

ON THE CONVERGENCE OF HE AND ZHU'S NEW SERIES SOLUTION FOR PRICING OPTIONS WITH THE HESTON MODEL

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ABSTRACT. In this paper, a modified formula for European options and a set of complete convergence proofs for the solution that cover the entire time horizon of a European option contract are presented under the Heston model with minimal entropy martingale measure. Although He & Zhu [5] worked on this model, they only provided a converged solution with a condition imposed on the time to expiry. The new solution presented here is only slightly modified in its form. But, it is accompanied with the proof of convergence of the solution for the entire span of the time horizon of an option.

1. INTRODUCTION

Despite the great success of the Black-Scholes model [1], their over-simplified assumptions to achieve the analytical tractability could lead to pricing biases. Therefore, many different models have been proposed to modify the Black-Scholes model. In particular, stochastic volatility models are one kind of the most popular modifications and many authors such as Hull & White [3] and Stein & Stein [4] worked on this category. Among all of these models, the Heston model [2] with a closed-form pricing formula is very popular. To a large extent, the popularity of the Heston model stems from not only nice features such as the mean-reverting property that have well captured some primary characteristics of market dynamics, but also the “close-formness” of the solution which has considerably facilitated model calibration. However, a fundamental assumption in the Heston model is that the underlying market is an incomplete one, and thus there exist different equivalent martingale measures. Thus, the attempts of deriving option pricing formulae, based on the Heston model but for other equivalent and meaningful martingale measures are pursued with the aim of preserving the unique feature of “close-formness” of the pricing formula.

Recently, He & Zhu [5] worked on the Heston stochastic volatility model under the minimal entropy martingale measure and derived a closed-form pricing formula for European options based on a series solution approach. Although their solution is in an infinite series form, which is different from Heston's original formula in the form of inverse Fourier transformation, they indeed preserved the “close-formness” of the pricing formula which is defined in [7], as they provided a convergence proof

Received November 8, 2016.

2010 *Mathematics Subject Classification.* Primary ????

Key words and phrases. European options; Minimal entropy martingale measure; Proof of convergence; Heston model.

of the solution [5]. However, a noticeably shortfall of their proof is that it is only valid for a particular time interval, rather than the entire time horizon that is defined in an option contract. As a supplementary paper, this short research article presents a set of complete convergence proofs for the solution that cover the entire time horizon of a European option contract. The slightly modified formulae for other time range are based on shifting the point at which the series is expanded sequentially until the entire time horizon of an option is covered.

Such a simple and yet important supplement is provided in the next section, which is accompanied by numerical experiments showing the accuracy of the newly derived formulae.

2. PRICING FORMULAE

In this section, a set of option pricing formulae for different time range under the Heston stochastic volatility model with the minimal entropy martingale measure that converges outside the converged area shown in [5] is derived. Numerical experiments are also carried out to show the accuracy of the new formulae.

2.1. Theoretical derivation

Let S_t and v_t be the underlying asset price and the volatility, respectively, the dynamics of the Heston model under the physical measure are specified as follows

$$\begin{aligned}\frac{dS_t}{S_t} &= \mu v_t dt + \sqrt{v_t} dB_t, \\ dv_t &= k(\theta - v_t) dt + \beta \sqrt{v_t} dW_t,\end{aligned}$$

where B_t and W_t are two standard Brownian motions with correlation ρ . k and θ represent the mean-reverting speed and level, respectively. β is the so-called volatility of volatility. According to the results in [5], if we set

$$(2.1) \quad W_t = \rho B_t + \sqrt{1 - \rho^2} C_t,$$

where B_t and C_t are two independent Brownian motions, the dynamics under the minimal entropy martingale measure Q can be obtained by the following transformation

$$\begin{aligned}dB_t^Q &= dB_t + \mu v \sqrt{v_t} dt, \\ dC_t^Q &= dC_t + \frac{1}{\beta \sqrt{1 - \rho^2}} \lambda(\tau) \sqrt{v_t} dt,\end{aligned}$$

where $\tau = T - t$, $\lambda(\tau) = 2\Delta \tanh(\Delta\tau + b) - k - \rho\beta\mu$ with $\Delta = \sqrt{\frac{1}{4}k^2 + \frac{1}{2}k\rho\beta\mu + \frac{1}{4}\beta^2\mu^2}$ and $b = \tanh^{-1}(\frac{1}{2} \frac{k + \beta\rho\mu}{\Delta})$. In this way, the corresponding expression for the dynamics under the minimal entropy martingale measure can be derived as

