SYMMETRIC $m$-CONVEX ALGEBRAS WITHOUT ALGEBRAIC ZERO DIVISORS AND RESULTS OF GELFAND-MAZUR TYPE

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Abstract. We show that in a symmetric $m$-convex algebra without algebraic zero divisors, any self-adjoint and invertible element is either positive or negative. As a consequence we obtain that a symmetric $m$-convex algebra containing no algebraic zero divisors and for which every positive element has a positive square root is isomorphic to $\mathbb{C} + \text{Rad}(A)$.

1. Introduction

In [5, Proposition 11.12], it is shown that every complete $Q$-$m$-convex algebra which does not contain topological zero-divisors is isomorphic to the algebra of complex numbers. The result is not true when considering only algebraic zero divisors [4]. Here, in the context of symmetric $m$-convex algebras and with just algebraic zero divisors, we give a characterization of the spectrum of self-adjoint elements. Let $A$ be a symmetric $m$-convex algebra, the subset $A_+ = \{ x \in A : x = x^* \text{ and } \text{Sp}(x) \subset \mathbb{R}^+ \}$ be its cone of positive elements and $A_0^+ = A_+ \cap G$ ($G$ is the subgroup of invertible elements). We show that if $A_+$ does not contain algebraic zero divisors, then $\text{Sym}(A) \cap G$ is the disjoint union of $A_0^+$ and $-A_0^+$ (Theorem 2). In other words, the spectrum of a self-adjoint element cannot contain both positive and negative real numbers unless it contains zero. As a consequence, we obtain that in this case, the spectrum of any self-adjoint element is an interval (Theorem 6). These two results, in fact, give us a method to check the existence of algebraic zero divisors in symmetric $m$-convex algebras. Finally, we show that a symmetric $m$-convex algebra containing no algebraic zero divisors and for which every positive element has a positive square root is isomorphic to $\mathbb{C} + \text{Rad}(A)$ (Theorem 3). If the second condition is satisfied for every locally $C^*$-algebra, we get that the algebra of complex numbers is the unique locally $C^*$-algebra without algebraic zero divisors [2, Theorem 4.4].
2. Preliminaries

An $m$-convex algebra $(A, (|·|_\lambda))$ is a locally convex algebra, the topology of which is defined by a family $(|·|_\lambda)$ of sub-multiplicative semi-norms, i.e., $|xy|_\lambda \leq |x|_\lambda |y|_\lambda$, for every $x, y$ in $A$ and every $\lambda$. A unital symmetric $m$-convex algebra is a $m$-convex algebra endowed with an involution $^*: A \to A^*$ such that $e + xx^*$ is invertible in $A$ for every $x$ where $e$ is the unit element. We denote by Sym$(A)$, $A_+$, $A_0^+$, and $G$, respectively, the subset of self-adjoint elements, positive elements, strictly positive elements and the open subgroup of invertible elements:

$$\text{Sym}(A) = \{x \in A : x = x^*\},$$

$$A_+ = \{x \in A : x = x^* \text{ and } \text{Sp}(x) \subset \mathbb{R}^+\},$$

$$A_0^+ = \{x \in A : x = x^* \text{ and } \text{Sp}(x) \subset \mathbb{R}^+ \setminus \{0\}\},$$

and

$$G = G_A = \{a \in A : a \text{ is invertible in } A\}.$$

A locally $C^*$-algebra is a complete $m$-convex algebra $(A, (|·|_\lambda))$ endowed with an involution $^*: A \to A^*$ such that $|xx^*| = |x|^2$, for every $x$ in $A$ and every $\lambda$. We denote by Sp$(x)$ and $\rho(x)$ the spectrum and the spectral radius of $x$. Two elements $a, b$ in $A$ are called algebraic zero divisors if $a \neq 0, b \neq 0$ and $ab = 0$. Throughout the next, the algebra $A$ will be supposed to be complete and unital.

3. Symmetric $m$-convex algebras without algebraic zero divisors

Let $(A, (|·|_\lambda))$ be a symmetric $m$-convex algebra. The convex cone $A_0^+$ does not contain algebraic zero divisors, indeed, $a^{-1} \in A_0^+$, for every $a \in A_0^+$. The convex cone $A_+$ contains in general, for example, the Banach algebra of square matrices with usual operations and involution. Let $K_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $K_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $K_1 \neq 0$, $K_2 \neq 0$ and $K_1 K_2 = 0$.

In the following, we show that if $A_+$ contains no algebraic zero divisors, then each element of Sym$(A) \cap G$ is either positive or negative with respect to the order defined by $A_0^+$.

We first prove that if $A_+$ contains no algebraic zero divisors, then Sym$(A)$ does not contain any too.

