v.

A *Roman dominating function* (RDF) on a graph $G = (V,E)$ is defined in [**16, 17**] as a function $f\colon V \to \{0,1,2\}$ satisfying the condition that every vertex v for which $f(v) = 0$ is adjacent to at least one vertex u for which $f(u) = 2$. The *weight* of an RDF f is the value $\omega(f) = \sum_{v\in V} f(v)$. The *Roman domination*

Received June 15, 2016; revised December 21, 2017.
2010 *Mathematics Subject Classification.* Primary 05C69.
Key words and phrases. Annihilation number; Roman dominating function; Roman domination number.

number of a graph G, denoted by $\gamma_R(G)$, equals the minimum weight of an RDF on G. A $\gamma_R(G)$-*function* is a Roman dominating function of G with weight $\gamma_R(G)$.

The definition of the Roman dominating function was given implicitly by Stewart [**17**] and ReVelle and Rosing [**16**]. Cockayne, Dreyer Jr., Hedetniemi and Hedetniemi [**3**] as well as Chambers, Kinnersley, Prince and West [**2**] gave a lot of results on Roman domination. For more information on Roman domination, we refer the reader to [**4, 9, 10, 11, 12, 13**].

Let $d_1, d_2, \dots, d_n$ be the degree sequence of a graph G arranged in non-decreasing order, so $d_1 \le d_2 \le \dots \le d_n$. Pepper [**15**] defined the annihilation number of G, denoted $a(G)$, to be the largest integer k such that the sum of the first k terms of the degree sequence is at most half the sum of the degrees in the sequence. Equivalently, the annihilation number is the largest integer k such that

$$\sum_{i=1}^{k} d_i \le \sum_{i=k+1}^{n} d_i.$$

We observe that if G has m edges and annihilation number k, then $\sum_{i=1}^{k} d_i \le m$.

The relation between annihilation number and domination parameters were studied in [**1, 5, 6, 7, 8, 14**].

Our purpose in this paper is to establish an upper bound on Roman domination number in term of annihilation number for trees.

Proposition A ([**3**]). *For $n \ge 2$,*

$$\gamma_R(P_n) = \left\lceil \frac{2n}{3} \right\rceil.$$

Proposition B ([**15**]). *For $n \ge 2$,*

$$a(P_n) = \left\lceil \frac{n}{2} \right\rceil.$$

Corollary 1. *For $n \ge 2$,*

$$\gamma_R(P_n) \le \frac{4a(P_n) + 2}{3}$$

with equality if and only if $n \equiv 2 \pmod 6$.

2. Main result

A *subdivision* of an edge uv is obtained by removing the edge uv, adding a new vertex w, and adding edges uw and wv. The *subdivision graph* $S(G)$ is the graph obtained from G by subdividing each edge of G. The subdivision star $S(K_{1,t})$ for $t \ge 2$, is called a *healthy spider* S_t. A *wounded spider* S_t is the graph formed by subdividing at most $t-1$ of the edges of a star $K_{1,t}$ for $t \ge 2$. Note that stars are wounded spiders. A *spider* is a healthy or wounded spider.

Lemma 2. *If T is a spider, then $\gamma_R(T) < \frac{4a(T)+2}{3}$.*

Proof. First let $T = S_t$ be a healthy spider for some $t \geq 2$. Then obviously $\gamma_R(T) = 2 + t$ and $a(T) = t + \lfloor \frac{t}{2} \rfloor$, and hence $\gamma_R(T) = 2 + t \leq \frac{4a(T)}{3} < \frac{4a(T)+2}{3}$.

Now let T be a wounded spider obtained from $K_{1,t}$ ($t \geq 2$) by subdividing $0 \leq s \leq t-1$ edges. If $s = 0$, then T is a star, and we have $\gamma_R(T) = 2$ and $a(T) = t$. Hence $\gamma_R(T) = 2 < \frac{4a(T)+2}{3}$. Now assume that $s > 0$. If $s = 1$ and $t = 2$, then $T = P_4$ and the result follows from Corollary 1. Let $s \geq 2$ or $s = 1$ and $t \geq 3$. Then $\gamma_R(T) = 2 + s$ and $a(T) = t + \lfloor \frac{s}{2} \rfloor$. It follows that $\gamma_R(T) < \frac{4a(T)+2}{3}$. This completes the proof. $\square$

Theorem 3. *If T is a tree of order $n \geq 2$, then $\gamma_R(T) \leq \frac{4a(T)+2}{3}$, and this bound is sharp.*

Proof. The proof is by induction on n. The statement holds for all trees of order $n = 2, 3, 4$. For the the induction hypothesis, let $n \geq 5$ and suppose that for every nontrivial tree T of order less than n the result is true. Let T be a tree of order n. We may assume that T is not a path otherwise the result follows by Corollary 1. If $\mathrm{diam}(T) = 2$, then T is a star, and we have $\gamma_R(T) < \frac{4a(T)+2}{3}$ by Lemma 2. If $\mathrm{diam}(T) = 3$, then T is a double star $S(r, s)$. In this case, $a(T) = r + s$ and $\gamma_R(T) \leq 4$. If $r + s = 3$, then $\gamma_R(T) = 3$ and so $\gamma_R(T) < \frac{4a(T)+2}{3}$. If $r + s \geq 4$, then $\gamma_R(T) \leq 4$ and we have $\gamma_R(T) < \frac{4a(T)+2}{3}$. Hence, we may assume that $\mathrm{diam}(T) \geq 4$.

