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SYMMETRIC m-CONVEX ALGEBRAS WITHOUT ALGEBRAIC ZERO DIVISORS AND RESULTS OF GELFAND-MAZUR TYPE

M. A. NEJJARI

ABSTRACT. We show that in a symmetric m-convex algebra without algebraic zero divisors, any self-adjoint and invertible element is either positive or negative. As a consequence we obtain that a symmetric m-convex algebra containing no algebraic zero divisors and for which every positive element has a positive square root is isomorphic to $\mathbb{C} + \text{Rad}(A)$.

1. INTRODUCTION

In [**5**, Proposition 11.12], it is shown that every complete Q-m-convex algebra which does not contain topological zero-divisors is isomorphic to the algebra of complex numbers. The result is not true when considering only algebraic zero divisors [**4**]. Here, in the context of symmetric m-convex algebras and with just algebraic zero divisors, we give a characterization of the spectrum of self-adjoint elements. Let A be a symmetric m-convex algebra, the subset $A_+ = \{x \in A : x = x^*$ and $\text{Sp}(x) \subset \mathbb{R}^+ \}$ be its cone of positive elements and $A_+^0 = A_+ \cap G$ (G is the subgroup of invertible elements). We show that if A_+ does not contain algebraic zero divisors, then $\text{Sym}(A) \cap G$ is the disjoint union of A_+^0 and $-A_+^0$ (Theorem 2). In other words, the spectrum of a self-adjoint element cannot contain both positive and negative real numbers unless it contains zero. As a consequence, we obtain that in this case, the spectrum of any self-adjoint element is an interval (Theorem 6). These two results, in fact, give us a method to check the existence of algebraic zero divisors in symmetric m-convex algebras. Finally, we show that a symmetric m-convex algebra containing no algebraic zero divisors and for which every positive element has a positive square root is isomorphic to $\mathbb{C} + \text{Rad}(A)$ (Theorem 3). If the second condition is satisfied for every locally C^*-algebra, we get that the algebra of complex numbers is the unique locally C^*-algebra without algebraic zero divisors [**2**, Theorem 4.4].

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2. Preliminaries

An m-convex algebra $(A, (|\cdot|_\lambda)_\lambda)$ is a locally convex algebra, the topology of which is defined by a family $(|\cdot|_\lambda)_\lambda$ of sub-multiplicative semi-norms, i.e., $|xy|_\lambda \leq |x|_\lambda\,|y|_\lambda$, for every x, y in A and every λ. A unital symmetric m-convex algebra is a m-convex algebra endowed with an involution $^*: x \mapsto x^*$ such that $e + xx^*$ is invertible in A for every x where e is the unit element. We denote by $\mathrm{Sym}(A)$, A_+, A_+^0 and G, respectively, the subset of self-adjoint elements, positive elements, strictly positive elements and the open subgroup of invertible elements:

$$\begin{aligned}\mathrm{Sym}(A) &= \{x \in A : x = x^*\}\,,\\ A_+ &= \{x \in A : x = x^* \text{ and } \ \mathrm{Sp}(x) \subset \mathbb{R}^+\},\\ A_+^0 &= \left\{x \in A : x = x^* \text{ and } \ \mathrm{Sp}(x) \subset \mathbb{R}^+ \setminus \{0\}\right\}\end{aligned}$$

and

$$G = G_A = \{a \in A : a \text{ is invertible in } A\}\,.$$

A locally C^*-algebra is a complete m-convex algebra $(A, (|\cdot|_\lambda)_\lambda)$ endowed with an involution $^*: x \mapsto x^*$ such that, $|xx^*| = |x|_\lambda^2$, for every x in A and every λ. We denote by $\mathrm{Sp}(x)$ and $\rho(x)$ the spectrum and the spectral radius of x. Two elements a, b in A are called algebraic zero divisors if $a \neq 0, b \neq 0$ and $ab = 0$. Throughout the next, the algebra A will be supposed to be complete and unital.

3. Symmetric m-convex algebras without algebraic zero divisors

Let $(A,(|\cdot|_\lambda)_\lambda)$ be a symmetric m-convex algebra. The convex cone A_+^0 does not contain algebraic zero divisors, indeed, $a^{-1} \in A_+^0$, for every $a \in A_+^0$. The convex cone A_+ contains in general, for example, the Banach algebra of square matrices with usual operations and involution. Let $K_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $K_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $K_1 \neq 0$, $K_2 \neq 0$ and $K_1K_2 = 0$.

In the following, we show that if A_+ contains no algebraic zero divisors, then each element of $\mathrm{Sym}(A) \cap G$ is either positive or negative with respect to the order defined by A_+^0.

We first prove that if A_+ contains no algebraic zero divisors, then $\mathrm{Sym}(A)$ does not contain any too.