**Proposition 1.** Let $A$ be a symmetric $m$-convex algebra. The set Sym$(A)$ contains no algebraic zero divisors if and only if $A_+$ does not contain any zero divisors.

**Proof.** Suppose that $A_+$ does not contain any algebraic zero divisors, and let $a, b \in \text{Sym}(A)$ such that $ab = 0$. Since $A$ is symmetric, $a^2, b^2 \in A_+$. We have $a^2 b^2 = aabb = 0$. Thus, according to the assumption, $a^2 = 0$ or $b^2 = 0$; and so Sp$(a) = \{0\}$ or Sp$(b) = \{0\}$. Now $a \in A_+$ and $a^2 = 0$, or $b \in A_+$ and $b^2 = 0$. Hence, by the assumption again, $a = 0$ or $b = 0$. \(\square\)
**Theorem 2.** Let $A$ be a symmetric $m$-convex algebra. If $A_+$ contains no algebraic zero divisors, then the two convex cones $A_+^0$ and $(-A_+^0)$ form a partition of $\text{Sym}(A) \cap G$; that is, $\text{Sym}(A) \cap G = (-A_+^0) \cup A_+^0$.

**Proof.** Let $h \in \text{Sym}(A) \cap G$. We have $\text{Sp}(h^2) \subseteq \mathbb{R}^+ \setminus \{0\}$, so $h^2 \in A_+^0$. Let $A_h$ be the maximal commutative sub-algebra containing $h$. According to the preceding theorem, we can conclude that there are necessarily an algebraic zero divisor in $A_h$.

Let $A$ be a symmetric $m$-convex algebra. If $A_+$ contains no algebraic zero divisors, then each element of $\text{Sym}(A) \cap G$ is either positive or negative with respect to the order defined by $A_+^0$.

**Corollary 3.** Let $A$ be a symmetric $m$-convex algebra. If $A_+$ contains no algebraic zero divisors, then each element of $\text{Sym}(A) \cap G$ is either positive or negative with respect to the order defined by $A_+^0$.

**Corollary 4.** Let $A$ be a symmetric $m$-convex algebra. If $A_+$ does not contain algebraic zero divisors, then for every $h \in \text{Sym}(A) \cap G$, there exists a unique $u \in A_+^0$ such that $h = u^2$ or $h = -u^2$.

**Remark 5.**

1. In a symmetric $m$-convex algebra, if there exists at least one invertible self-adjoint element whose spectrum contains both positive and negative real numbers, then the algebra necessarily contains zero divisors.

2. In a symmetric $m$-convex algebra $A$, we don’t have in general, $\text{Sym}(A) \cap G = (-A_+^0) \cup A_+^0$. Indeed, in the previous example, the matrix $L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is self-adjoint and invertible but does not belong to $A_+^0$ or $(-A_+^0)$. Moreover, according to the preceding theorem, we can conclude that there are necessarily algebraic zero divisors in $A_+$.

3. The condition $\text{Sym}(A) \cap G = (-A_+^0) \cup A_+^0$ in the previous theorem is necessary but not sufficient to imply the non-existence of algebraic zero divisors. Indeed, consider the symmetric Banach algebra $(C([0,1]), \|\cdot\|_\infty)$ endowed with usual operations and involution. We have $\text{Sym}(A) \cap G = (-A_+^0) \cup A_+^0$. Now take $f$ and $g$ such that $f(x) = \frac{1}{2} - x$, $g(x) = 0$ for $x \in [0 \frac{1}{2}]$ and $f(x) = 0$, $g(x) = x - \frac{1}{2}$ for $x \in \left[\frac{1}{2}, 1\right]$. $f \neq 0$ and $g \neq 0$ but $fg = 0$. 

**Proof.** Let $h \in \text{Sym}(A) \cap G$. We have $\text{Sp}(h^2) \subseteq \mathbb{R}^+ \setminus \{0\}$, so $h^2 \in A_+^0$. Let $A_h$ be the maximal commutative sub-algebra containing $h$. According to the preceding theorem, we can conclude that there are necessarily an algebraic zero divisor in $A_h$. 

Let $A$ be a symmetric $m$-convex algebra. If $A_+$ contains no algebraic zero divisors, then each element of $\text{Sym}(A) \cap G$ is either positive or negative with respect to the order defined by $A_+^0$.
4. The condition $a^2 = 0 \Rightarrow a = 0$ for every $a \in A_+$, is weaker than that considered in Theorem 2, but it cannot replace it as shown by the following example. Consider the symmetric Banach algebra $\mathbb{C}^2$ with usual operations. It satisfies the first condition but there exist self-adjoint and invertible elements whose spectrum contains positive and negative numbers.