In what follows, we consider trees T' formed from T by removing a set of vertices. For such a tree T' of order n', let $d'_1, d'_2, \ldots, d'_{n'}$ be the non-decreasing degree sequence of T', and let S' be a set of vertices corresponding to the first $a(T')$ terms in the degree sequence of T'. In fact, if $u_1, u_2, \ldots, u_{n'}$ are the vertices of T' such that $\deg(u_i) = d'_i$ for each $1 \leq i \leq n'$, then $S' = \{u_1, u_2, \ldots, u_{a(T')}\}$. We denote the size of T' by m'. We proceed further with a series of claims that we may assume satisfied by the tree.

Claim 1. *T has no strong support vertex such as u that the graph obtained from T by removing u and the leaves adjacent to u is connected.*

Proof. Let T have a strong support vertex u such that the graph obtained from T by removing u and the leaves adjacent to u is connected. Suppose w is a vertex in T with maximum distance from u. Root T at w and let v be the parent of u. Assume $T' = T - T_u$. It is easy to see that $\gamma_R(T) \leq \gamma_R(T') + 2$.

$$\sum(S', T) = \begin{cases} \sum(S', T') & \text{if } v \notin S', \\ \sum(S', T') + 1 & \text{if } v \in S'. \end{cases}$$

Thus,

$$\sum(S', T) - 1 \leq \sum(S', T') \leq m' \leq m - 3.$$

and hence $\sum(S', T) \leq m - 2$. Let z_1, z_2 be two leaves adjacent to u and assume $S = S' \cup \{z_1, z_2\}$. Then $\sum(S, T) = \sum(S', T) + 2 \leq m$, implying that $a(T) \geq a(T') + 2$.

By the induction hypothesis, we obtain

$$\gamma_R(T) \le \gamma_R(T') + 2 \le \frac{4a(T')+2}{3} + 2 \le \frac{4(a(T)-2)+2}{3} + 2 < \frac{4a(T)+2}{3},$$

as desired. □

Let $v_1v_2\dots v_D$ be a diametral path in T and root T at v_D (v_1, respectively). By Claim 1, we may assume any support vertex on a diametral path has degree 2. In particular, $\deg_T(v_2) = \deg_T(v_{D-1}) = 2$. If $\mathrm{diam}(T) = 4$, then T is a spider and so $\gamma_R(T) < \frac{4a(T)+2}{3}$ by Lemma 2. Assume $\mathrm{diam}(T) \ge 5$. It follows from Claim 1 that T_{v_3} ($T_{v_{D-2}}$, respectively) is a spider.

Claim 2. $\deg_T(v_3) \le 3$.

Proof. Let $\deg_T(v_3) \ge 4$. Let $T' = T - \{v_1, v_2\}$. Then obviously there exists a $\gamma_R(T')$-function that assigns 2 to v_3 and hence can be extended to an RDF of T by assigning 1 to v_1 and 0 to v_2. Thus $\gamma_R(T) \le \gamma_R(T') + 1$.

$$\sum(S', T) = \begin{cases} \sum(S', T') & \text{if } v_3 \notin S', \\ \sum(S', T') + 1 & \text{if } v_3 \in S'. \end{cases}$$

Thus,

$$\sum(S', T) \le \sum(S', T') + 1 \le m' + 1 = m - 1.$$

Let $S = S' \cup \{v_1\}$. Then $\sum(S, T) = \sum(S', T) + \deg_T(v_1) \le m$, and hence $a(T) \ge |S| = |S'| + 1 = a(T') + 1$. By the induction hypothesis, we obtain

$$\gamma_R(T) \le \gamma_R(T') + 1 \le \frac{4a(T')+2}{3} + 1 \le \frac{4(a(T)-1)+2}{3} + 1 < \frac{4a(T)+2}{3},$$

as desired. □

Claim 3. $\deg_T(v_3) = 2$.

Proof. Assume that $\deg_T(v_3) = 3$. First let v_3 be adjacent to a support vertex $z_2 \notin \{v_2, v_4\}$. By Claim 1, we may assume $\deg_T(z_2) = 2$. Suppose z_1 is the leaf adjacent to z_2 and let $T' = T - T_{v_3}$. Then every $\gamma_R(T')$-function can be extended to an RDF of T by assigning 1 to v_1, z_1, 0 to v_2, z_2 and 2 to v_3. Thus $\gamma_R(T) \le \gamma_R(T') + 4$. As above, we have $\sum(S', T) \le \sum(S', T') + 1 \le m' + 1 = m - 4$. Let $S = S' \cup \{v_1, v_2, z_1\}$. Then

$$\sum(S, T) = \sum(S', T) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(z_1) \le m,$$

implying that $a(T) \ge |S| = |S'| + 3 = a(T') + 3$. It follows from the induction hypothesis that

$$\gamma_R(T) \le \gamma_R(T') + 4 \le \frac{4a(T')+2}{3} + 4 \le \frac{4(a(T)-3)+2}{3} + 4 = \frac{4a(T)+2}{3}.$$

Now let v_3 be adjacent to a leaf w. Considering Claims 1, 2 and the first part of Claim 3, we distinguish the following cases.

<u>*Case 3.1.*</u> $\deg_T(v_4) \geq 4$.

Let $T' = T - T_{v_3}$. Then every $\gamma_R(T')$-function can be extended to an RDF of T by assigning 1 to v_1, 0 to v_2, w and 2 to v_3. Hence, $\gamma_R(T) \leq \gamma_R(T') + 3$. Suppose that $v_4 \notin S'$. In this case, let $S = S' \cup \{v_1, v_2, w\}$. Then

$$\begin{aligned}\sum(S,T) &= \sum(S',T) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(w)\\ &= \sum(S',T') + 4 \leq m' + 4 = m,\end{aligned}$$

implying that $a(T) \geq a(T') + 3$. By the induction hypothesis, we have

$$\gamma_R(T) \leq \gamma_R(T') + 3 \leq \frac{4a(T')+2}{3} + 3 \leq \frac{4(a(T)-3)+2}{3} + 3 < \frac{4a(T)+2}{3}.$$

Now let $v_4 \in S'$. In this case, let $S = (S' - \{v_4\}) \cup \{v_1, v_2, v_3, w\}$. Then

$$\sum(S,T) = \sum(S',T') - \deg_{T'}(v_4) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(v_3) + \deg_T(w) \leq m,$$

which implies that $a(T) \geq |S| = |S'| + 3 = a(T') + 3$. As above, it follows from the induction hypothesis that $\gamma_R(T) < \frac{4a(T)+2}{3}$.