Proposition 1. *Let A be a symmetric m-convex algebra. The set* $\mathrm{Sym}(A)$ *contains no algebraic zero divisors if and only if A_+ does not contain any zero divisors.*

Proof. Suppose that A_+ does not contain any algebraic zero divisors, and let $a, b \in \mathrm{Sym}(A)$ such that $ab = 0$. Since A is symmetric, a^2, $b^2 \in A_+$. We have $a^2b^2 = aabb = 0$. Thus, according to the assumption, $a^2 = 0$ or $b^2 = 0$; and so $\mathrm{Sp}(a) = \{0\}$ or $\mathrm{Sp}(b) = \{0\}$. Now $a \in A_+$ and $a^2 = 0$, or $b \in A_+$ and $b^2 = 0$. Hence, by the assumption again, $a = 0$ or $b = 0$. □

Theorem 2. *Let A be a symmetric m-convex algebra. If A_+ contains no algebraic zero divisors, then the two convex cones A_+^0 and $(-A_+^0)$ form a partition of* $\mathrm{Sym}(A) \cap G$; *that is,* $\mathrm{Sym}(A) \cap G = (-A_+^0) \cup A_+^0$.

Proof. Let $h \in \mathrm{Sym}(A) \cap G$. We have $\mathrm{Sp}(h^2) \subset \mathbb{R}^+ \setminus \{0\}$, so $h^2 \in A_+^0$. Let A_h be the maximal commutative sub-algebra containing h and e [**3**, Theorem 4.13]. There exists $u \in A_+^0 \cap A_h$ such that $h^2 = u^2$ [**3**, Theorem 5.8]. Then, by [**3**, Theorem 4.18],

$$\mathrm{Sp}_A(u \pm h) = \mathrm{Sp}_{A_h}(u \pm h) = \{\phi(u) \pm \phi(h) : \phi \in \Phi_{A_h}\},$$

where Φ_{A_h} designates the set of all characters on A_h. For $\phi \in \Phi_{A_h}$, we have

$$\phi(h)^2 = \phi(h^2) = \phi(u^2) = \phi(u)^2, \qquad \phi(h) = \pm\phi(u).$$

It follows that $\mathrm{Sp}_A(u \pm h) \subset \mathbb{R}^+$, and so $u \pm h \in A_+$. Let $p = \frac{1}{2}(u+h)$ and $q = \frac{1}{2}(u-h)$, we have

$$h = p - q \qquad \text{and} \qquad pq = qp = 0.$$

Since $p, q \in A_+$ and A_+ is supposed to contain no algebraic zero divisors, one has $p = 0$ or $q = 0$. Hence $h = p \in A_+^0$ or $h = -q \in (-A_+^0)$. □

The following corollaries are equivalent forms of Theorem 2.

Corollary 3. *Let A be a symmetric m-convex algebra. If A_+ contains no algebraic zero divisors, then each element of* $\mathrm{Sym}(A) \cap G$ *is either positive or negative with respect to the order defined by A_+^0.*

Corollary 4. *Let A be a symmetric m-convex algebra. If A_+ does not contain algebraic zero divisors, then for every $h \in$* $\mathrm{Sym}(A) \cap G$, *there exists a unique $u \in A_+^0$ such that $h = u^2$ or $h = -u^2$.*

Remark 5.

1. In a symmetric m-convex algebra, if there exists at least one invertible and self-adjont element whose spectrum contains both positive and negative real numbers, then the algebra necessarily contains zero divisors.
2. In a symmetric m-convex algebra A, we don't have in general, $\mathrm{Sym}(A) \cap G = (-A_+^0) \cup A_+^0$. Indeed, in the previous example, the matrix $L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is self-adjoint and invertible but does not belong to A_+^0 or $(-A_+^0)$. Moreover, according to the preceding theorem, we can conclude that there are necessarily algebraic zero divisors in A_+.
3. The condition $\mathrm{Sym}(A) \cap G = (-A_+^0) \cup A_+^0$ in the previous theorem is necessary but not sufficient to imply the non-existence of algebraic zero divisors. Indeed, consider the symmetric Banach algebra $(C([0,1]), \|\cdot\|_\infty)$ endowed with usual operations and involution. We have $\mathrm{Sym}(A) \cap G = (-A_+^0) \cup A_+^0$. Now take f and g such that $f(x) = \frac{1}{2} - x$, $g(x) = 0$ for $x \in \left[0\ \frac{1}{2}\right]$ and $f(x) = 0$, $g(x) = x - \frac{1}{2}$ for $x \in \left]\frac{1}{2}\ 1\right]$, $f \neq 0$ and $g \neq 0$ but $fg = 0$.

4. The condition $a^2 = 0 \implies a = 0$ for every $a \in A_+$, is weaker than that considered in Theorem 2, but it can not replace it as shown by the following example. Consider the symmetric Banach algebra $\mathbb{C}^2$ with usual operations. It satisfies the first condition but there exist self-adjoint and invertible elements whose spectrum contains positive and negative numbers.