In the next, we give some characterisations of the spectrum of self-adjoint elements.

**Theorem 6.** Let $A$ be a symmetric $m$-convex algebra. If $A_+$ does not contain algebraic zero divisors, then the spectrum of each element of $\text{Sym}(A)$ is an interval.

**Proof.** Let $a \in \text{Sym}(A)$. Suppose that $\text{Sp}(a)$ is not an interval. There exist $\alpha, \beta \in \text{Sp}(a)$ and $\gamma \notin \text{Sp}(a)$ such that $\alpha < \gamma < \beta$. Then $a - \gamma \in \text{Sym}(A) \cap G$ and $\text{Sp}(a - \gamma)$ contains positive and negative real numbers. This contradicts the result of Theorem 2. $\square$

**Corollary 7.** Let $A$ be a symmetric $m$-convex algebra. If $A_+$ does not contain algebraic zero divisors, then the spectrum of each element of $\text{Sym}(A)$ is either infinite or reduced to a single element.

**Corollary 8.** Let $A$ be a symmetric $m$-convex algebra without algebraic zero divisors. A self-adjoint element whose spectrum contains both positive and negative real numbers is necessarily not invertible.

**Remark 9.** The result of Theorem 2 remains true for every symmetric $m$-convex algebra satisfying:
1. For every $x \in A_+^0$, there exists $u \in A_+^0$ such that $x = u^2$.
2. $A_+$ contains no algebraic zero divisors.

If we suppose that the first condition is satisfied by $A_+$ (every positive element has a positive square root), then we can state a theorem of a Gelfand-Mazur type as follows.

**Theorem 10.** Let $A$ be a symmetric $m$-convex algebra. If $A_+$ satisfies the two conditions:
1. For every $x \in A_+$, there exists $u \in A_+$ such that $x = u^2$.
2. $A_+$ contains no algebraic zero divisors.

Then $A$ is isomorphic to $\mathbb{C} + \text{Rad}(A)$, where $\text{Rad}(A)$ is the Jacobson radical of $A$. In addition, if the algebra $A$ is semi-simple or if the cone $A_+$ is salient ($(\mathbb{C} - A_+) \cap A_+ = \{0\}$), then $A$ is isomorphic to $\mathbb{C}$.

**Proof.** By replacing $A_+^0$ and $\text{Sym}(A) \cap G$, respectively, by $A_+$ and $\text{Sym}(A)$ in the proof of Theorem 2, we obtain that $\text{Sym}(A) = (-A_+ \cup A_+$. Now for $x \in \text{Sym}(A)$, suppose that $\text{Sp}(x)$ contains more than one element; $\alpha$ and $\beta$ for example. We have that $x - \frac{\alpha + \beta}{2}$ belongs to $\text{Sym}(A)$ but not to $(-A_+ \cup A_+$, a contradiction. Let $x \in \text{Sym}(A)$ and let $\mu \in \mathbb{R}$ such that $\text{Sp}(x) = \{\mu\}$, we have that $p_A(x - \mu) = \rho(x - \mu) = 0$, where $p_A(x) := \rho(xx^*)^{1/2}$ for all $x \in A$, is the “Pták function” [3, (22.1) p. 279]. Since the algebra $A$ is symmetric and satisfies $\rho |_{\text{Sym}(A)} < \infty$, it
is a spectral algebra [1, Theorem 4.4]. Hence, by [3, Corollary 22.15, p. 291],
\[ x - \mu \in \text{Ker} \, p_A = \text{Rad}(A). \]
Finally, \[ A = \text{Sym}(A) + i \text{Sym}(A) \cong \mathbb{C} + \text{Rad}(A). \]

Since any locally \( \mathbb{C}^* \)-algebra is semi-simple and satisfies the first condition, we obtain [2, Theorem 4.4] as corollary.

**Corollary 11.** ([2, Theorem 4.4]) The algebra of complex numbers is the unique locally \( \mathbb{C}^* \)-algebra for which the cone of positive elements contains no algebraic zero divisors.

As a consequence of Corollary 7, we obtain another result of Gelfand-Mazur type.

**Theorem 12.** Let \( A \) be a symmetric \( m \)-convex algebra. If \( A_+ \) satisfies the two conditions:
1. Each element of \( A_+ \) has a finite spectrum,
2. \( A_+ \) contains no algebraic zero divisors,
then \( A \) is isomorphic to \( \mathbb{C} + \text{Rad}(A) \).

**Proof.** According to Corollary 7, the spectrum of each element of \( \text{Sym}(A) \) is reduced to a single element. We conclude with the same arguments used in the proof of Theorem 10. \( \square \)

**References**


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