<u>*Case 3.2.*</u> $\deg_T(v_4) = 2$.

Assume $T' = T - T_{v_4}$. Then every $\gamma_R(T')$-function can be extended to an RDF of T by assigning 1 to v_1, 0 to v_2, v_4, w and 2 to v_3. Hence, $\gamma_R(T) \leq \gamma_R(T') + 3$.

$$\sum(S',T) = \begin{cases} \sum(S',T') & \text{if } v_5 \notin S',\\ \sum(S',T') + 1 & \text{if } v_5 \in S'.\end{cases}$$

Thus, $\sum(S',T) \leq \sum(S',T') + 1 \leq m' + 1 = m - 4$.

Let $S = S' \cup \{v_1, v_2, w\}$. Then

$$\sum(S,T) = \sum(S',T) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(w) \leq m.$$

Therefore, $a(T) \geq |S| = |S'| + 3 = a(T') + 3$. As above, it follows from the induction hypothesis that $\gamma_R(T) < \frac{4a(T)+2}{3}$.

<u>*Case 3.3.*</u> $\deg_T(v_4) = 3$ and v_4 is adjacent to a leaf, say w'.

Let $T' = T - T_{v_4}$. Then every $\gamma_R(T')$-function can be extended to an RDF of T by assigning 1 to v_1, w', 0 to v_2, v_4, w and 2 to v_3. Hence, $\gamma_R(T) \leq \gamma_R(T') + 4$. Clearly, we have $\sum(S',T) \leq \sum(S',T') + 1 \leq m' + 1 = m - 5$.

Let $S = S' \cup \{v_1, v_2, w, w'\}$. Then

$$\sum(S,T) = \sum(S',T) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(w) + \deg_T(w') \leq m,$$

implying that $a(T) \geq |S| = |S'| + 4 = a(T') + 4$. It follows from the induction hypothesis that

$$\gamma_R(T) \leq \gamma_R(T') + 4 \leq \frac{4a(T')+2}{3} + 4 \leq \frac{4(a(T)-4)+2}{3} + 4 < \frac{4a(T)+2}{3}.$$

<u>*Case 3.4.*</u> $\deg_T(v_4) = 3$ and v_4 is adjacent to a support vertex other than v_5, say w_2.

By Claim 1, we may assume $\deg_T(w_2) = 2$. Let w_1 be the leaf adjacent to w_2 and let $T' = T - T_{v_4}$. Then every $\gamma_R(T')$-function can be extended to an RDF of T by assigning 1 to v_1, w_1, w_2, 0 to v_2, v_4, w and 2 to v_3. Hence, $\gamma_R(T) \leq \gamma_R(T') + 5$. Obviously, $\sum(S', T) \leq \sum(S', T')+1 \leq m'+1 = m-6$. Let $S = S' \cup \{v_1, v_2, w, w_1\}$. Then

$$\sum(S,T) = \sum(S',T) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(w) + \deg_T(w_1) \leq m,$$

which implies that $a(T) \geq |S| = |S'| + 4 = a(T') + 4$. It follows from the induction hypothesis that

$$\gamma_R(T) \leq \gamma_R(T') + 5 \leq \frac{4a(T')+2}{3} + 5 \leq \frac{4(a(T)-4)+2}{3} + 5 < \frac{4a(T)+2}{3}.$$

Case 3.5. $\deg_T(v_4) = 3$ and there is a path $v_4w_3w_2w_1$ in T such that $\deg_T(w_3) = \deg_T(w_2) = 2$, $\deg_T(w_1) = 1$ and $w_3 \neq v_5$.
Let $T' = T - T_{v_4}$. Then every $\gamma_R(T')$-function can be extended to an RDF of T by assigning 1 to v_1, 0 to v_2, v_4, w, w_3, w_1 and 2 to v_3, w_2. Hence, $\gamma_R(T) \leq \gamma_R(T')+5$. If $v_5 \notin S'$, then $\sum(S', T) = \sum(S', T')$, and if $v_5 \in S'$, then $\sum(S', T) = \sum(S', T') + 1$. Thus, $\sum(S', T) \leq \sum(S', T') + 1 \leq m' + 1 = m - 7$. Let $S = S' \cup \{v_1, v_2, w, w_1, w_2\}$. Then

$$\sum(S,T) = \sum(S',T)+\deg_T(v_1)+\deg_T(v_2)+\deg_T(w)+\deg_T(w_1)+\deg_T(w_2) \leq m,$$

implying that $a(T) \geq |S| = |S'| + 5 = a(T') + 5$. By the induction hypothesis, we have

$$\gamma_R(T) \leq \gamma_R(T') + 5 \leq \frac{4a(T')+2}{3} + 5 \leq \frac{4(a(T)-5)+2}{3} + 5 < \frac{4a(T)+2}{3}.$$

