SYMMETRIC $m$-CONVEX ALGEBRAS WITHOUT ALGEBRAIC ZERO DIVISORS AND RESULTS OF GELFAND-MAZUR TYPE

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Abstract. We show that in a symmetric $m$-convex algebra without algebraic zero divisors, any self-adjoint and invertible element is either positive or negative. As a consequence we obtain that a symmetric $m$-convex algebra containing no algebraic zero divisors and for which every positive element has a positive square root is isomorphic to $\mathbb{C} + \text{Rad}(A)$.

1. Introduction

In [5, Proposition 11.12], it is shown that every complete $Q$-$m$-convex algebra which does not contain topological zero-divisors is isomorphic to the algebra of complex numbers. The result is not true when considering only algebraic zero divisors [4]. Here, in the context of symmetric $m$-convex algebras and with just algebraic zero divisors, we give a characterization of the spectrum of self-adjoint elements. Let $A$ be a symmetric $m$-convex algebra, the subset $A_+ = \{ x \in A : x = x^* \text{ and } \text{Sp}(x) \subset \mathbb{R}^+ \}$ be its cone of positive elements and $A_+^0 = A_+ \cap G$ ($G$ is the subgroup of invertible elements). We show that if $A_+$ does not contain algebraic zero divisors, then $\text{Sym}(A) \cap G$ is the disjoint union of $A_+^0$ and $-A_+^0$ (Theorem 2). In other words, the spectrum of a self-adjoint element cannot contain both positive and negative real numbers unless it contains zero. As a consequence, we obtain that in this case, the spectrum of any self-adjoint element is an interval (Theorem 6). These two results, in fact, give us a method to check the existence of algebraic zero divisors in symmetric $m$-convex algebras. Finally, we show that a symmetric $m$-convex algebra containing no algebraic zero divisors and for which every positive element has a positive square root is isomorphic to $\mathbb{C} + \text{Rad}(A)$ (Theorem 3). If the second condition is satisfied for every locally $C^*$-algebra, we get that the algebra of complex numbers is the unique locally $C^*$-algebra without algebraic zero divisors [2, Theorem 4.4].

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2. Preliminaries

An $m$-convex algebra $(A, (| \cdot |_\lambda)_{\lambda})$ is a locally convex algebra, the topology of which is defined by a family $(| \cdot |_\lambda)_{\lambda}$ of sub-multiplicative semi-norms, i.e., $|xy|_\lambda \leq |x|_\lambda |y|_\lambda$, for every $x, y$ in $A$ and every $\lambda$. A unital symmetric $m$-convex algebra is a $m$-convex algebra endowed with an involution $^*: x \mapsto x^*$ such that $e + xx^*$ is invertible in $A$ for every $x$ where $e$ is the unit element. We denote by $\text{Sym}(A)$, $A_+$, $A_+^0$ and $G$, respectively, the subset of self-adjoint elements, positive elements, strictly positive elements and the open subgroup of invertible elements:

$$\text{Sym}(A) = \{ x \in A : x = x^* \},$$

$$A_+ = \{ x \in A : x = x^* \text{ and } \text{Sp}(x) \subset \mathbb{R}^+ \},$$

$$A_+^0 = \{ x \in A : x = x^* \text{ and } \text{Sp}(x) \subset \mathbb{R}^+ \setminus \{0\} \}$$

and

$$G = G_A = \{ a \in A : a \text{ is invertible in } A \}.$$

A locally $C^*$-algebra is a complete $m$-convex algebra $(A, (| \cdot |_\lambda)_{\lambda})$ endowed with an involution $^*: x \mapsto x^*$ such that $|xx^*| = |x|^2$, for every $x$ in $A$ and every $\lambda$. We denote by Sp$(x)$ and $\rho(x)$ the spectrum and the spectral radius of $x$. Two elements $a, b$ in $A$ are called algebraic zero divisors if $a \neq 0, b \neq 0$ and $ab = 0$.

Throughout the next, the algebra $A$ will be supposed to be complete and unital.

3. Symmetric $m$-convex algebras without algebraic zero divisors

Let $(A, (| \cdot |_\lambda)_{\lambda})$ be a symmetric $m$-convex algebra. The convex cone $A_+^0$ does not contain algebraic zero divisors, indeed, $a^{-1} \in A_+^0$, for every $a \in A_+^0$. The convex cone $A_+$ contains in general, for example, the Banach algebra of square matrices with usual operations and involution. Let $K_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $K_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $K_1 \neq 0, K_2 \neq 0$ and $K_1 K_2 = 0$.

In the following, we show that if $A_+$ contains no algebraic zero divisors, then each element of $\text{Sym}(A) \cap G$ is either positive or negative with respect to the order defined by $A_+^0$.

We first prove that if $A_+$ contains no algebraic zero divisors, then $\text{Sym}(A)$ does not contain any too.