   In the next, we give some characterisations of the spectrum of self-adjoint elements.

Theorem 6. *Let A be a symmetric m-convex algebra. If A_+ does not contain algebraic zero divisors, then the spectrum of each element of* $\mathrm{Sym}(A)$ *is an interval.*

Proof. Let $a \in \mathrm{Sym}(A)$. Suppose that $\mathrm{Sp}(a)$ is not an interval. There exist $\alpha, \beta \in \mathrm{Sp}(a)$ and $\gamma \notin \mathrm{Sp}(a)$ such that $\alpha < \gamma < \beta$. Then $a - \gamma \in \mathrm{Sym}(A) \cap G$ and $\mathrm{Sp}(a-\gamma)$ contains positive and negative real numbers. This contradicts the result of Theorem 2. □

Corollary 7. *Let A be a symmetric m-convex algebra. If A_+ does not contain algebraic zero divisors, then the spectrum of each element of* $\mathrm{Sym}(A)$ *is either infinite or reduced to a single element.*

Corollary 8. *Let A be a symmetric m-convex algebra without algebraic zero divisors. A self-adjoint element whose spectrum contains both positive and negative real numbers is necessarily not invertible.*

Remark 9. The result of Theorem 2 remains true for every symmetric m-convex algebra satisfying:

1. For every $x \in A_+^0$, there exists $u \in A_+^0$ such that $x = u^2$.
2. A_+ contains no algebraic zero divisors.

If we suppose that the first condition is satisfied by A_+ (every positive element has a positive square root), then we can state a theorem of a Gelfand-Mazur type as follows.

Theorem 10. *Let A be a symmetric m-convex algebra. If A_+ satisfies the two conditions:*

1. *For every $x \in A_+$, there exists $u \in A_+$ such that $x = u^2$.*
2. *A_+ contains no algebraic zero divisors.*

Then A is isomorphic to $\mathbb{C} + \mathrm{Rad}(A)$, where $\mathrm{Rad}(A)$ *is the Jacobson radical of A. In addition, if the algebra A is semi-simple or if the cone A_+ is salient* $((-A_+) \cap A_+ = \{0\})$*, then A is isomorphic to $\mathbb{C}$.*

Proof. By replacing A_+^0 and $\mathrm{Sym}(A) \cap G$, respectively, by A_+ and $\mathrm{Sym}(A)$ in the proof of Theorem 2, we obtain that $\mathrm{Sym}(A) = (-A_+) \cup A_+$. Now for $x \in \mathrm{Sym}(A)$, suppose that $\mathrm{Sp}(x)$ contains more than one element; α and β for example. We have that $x - \frac{\alpha+\beta}{2}$ belongs to $\mathrm{Sym}(A)$ but not to $(-A_+) \cup A_+$, a contradiction. Let $x \in \mathrm{Sym}(A)$ and let $\mu \in \mathbb{R}$ such that $\mathrm{Sp}(x) = \{\mu\}$, we have that $p_A(x-\mu) = \rho(x-\mu) = 0$, where $p_A(x) := \rho(xx^*)^{\frac{1}{2}}$ for all $x \in A$, is the "Pták function" [**3**, (22.1) p. 279]. Since the algebra A is symmetric and satisfies $\rho\,|_{\mathrm{Sym}(A)} < \infty$, it is a spectral algebra [**1**, Theorem 4.4.]. Hence, by [**3**, Corollary 22.15, p. 291], $x - \mu \in \operatorname{Ker} p_A = \operatorname{Rad}(A)$. Finally, $A = \operatorname{Sym}(A) + \mathrm{i}\operatorname{Sym}(A) \simeq \mathbb{C} + \operatorname{Rad}(A)$. □

Since any locally $\mathbb{C}^*$-algebra is semi-simple and satisfies the first condition, we obtain [**2**, Theorem 4.4] as corollary.

Corollary 11. ([**2**, Theorem 4.4]) *The algebra of complex numbers is the unique locally $\mathbb{C}^*$-algebra for which the cone of positive elements contains no algebraic zero divisors.*

As a consequence of Corollary 7, we obtain another result of Gelfand-Mazur type.

Theorem 12. *Let A be a symmetric m-convex algebra. If A_+ satisfies the two conditions:*

1. *Each element of A_+ has a finite spectrum,*
2. *A_+ contains no algebraic zero divisors,*

then A is isomorphic to $\mathbb{C} + \operatorname{Rad}(A)$.

Proof. According to Corollary 7, the spectrum of each element of $\operatorname{Sym}(A)$ is reduced to a single element. We conclude with the same arguments used in the proof of Theorem 10. □

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M. A. Nejjari, CRMEF Oujda, Morocco, *e-mail*: m_a_nejjari@hotmail.com