Case 3.6. $\deg_T(v_4) = 3$ and there is a path $w_4w_3w_2w_1$ in T such that $v_4w_3 \in E(T)$, $\deg_T(w_3) = 3, \deg_T(w_2) = 2$, $\deg_T(w_1) = \deg_T(w_4) = 1$ and $w_3 \neq v_5$.
Assume $T' = T - T_{v_4}$. Then every $\gamma_R(T')$-function can be extended to an RDF of T by assigning 1 to v_1, w_1, 0 to v_2, v_4, w, w_2, w_4 and 2 to v_3, w_3. Hence, $\gamma_R(T) \leq \gamma_R(T') + 6$. As above, we have $\sum(S', T) \leq \sum(S', T') + 1 \leq m' + 1 = m - 8$. Suppose $S = S' \cup \{v_1, v_2, w, w_1, w_2, w_4\}$. Then

$$\sum(S,T) = \sum(S',T) + 8 \leq m,$$

implying that $a(T) \geq |S| = |S'| + 6 = a(T') + 6$. By the induction hypothesis,

$$\gamma_R(T) \leq \gamma_R(T') + 6 \leq \frac{4a(T')+2}{3} + 6 \leq \frac{4(a(T)-6)+2}{3} + 6 < \frac{4a(T)+2}{3}.$$

□

So far, we have proved that we can assume $\deg(v_2) = \deg(v_3) = 2$. Similarly, we can assume that $\deg(v_{D-1}) = \deg(v_{D-2}) = 2$. Since T is not a path, we must have $\mathrm{diam}(T) \geq 6$.

Claim 4. $\deg_T(v_4) = 2$.

Proof. Assume $\deg_T(v_4) \geq 3$ and let $T' = T - T_{v_3}$. Then every $\gamma_R(T')$-function can be extended to an RDF of T by assigning 0 to v_1, v_3 and 2 to v_2. Thus

$\gamma_R(T) \leq \gamma_R(T') + 2$. Suppose that $v_4 \notin S'$. Then $\sum(S', T) = \sum(S', T')$. In this case, let $S = S' \cup \{v_1, v_2\}$. Then

$$\sum(S,T) = \sum(S',T) + \deg_T(v_1) + \deg_T(v_2) \leq m' + 3 = m,$$

implying that $a(T) \geq |S| = |S'|+2 = a(T')+2$. Applying the induction hypothesis, we obtain

$$\gamma_R(T) \leq \gamma_R(T') + 2 \leq \frac{4a(T')+2}{3} + 2 \leq \frac{4(a(T)-2)+2}{3} + 2 < \frac{4a(T)+2}{3},$$

as desired.

Now we assume $v_4 \in S'$. In this case, let $S = (S' - \{v_4\}) \cup \{v_1, v_2, v_3\}$. Since $\deg_T(v_3) = 2 \leq \deg_{T'}(v_4)$, we have

$$\sum(S,T) = \sum(S',T) - \deg_{T'}(v_4) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(v_3) \leq m.$$

Therefore, $a(T) \geq |S| = |S'| + 2 = a(T') + 2$. As above, it follows from the the induction hypothesis that

$$\gamma_R(T) < \frac{4a(T)+2}{3},$$

as desired. □

Similarly, we may assume that $\deg_T(v_{D-3}) = 2$. Since T is not a path, we have $\mathrm{diam}(T) \geq 8$.

Claim 5. $\deg_T(v_5) = 2$.

Proof. Assume that $\deg_T(v_5) \geq 3$. By Claims 1, 2, 3 and 4, we distinguish the following cases.

<u>*Case 5.1*</u> $\deg_T(v_5) \geq 4$.

Assume that $T' = T - (T_{v_4} \cup T_{v_{D-2}})$, where $T_{v_{D-2}}$ is the maximal subtree at v_{D-2} when T is rooted at v_1. Then every $\gamma_R(T')$-function can be extended to an RDF of T by assigning 1 to v_1, 0 to v_2, v_4, v_D, v_{D-2} and 2 to v_3, v_{D-1}. Thus $\gamma_R(T) \leq \gamma_R(T')+5$. If $v_5 \notin S'$, then obviously $\sum(S', T) \leq \sum(S', T')+1 \leq m-6$. In this case, let $S = S' \cup \{v_1, v_2, v_3, v_D\}$. Then

$$\sum(S,T) = \sum(S',T) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(v_3) + \deg_T(v_D) \leq m,$$

implying that $a(T) \geq |S| = |S'| + 4 = a(T') + 4$. It follows from the induction hypothesis that

$$\gamma_R(T) \leq \gamma_R(T') + 5 \leq \frac{4a(T')+2}{3} + 5 \leq \frac{4(a(T)-4)+2}{3} + 5 < \frac{4a(T)+2}{3}.$$

Let $v_5 \in S'$. Then $\sum(S', T) \leq \sum(S', T') + 2$. Suppose $S = (S' - \{v_5\}) \cup \{v_1, v_2, v_3, v_4, v_D\}$. Then

$$\begin{aligned}\sum(S,T) \leq \sum(S',T) &- \deg_{T'}(v_5) + \deg_T(v_1) + \deg_T(v_2) \\ &+ \deg_T(v_3) + \deg_T(v_4) + \deg_T(v_D) \leq m,\end{aligned}$$

which implies that $a(T) \geq |S| = |S'|+4 = a(T')+4$. By the induction hypothesis,

$$\gamma_R(T) < \frac{4a(T)+2}{3}$$

as above.