**Proposition 1.** Let $A$ be a symmetric $m$-convex algebra. The set $\text{Sym}(A)$ contains no algebraic zero divisors if and only if $A_+$ does not contain any zero divisors.

**Proof.** Suppose that $A_+$ does not contain any algebraic zero divisors, and let $a, b \in \text{Sym}(A)$ such that $ab = 0$. Since $A$ is symmetric, $a^2, b^2 \in A_+$. We have $a^2 b^2 = aabb = 0$. Thus, according to the assumption, $a^2 = 0$ or $b^2 = 0$; and so $\text{Sp}(a) = \{0\}$ or $\text{Sp}(b) = \{0\}$. Now $a \in A_+$ and $a^2 = 0$, or $b \in A_+$ and $b^2 = 0$. Hence, by the assumption again, $a = 0$ or $b = 0$. □
Theorem 2. Let $A$ be a symmetric $m$-convex algebra. If $A_+$ contains no algebraic zero divisors, then the two convex cones $A_0^0$ and $(-A_0^0)$ form a partition of $\text{Sym}(A) \cap G$; that is, $\text{Sym}(A) \cap G = (-A_0^0) \cup A_0^0$.

Proof. Let $h \in \text{Sym}(A) \cap G$. We have $\text{Sp}(h^2) \subset \mathbb{R}^+ \setminus \{0\}$, so $h^2 \in A_0^0$. Let $A_h$ be the maximal commutative sub-algebra containing $h$ and $e$ [3, Theorem 4.13]. There exists $u \in A_+^0 \cap A_h$ such that $h^2 = u^2$ [3, Theorem 5.8]. Then, by [3, Theorem 4.18],

$$\text{Sp}_A(u \pm h) = \text{Sp}_{A_h}(u \pm h) = \{\phi(u) \pm \phi(h) : \phi \in \Phi_{A_h}\},$$

where $\Phi_{A_h}$ designates the set of all characters on $A_h$. For $\phi \in \Phi_{A_h}$, we have

$$\phi(h)^2 = \phi(h^2) = \phi(u^2) = \phi(u)^2, \quad \phi(h) = \pm \phi(u).$$

It follows that $\text{Sp}_A(u \pm h) \subset \mathbb{R}^+$, and so $u \pm h \in A_+$. Let $p = \frac{1}{2}(u + h)$ and $q = \frac{1}{2}(u - h)$, we have

$$h = p - q \quad \text{and} \quad pq = qp = 0.$$ 

Since $p, q \in A_+$ and $A_+$ is supposed to contain no algebraic zero divisors, one has $p = 0$ or $q = 0$. Hence $h = p \in A_0^0$ or $h = -q \in (-A_0^0)$. \qed

The following corollaries are equivalent forms of Theorem 2.

Corollary 3. Let $A$ be a symmetric $m$-convex algebra. If $A_+$ contains no algebraic zero divisors, then each element of $\text{Sym}(A) \cap G$ is either positive or negative with respect to the order defined by $A_0^0$.

Corollary 4. Let $A$ be a symmetric $m$-convex algebra. If $A_+$ does not contain algebraic zero divisors, then for every $h \in \text{Sym}(A) \cap G$, there exists a unique $u \in A_0^0$ such that $h = u^2$ or $h = -u^2$.

Remark 5.
1. In a symmetric $m$-convex algebra, if there exists at least one invertible and self-adjoint element whose spectrum contains both positive and negative real numbers, then the algebra necessarily contains zero divisors.
2. In a symmetric $m$-convex algebra $A$, we don’t have in general, $\text{Sym}(A) \cap G = (-A_0^0) \cup A_0^0$. Indeed, in the previous example, the matrix $L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is self-adjoint and invertible but does not belong to $A_0^0$ or $(-A_0^0)$. Moreover, according to the preceding theorem, we can conclude that there are necessarily algebraic zero divisors in $A_+$. 
3. The condition $\text{Sym}(A) \cap G = (-A_0^0) \cup A_0^0$ in the previous theorem is necessary but not sufficient to imply the non-existence of algebraic zero divisors. Indeed, consider the symmetric Banach algebra $(C([0,1]), \|\cdot\|_\infty)$ endowed with usual operations and involution. We have $\text{Sym}(A) \cap G = (-A_0^0) \cup A_0^0$. Now take $f$ and $g$ such that $f(x) = \frac{1}{2} - x$, $g(x) = 0$ for $x \in [0, \frac{1}{2}]$ and $f(x) = 0$, $g(x) = x - \frac{1}{2}$ for $x \in [\frac{1}{2}, 1]$, $f \neq 0$ and $g \neq 0$ but $fg = 0$. 

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4. The condition \( a^2 = 0 \implies a = 0 \) for every \( a \in A_+ \), is weaker than that considered in Theorem 2, but it can not replace it as shown by the following example. Consider the symmetric Banach algebra \( \mathbb{C}^2 \) with usual operations. It satisfies the first condition but there exist self-adjoint and invertible elements whose spectrum contains positive and negative numbers.