Case 5.2. $\deg_T(v_5) = 3$ and v_5 is adjacent to a support vertex $w_2 \neq v_6$.
By Claim 1, we may assume $\deg_T(w_2) = 2$. Suppose w_1 is the leaf adjacent to w_2 and let $T' = T - T_{v_5}$. Then every $\gamma_R(T')$-function can be extended to an RDF of T by assigning 1 to v_1, 0 to v_2, v_4, v_5, w_1 and 2 to v_3, w_2. Thus $\gamma_R(T) \leq \gamma_R(T') + 5$. Clearly, $\sum(S', T) \leq \sum(S', T') + 1$. Let $S = S' \cup \{v_1, v_2, v_3, w_1\}$. Then

$$\begin{aligned}\sum(S,T) &= \sum(S',T) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(v_3) + \deg_T(w_1)\\ &= \sum(S',T') + 7 \leq m' + 7 = m,\end{aligned}$$

implying that $a(T) \geq a(T') + 4$. It follows from the induction hypothesis that

$$\gamma_R(T) \leq \gamma_R(T') + 5 \leq \frac{4a(T')+2}{3} + 5 \leq \frac{4(a(T)-4)+2}{3} + 5 < \frac{4a(T)+2}{3}.$$

Case 5.3 $\deg_T(v_5) = 3$ and there is a path $w_4w_3w_2w_1$ in T such that $v_5w_3 \in E(T)$, $\deg_T(w_3) \geq 3, \deg_T(w_2) = 2$, $\deg_T(w_1) = \deg_T(w_4) = 1$ and $w_3 \neq v_6$.
Using an argument similar to that described in Claims 2 and 3, we may assume that $\deg_T(w_3) = 3$. Let $T' = T - (T_{v_4} \cup T_{w_3})$. Then every $\gamma_R(T')$-function can be extended to an RDF of T by assigning 1 to v_1, w_1, 0 to v_2, v_4, w_2, w_4 and 2 to v_3, w_3. Thus $\gamma_R(T) \leq \gamma_R(T') + 6$. Suppose that $v_5 \notin S'$. Then $\sum(S', T) = \sum(S', T')$. In this case, let $S = S' \cup \{w_1, w_4, v_1, v_2, v_3\}$. Then

$$\sum(S,T) = \sum(S',T) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(v_3) + \deg_T(w_1) + \deg_T(w_4) \leq m,$$

implying that $a(T) \geq |S| = |S'| + 5 = a(T') + 5$. It follows from the induction hypothesis that

$$\gamma_R(T) \leq \gamma_R(T') + 6 \leq \frac{4a(T')+2}{3} + 6 \leq \frac{4(a(T)-5)+2}{3} + 6 < \frac{4a(T)+2}{3}.$$

Assume now that $v_5 \in S'$. Then $\sum(S', T) = \sum(S', T') + 1$. Suppose $S = (S' - \{v_5\}) \cup \{v_1, v_2, v_3, w_1, w_2, w_4\}$. Then

$$\begin{aligned}\sum(S,T) = \sum(S',T) &- \deg_{T'}(v_5) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(v_3)\\ &+ \deg_T(w_1) + \deg_T(w_2) + \deg_T(w_4) \leq m.\end{aligned}$$

Therefore, $a(T) \geq |S| = |S'| + 5 = a(T') + 5$ and the result follows as above.

Case 5.4 $\deg_T(v_5) = 3$ and there is a path $v_5w_3w_2w_1$ in T such that $\deg_T(w_3) = \deg_T(w_2) = 2$, $\deg_T(w_1) = 1$ and $w_3 \neq v_6$.
Suppose $T' = T - (T_{v_4} \cup T_{w_3})$. Then every $\gamma_R(T')$-function can be extended to an RDF of T by assigning 1 to v_1, 0 to v_2, v_4, w_1, w_3 and 2 to v_3, w_2. Thus $\gamma_R(T) \leq \gamma_R(T') + 5$. If $v_5 \notin S'$, then let $S = S' \cup \{w_1, v_1, v_2, v_3\}$, and if $v_5 \in S'$, then let $S = (S' - \{v_5\}) \cup \{v_1, v_2, v_3, w_1, w_2\}$. In both cases, it is easy to see that

$\sum(S,T) \leq m$, which implies that $a(T) \geq |S| = |S'| + 4 = a(T') + 4$. It follows from the induction hypothesis that

$$\gamma_R(T) \leq \gamma_R(T') + 5 \leq \frac{4a(T')+2}{3} + 5 \leq \frac{4(a(T)-4)+2}{3} + 5 < \frac{4a(T)+2}{3}.$$

<u>*Case 5.5*</u> $\deg_T(v_5) = 3$ and there is a path $v_5w_4w_3w_2w_1$ in T such that $\deg_T(w_4) = \deg_T(w_3) = \deg_T(w_2) = 2$, $\deg_T(w_1) = 1$ and $w_4 \neq v_6$.
Let $T' = T - T_{v_5}$. Then every $\gamma_R(T')$-function can be extended to an RDF of T by assigning 0 to $v_1, v_3, v_4, w_1, w_3, w_4$ and 2 to v_2, v_5, w_2. Thus $\gamma_R(T) \leq \gamma_R(T') + 6$. Obviously, $\sum(S',T) \leq \sum(S',T') + 1$. Let $S = S' \cup \{w_1, v_1, v_2, v_3, v_4\}$. Then

$$\sum(S,T) = \sum(S',T) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(v_3) + \deg_T(v_4) + \deg_T(w_1) \leq m,$$

implying that $a(T) \geq |S| = |S'| + 5 = a(T') + 5$. It follows from the induction hypothesis that

$$\gamma_R(T) \leq \gamma_R(T') + 6 \leq \frac{4a(T')+2}{3} + 6 \leq \frac{4(a(T)-5)+2}{3} + 6 < \frac{4a(T)+2}{3}.$$

<u>*Case 5.6.*</u> $\deg_T(v_5) = 3$ and v_5 is adjacent to a leaf w.
If $\mathrm{diam}(T) = 8$, then T is the tree obtained from a path P_9 by adding a pendant edge to its center. In this case, we have $\gamma_R(T) = a(T) = 6$, and hence $\gamma_R(T) < \frac{4a(T)+2}{3}$. Let $\mathrm{diam}(T) \geq 9$ and let $T' = T - (T_{v_5} \cup T_{v_{D-2}})$, where $T_{v_{D-2}}$ is the maximal subtree at v_{D-2} when T is rooted at v_1. Then every $\gamma_R(T')$-function can be extended to an RDF of T by assigning 0 to $w, v_1, v_3, v_4, v_D, v_{D-2}$ and 2 to v_2, v_5, v_{D-1}. Thus $\gamma_R(T) \leq \gamma_R(T') + 6$. Clearly, $\sum(S',T) \leq \sum(S',T') + 2$. Let $S = S' \cup \{v_D, v_1, v_2, v_3, w\}$. Then

$$\sum(S,T) = \sum(S',T) + \deg_T(v_D) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(v_3) + \deg_T(w) \leq m,$$

implying that $a(T) \geq a(T') + 5$. It follows from the induction hypothesis that

$$\gamma_R(T) \leq \gamma_R(T') + 6 \leq \frac{4a(T')+2}{3} + 6 \leq \frac{4(a(T)-5)+2}{3} + 6 \leq \frac{4a(T)+2}{3}.$$

□

Claim 6. $\deg_T(v_6) = 2$.

Proof. Assume that $\deg_T(v_6) \geq 3$ and let $T' = T - T_{v_5}$. Then every $\gamma_R(T')$-function can be extended to an RDF of T by assigning 0 to v_1, v_3, v_4 and 2 to v_2, v_5. Thus $\gamma_R(T) \leq \gamma_R(T') + 4$. Suppose that $v_6 \notin S'$. Then $\sum(S',T) = \sum(S',T')$. In this case, let $S = S' \cup \{v_1, v_2, v_3\}$. Then

$$\sum(S,T) = \sum(S',T) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(v_3) \leq m' + 5 = m,$$

implying that $a(T) \geq |S| = |S'| + 3 = a(T') + 3$. Applying the induction hypothesis,

$$\gamma_R(T) \leq \gamma_R(T') + 4 \leq \frac{4a(T')+2}{3} + 4 \leq \frac{4(a(T)-3)+2}{3} + 4 = \frac{4a(T)+2}{3},$$

as desired.

Now assume $v_6 \in S'$. In this case, let $S = (S' - \{v_6\}) \cup \{v_1, v_2, v_3, v_4\}$. Then

$$\sum(S,T)=\sum(S',T')-\deg_{T'}(v_6)+\deg_T(v_1)+\deg_T(v_2)+\deg_T(v_3)+\deg_T(v_4)\leq m.$$

Therefore, $a(T) \geq |S| = |S'| + 3 = a(T') + 3$. Again we obtain

$$\gamma_R(T) \leq \frac{4a(T)+2}{3},$$

as desired. □

We now return to the proof of Theorem. Let $T' = T - T_{v_6}$, hence $m' = m - 6$. Every $\gamma_R(T')$-function can be extended to an RDF of T by assigning 0 to v_1, v_3, v_4, v_6 and 2 to v_2, v_5. Thus $\gamma_R(T) \leq \gamma_R(T') + 4$. Let $S = S' \cup \{v_1, v_2, v_3\}$. Then

$$\sum(S,T) \leq \sum(S',T') + 1 + \deg_T(v_1) + \deg_T(v_2) + \deg_T(v_3) \leq m' + 6 = m,$$

implying that $a(T) \geq |S| = |S'|+3 = a(T')+3$. Applying the induction hypothesis,

$$\gamma_R(T) \leq \gamma_R(T') + 4 \leq \frac{4a(T')+2}{3} + 4 \leq \frac{4(a(T)-3)+2}{3} + 4 = \frac{4a(T)+2}{3},$$

as desired. This completes the proof. □

For the rest of this section, we prove that for a tree T of order $n \geq 2$, if $\gamma_R(T) = \frac{4a(T)+2}{3}$, then $T = P_2$ or both ends of each diametral path in T are paths of length at least four.

Proposition 4. *Let T be a tree of order $n \geq 2$. If $\gamma_R(T) = \frac{4a(T)+2}{3}$, then $T = P_2$ or* $\mathrm{diam}(T) \geq 5$.

Proof. If $T = P_2$, then $\gamma_R(T) = 2 = \frac{4a(T)+2}{3}$. Assume next that $n \geq 3$ and $\gamma_R(T) = \frac{4a(T)+2}{3}$. By the proof of Claim 1 in Theorem 3, we may assume that the degree of each support vertex on a diametral path of T is two. If $\mathrm{diam}(T) \leq 4$, then clearly T is a spider, which leads to a contradiction to Lemma 2. □

Proposition 5. *If T is a tree of order n with* $\mathrm{diam}(T) = 5$*, then*

$$\gamma_R(T) < \frac{4a(T)+2}{3}.$$

Proof. If T is a path, then the result follows by Corollary 1. Suppose that T is not a path and let $v_1v_2\ldots v_6$ be a diametral path in T and root T at v_6 (at v_1, respectively). By a closer look at the proof of Theorem 3, we may assume that T_{v_3} (if T is rooted at v_6) and T_{v_4} (if T is rooted at v_1) are paths P_5. It is easy to see that $\gamma_R(T) = 8$ and $a(T) = 6$ yielding $\gamma_R(T) = \frac{4a(T)}{3} < \frac{4a(T)+2}{3}$, as desired. □

Proposition 6. *If T is a tree of order n with* $\mathrm{diam}(T) = 6, 7$*, then*

$$\gamma_R(T) < \frac{4a(T)+2}{3}.$$

Proof. As in Proposition 5, we assume T is not a path. Let $v_1v_2\dots v_D$ be a diametral path in T and root T at v_D (at v_1, respectively). By the proof of Theorem 3, we may assume that T_{v_3} and $T_{v_{D-2}}$ are P_5 and $\deg(v_4) = 2$ if $\operatorname{diam}(T) = 6$ and $\deg(v_4) = \deg(v_5) = 2$ when $\operatorname{diam}(T) = 7$. Then obviously $\gamma_R(T) = 8$ and $a(T) = 7$, implying that $\gamma_R(T) < \frac{4a(T)+2}{3}$. □

Proposition 7. *If T is a tree of order n with $\operatorname{diam}(T) \geq 8$, then*

$$\gamma_R(T) < \frac{4a(T)+2}{3}$$

unless both ends of each diametral path in T are paths of length at least four.