In the next, we give some characterisations of the spectrum of self-adjoint elements.

**Theorem 6.** Let \( A \) be a symmetric \( m \)-convex algebra. If \( A_+ \) does not contain algebraic zero divisors, then the spectrum of each element of \( \text{Sym}(A) \) is an interval.

**Proof.** Let \( a \in \text{Sym}(A) \). Suppose that \( \text{Sp}(a) \) is not an interval. There exist \( \alpha, \beta \in \text{Sp}(a) \) and \( \gamma \notin \text{Sp}(a) \) such that \( \alpha < \gamma < \beta \). Then \( a - \gamma \in \text{Sym}(A) \cap G \) and \( \text{Sp}(a - \gamma) \) contains positive and negative real numbers. This contradicts the result of Theorem 2. \( \square \)

**Corollary 7.** Let \( A \) be a symmetric \( m \)-convex algebra. If \( A_+ \) does not contain algebraic zero divisors, then the spectrum of each element of \( \text{Sym}(A) \) is either infinite or reduced to a single element.

**Corollary 8.** Let \( A \) be a symmetric \( m \)-convex algebra without algebraic zero divisors. A self-adjoint element whose spectrum contains both positive and negative real numbers is necessarily not invertible.

**Remark 9.** The result of Theorem 2 remains true for every symmetric \( m \)-convex algebra satisfying:

1. For every \( x \in A_0^+ \), there exists \( u \in A_+ \) such that \( x = u^2 \).
2. \( A_+ \) contains no algebraic zero divisors.

If we suppose that the first condition is satisfied by \( A_+ \) (every positive element has a positive square root), then we can state a theorem of a Gelfand-Mazur type as follows.

**Theorem 10.** Let \( A \) be a symmetric \( m \)-convex algebra. If \( A_+ \) satisfies the two conditions:

1. For every \( x \in A_+ \), there exists \( u \in A_+ \) such that \( x = u^2 \).
2. \( A_+ \) contains no algebraic zero divisors.

Then \( A \) is isomorphic to \( \mathbb{C} + \text{Rad}(A) \), where \( \text{Rad}(A) \) is the Jacobson radical of \( A \). In addition, if the algebra \( A \) is semi-simple or if the cone \( A_+ \) is salient ((\( -A_+ \times A_+ = \{ 0 \} \)), then \( A \) is isomorphic to \( \mathbb{C} \).

**Proof.** By replacing \( A_0^+ \) and \( \text{Sym}(A) \cap G \), respectively, by \( A_+ \) and \( \text{Sym}(A) \) in the proof of Theorem 2, we obtain that \( \text{Sym}(A) = ( -A_+ ) \cup A_+ \). Now for \( x \in \text{Sym}(A) \), suppose that \( \text{Sp}(x) \) contains more than one element; \( \alpha \) and \( \beta \) for example. We have that \( x - \frac{\alpha + \beta}{2} \) belongs to \( \text{Sym}(A) \) but not to \( ( -A_+ ) \cup A_+ \), a contradiction. Let \( x \in \text{Sym}(A) \) and let \( \mu \in \mathbb{R} \) such that \( \text{Sp}(x) = \{ \mu \} \), we have that \( p_A(x - \mu) = \rho(x - \mu) = 0 \), where \( p_A(x) := \rho(xx^*)^{1/2} \) for all \( x \in A \), is the “Pták function” [3, (22.1) p. 279]. Since the algebra \( A \) is symmetric and satisfies \( \rho \mid_{\text{Sym}(A)} < \infty \), it
is a spectral algebra \([1, \text{ Theorem 4.4}].\) Hence, by \([3, \text{ Corollary 22.15, p. 291}],\)
\[ x - \mu \in \text{Ker } p_A = \text{Rad}(A). \]
Finally, \(A = \text{Sym}(A) + i \text{Sym}(A) \simeq \mathbb{C} + \text{Rad}(A).\) \(\square\)

Since any locally \(C^*\)-algebra is semi-simple and satisfies the first condition, we obtain \([2, \text{ Theorem 4.4}]\) as corollary.

**Corollary 11.** (\([2, \text{ Theorem 4.4}]\)) The algebra of complex numbers is the unique locally \(C^*\)-algebra for which the cone of positive elements contains no algebraic zero divisors.

As a consequence of Corollary 7, we obtain another result of Gelfand-Mazur type.

**Theorem 12.** Let \(A\) be a symmetric \(m\)-convex algebra. If \(A_+\) satisfies the two conditions:
1. Each element of \(A_+\) has a finite spectrum,
2. \(A_+\) contains no algebraic zero divisors,
then \(A\) is isomorphic to \(\mathbb{C} + \text{Rad}(A)\).

**Proof.** According to Corollary 7, the spectrum of each element of \(\text{Sym}(A)\) is reduced to a single element. We conclude with the same arguments used in the proof of Theorem 10. \(\square\)

**References**


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