Proof. Let $v_1v_2\dots v_D$ be a diametral path in T and root T at v_D (at v_1, respectively). By a closer look at the proof of Theorem 3, we need to consider three Cases.

Case 1. T_{v_3} is $P_5 = v_1v_2v_3w_2w_1$.
First let $\deg_T(v_4) = 2$. If $\deg_T(v_5) = 2$, then let $T' = T - T_{v_5}$. It is easy to see that $\gamma_R(T) \leq \gamma_R(T') + 5$ and $a(T) \geq a(T') + 4$. Hence, the result follows by Theorem 3. Let $\deg_T(v_5) \geq 3$ and let $T' = T - T_{v_4}$. Then obviously $\gamma_R(T) \leq \gamma_R(T') + 4$. If $v_5 \notin S'$, then let $S = S' \cup \{v_1, v_2, w_1, w_2\}$, and if $v_5 \in S'$, then let $S = (S' - \{v_5\}) \cup \{v_1, v_2, v_4, w_1, w_2\}$. In both cases, we have $\sum(S,T) \leq m$, and hence $a(T) \geq a(T') + 4$. It follows from Theorem 3 that $\gamma_R(T) \leq \gamma_R(T') + 5 \leq \frac{4a(T')+2}{3} + 5 \leq \frac{4(a(T)-4)+2}{3} + 5 < \frac{4a(T)+2}{3}$.

Now let $\deg_T(v_4) \geq 3$. We consider the following subcases.

Subcase 1.1. $\deg_T(v_4) = 3$ and there is a path $z_1z_2z_3z_4z_5$ in T such that $v_4z_3 \in E(T)$, $\deg_T(z_3) = 3, \deg_T(z_2) = \deg_T(z_4) = 2$ and $\deg_T(z_1) = \deg_T(z_5) = 1$.
Let $T' = T - T_{v_4}$. It is easy to see that $\gamma_R(T) \leq \gamma_R(T') + 8$ and $a(T) \geq a(T') + 7$, implying that $\gamma_R(T) < \frac{4a(T)+2}{3}$ by Theorem 3.

Subcase 1.2. $\deg_T(v_4) \geq 4$ and there is a path $z_1z_2z_3z_4z_5$ in T such that $v_4z_3 \in E(T)$, $\deg_T(z_3) = 3, \deg_T(z_2) = \deg_T(z_4) = 2$ and $\deg_T(z_1) = \deg_T(z_5) = 1$.
Assume $T' = T - (T_{v_3} \cup T_{z_3})$. As above, we have $\gamma_R(T) \leq \gamma_R(T') + 8$. If $v_4 \notin S'$, then let $S = S' \cup \{v_1, v_2, w_1, w_2, z_1, z_2, z_5\}$, and if $v_4 \in S'$, then let $S = (S' - \{v_4\}) \cup \{v_1, v_2, w_1, w_2, z_1, z_2, z_4, z_5\}$. Then clearly $\sum(S,T) \leq m$, implying that $a(T) \geq a(T') + 7$. Now the result follows by Theorem 3.

Subcase 1.3. $\deg_T(v_4) \geq 3$ and there is a path $z_1z_2z_3v_4$ in T such that $\deg_T(z_3) = \deg_T(z_2) = 2$ and $\deg_T(z_1) = 1$.
Using Theorem 3 and an argument similar to that described in the proof of Claim 4, show that $\gamma_R(T) < \frac{4a(T)+2}{3}$.

Subcase 1.4. $\deg_T(v_4) = 3$ and there is a path $v_4z_2z_1$ in T such that $\deg_T(z_2) = 2$ and $\deg_T(z_1) = 1$.
Let $T' = T - T_{v_4}$. It is easy to check that $\gamma_R(T) \leq \gamma_R(T') + 6$ and $a(T) \geq a(T') + 5$, implying that $\gamma_R(T) < \frac{4a(T)+2}{3}$ by Theorem 3.

Subcase 1.5. $\deg_T(v_4) \geq 4$ and there is a path $v_4z_2z_1$ in T such that $\deg_T(z_2) = 2$ and $\deg_T(z_1) = 1$.

Let $T' = T - (T_{v_3} \cup T_{z_2})$. Clearly, $\gamma_R(T) \leq \gamma_R(T') + 6$. If $v_4 \notin S'$, then let $S = S' \cup \{v_1, v_2, w_1, w_2, z_1\}$, and if $v_4 \in S'$, then let $S = (S' - \{v_4\}) \cup \{v_1, v_2, w_1, w_2, z_1, z_2\}$. Obviously, $\sum(S,T) \leq m$ and so $a(T) \geq a(T') + 5$. It follows from Theorem 3 that $\gamma_R(T) < \frac{4a(T)+2}{3}$.

Subcase 1.6. $\deg_T(v_4) \geq 4$ and all neighbors of v_4, except v_3, v_5, are leaves. Suppose $T' = T - T_{v_4}$. It is easy to see that $\gamma_R(T) \leq \gamma_R(T') + 6$ and $a(T) \geq a(T') + 5$. Now the result follows as above.

Subcase 1.7. $\deg_T(v_4) = 3$, v_4 is adjacent to a leaf, say w, and $\deg_T(v_5) = 2$. Let $T' = T - T_{v_5}$. It is easy to check that $\gamma_R(T) \leq \gamma_R(T') + 6$ and $a(T) \geq a(T') + 5$, implying that $\gamma_R(T) < \frac{4a(T)+2}{3}$ by Theorem 3.

Subcase 1.8. $\deg_T(v_4) = 3$, v_4 is adjacent to a leaf, say w, and $\deg_T(v_5) \geq 3$. Let $T' = T - (T_{v_4} \cup T_{v_{D-2}})$, where $T_{v_{D-2}}$ is the maximal subtree at v_{D-2} when T is rooted at v_1. First let $T_{v_{D-2}} = P_3$. It is easy to see that $\gamma_R(T) \leq \gamma_R(T') + 7$ and $a(T) \geq a(T') + 6$, implying that $\gamma_R(T) < \frac{4a(T)+2}{3}$ by Theorem 3. Now let $T_{v_{D-2}} = P_5 = v_D v_{D-1} v_{D-2} u_2 u_1$. Clearly, $\gamma_R(T) \leq \gamma_R(T') + 9$. If $v_5 \notin S'$, then let $S = S' \cup \{v_1, v_2, w_1, w_2, w, u_1, v_D, v_{D-1}\}$, and if $v_5 \in S'$, then let $S = (S' - \{v_5\}) \cup \{v_1, v_2, w_1, w_2, w, u_1, u_2, v_D, v_{D-1}\}$. Then $\sum(S,T) \leq m$ and hence $a(T) \geq a(T') + 8$. Now the result follows by Theorem 3.

<u>*Case 2.*</u> T_{v_4} is a path and v_5 is adjacent to a leaf, say w.
Considering Case 1, we may assume that $T_{v_{D-2}}$ is a path P_3 in the rooted tree T at v_1. Let $T' = T - (T_{v_4} \cup T_{v_{D-2}})$. It is easy to see that $\gamma_R(T) \leq \gamma_R(T') + 5$. If $v_5 \notin S'$, then let $S = S' \cup \{v_1, v_2, v_3, v_D\}$ and if $v_5 \in S'$, then let $S = (S' - \{v_5\}) \cup \{v_1, v_2, v_3, v_4, v_D\}$. Then $\sum(S,T) \leq m$ and so $a(T) \geq a(T') + 4$. It follows from Theorem 3 that $\gamma_R(T) < \frac{4a(T)+2}{3}$.

Hence, T_{v_5} is a path when T is rooted at v_D. In a similar fashion, we can prove $T_{v_{D-4}}$ is a path when T is rooted at v_1. This completes the proof. $\square$

We conclude this paper with an open problem.

Problem. Characterize all trees achieving the bound in Theorem 3.

References

1. Amjadi J., *An upper bound on the double domination number of trees*, Kragujevac J. Math. **39** (2015), 133–139.
2. Chambers E. W., Kinnersley B., Prince N. and West D. B., *Extremal problems for Roman domination*, SIAM J. Discrete Math. **23** (2009), 1575–1586.
3. Cockayne E. J., Dreyer Jr. P. M., Hedetniemi S. M. and Hedetniemi S. T., *On Roman domination in graphs*, Discrete Math. **278** (2004), 11–22.
4. Cockayne E. J., Favaron O. and Mynhardt C. M., *Secure domination, weak Roman domination and forbidden subgraphs*, Bull. Inst. Combin. Appl. **39** (2003), 87–100.
5. Dehgardi N., Khodkar A. and Sheikholeslami S. M., *Bounding the rainbow domination number of a tree in terms of its annihilation number*, Trans. Comb. **2** (2013), 21–32.
6. Dehgardi N., Khodkar A. and Sheikholeslami S. M., *Bounding the paired-domination number of a tree in terms of its annihilation number*, Filomat **28** (2014), 523–529.
7. Dehgardi N., Norouzian S. and Sheikholeslami S. M., *Bounding the domination number of a tree in terms of its annihilation number*, Trans. Comb. **2** (2013), 9–16.

8. Desormeaux W. J., Haynes T. W. and Henning M. A., *Relating the annihilation number and the total domination number of a tree*, Discrete Appl. Math. **161** (2013) 349–354.
9. Favaron O., Karami H. and Sheikholeslami S. M., *On the Roman domination number in graphs*, Discrete Math. **309** (2009), 3447–3451.
10. Hansberg A. and Volkmann L., *Upper bounds on the k-domination number and the k-Roman domination number*, Discrete Appl. Math. **157** (2009), 1634–1639.
11. Henning M. A., *A characterization of Roman trees*, Discuss. Math. Graph Theory **22** (2002), 225–234.
12. Henning M. A., *Defending the Roman Empire from multiple attacks*, Discrete Math. **271** (2003), 101–115.
13. Henning M. A. and Hedetniemi S. T., *Defending the Roman Empire – A new strategy*, Discrete Math. **266** (2003), 239–251.
14. Larson C. E. and Pepper R., *Graphs with equal independence and annihilation numbers*, the electronic journal of combinatorics 18 (2011), #P180.
15. Pepper R., *On the annihilation number of a graph*, Recent Advances In Electrical Engineering: Proceedings of the 15th American Conference on Applied Mathematics (2009) 217–220.
16. ReVelle C. S. and Rosing K. E., *Defendens imperium romanum: a classical problem in military strategy*, Amer. Math. Monthly **107** (2000), 585–594.
17. Stewart I., *Defend the Roman Empire*, Sci. Amer. **281** (6) (1999), 136–139.

H. Aram, Department of Mathematics, Gareziaeddin Center, Khoy Branch, Islamic Azad University, Khoy, Iran, *e-mail*: `hamideh.aram@gmail.com`

R. Khoeilar, Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, I.R. Iran, *e-mail*: `khoeilar@azaruniv.edu`

S. M. Sheikholeslami, Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, I.R. Iran, *e-mail*: `s.m.sheikholeslami@azaruniv.edu`

L. Volkmann, Lehrstuhl II für Mathematik, RWTH Aachen University, 52056 Aachen, Germany, *e-mail*: `volkm@math2.rwth-aachen.de`