$$(2.2) \quad \begin{aligned}\frac{dS_t}{S_t} &= \sqrt{v_t} dB_t^Q, \\ dv_t &= [k(\theta - v_t) - \beta\rho\mu v_t - \lambda(\tau)v_t] dt + \beta \sqrt{v_t} dW_t^Q,\end{aligned}$$

and we can also find that the European call option price $U(S, v, t)$ satisfies the following PDE (partial differential equation)

$$(2.3) \quad \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \rho\sigma vS\frac{\partial^2 U}{\partial S\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} + [k(\theta - v) - \beta\rho\mu v - \lambda(\tau)v]\frac{\partial U}{\partial v} + \frac{\partial U}{\partial t} = 0$$

with terminal condition

$$(2.4) \quad U(S, v, T) = \max(S_T - K, 0),$$

and boundary conditions:

$$\begin{aligned} U(0, v, t) &= 0, & \lim_{S \rightarrow +\infty} U(S, v, t) &= S, \\ U(S, \infty, t) &= S, & \lim_{v \rightarrow 0} U(S, v, t) &= \max(S - K, 0). \end{aligned}$$

If we make the assumption that $U(S, v, t)$ takes the form of

$$(2.5) \quad U(S, v, t) = S_t P_1(S, v, t) - K e^{-r(T-t)} P_2(S, v, t)$$

and make the transformation of $x = \ln(S)$, it is not difficult to find that $P_j, j = 1, 2$ can be calculated with the conditional characteristic function of the log-returns $f_j(x, v, t; \phi)$

$$(2.6) \quad P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \operatorname{Re} \left[\frac{e^{-i\phi \ln[K]} f_j}{i\phi} \right] d\phi.$$

If we further assume that the conditional characteristic function of the log-returns $f_j(x, v, t; \phi)$ is in the form of

$$(2.7) \quad f_j = e^{C(\tau; \phi) + D(\tau; \phi)v + i\phi x},$$

the pricing problem can be reduced to solve the following two ODEs (ordinary differential equations)

$$(2.8) \quad D'(\tau) = \frac{1}{2}\sigma^2 D^2 + [i\rho\sigma\phi - k + m_j - \beta\rho\mu - \lambda(\tau)]D + \left(l_j i\phi - \frac{1}{2}\phi^2\right),$$

$$(2.9) \quad C'(\tau) = k\theta D,$$

with the initial condition $C(0) = D(0) = 0$. Here $l_1 = \frac{1}{2}$, $l_2 = -\frac{1}{2}$, $m_1 = \rho\sigma$, $m_2 = 0$. It is obvious that once D is derived, C can be worked out straightforwardly, and thus what we need to do is to solve the ODE for D , which is actually a Riccati equation with variable coefficients. By making the transformation of $D = -\frac{u'}{q_2 u}$, it can be transformed into a second-order linear ODE with variable coefficients as

$$(2.10) \quad u'' - q_1 u' + q_0 q_2 u = 0,$$

where $q_0 = l_j i\phi - \frac{1}{2}\phi^2$, $q_1 = i\rho\sigma\phi + m_j - 2\Delta \tanh(\Delta\tau + b)$ and $q_2 = \frac{1}{2}\sigma^2$. Also, the boundary condition for ODE (2.10) is $u' = 0$.

It should be noticed that to seek a series solution of Equation (2.10), He & Zhu [5] expanded the solution at $\tau = 0$, which converges in the region of $\tau \leq$

$\frac{1}{\Delta} \sqrt{b^2 + \frac{\pi^2}{4}}$. Obviously, when τ is larger than this radius, the convergence of the solution can not be guaranteed. As a result, other solutions should be found in this case. In particular, if we assume that

$$t_m = \sum_{k=1}^m \frac{1}{\Delta} \sqrt{b^2 + \frac{\pi^2}{4}} * k - \frac{b}{\Delta},$$

then we can expand the solution u at the point $\tau = t_m$ when $t_m \leq \tau \leq t_{m+1}$, i.e.,

$$(2.11) \quad u = \sum_{n=0}^{\infty} a_n (\tau - t_m)^n.$$

As $\tanh(x)$ can be expressed as $\tanh(x) = \frac{e^{2x} - 1}{e^{2x} + 1}$, and $e^{2\Delta\tau}$ can be expanded as $e^{2\Delta\tau} = e^{2\Delta t_m} \sum_{n=0}^{+\infty} c_n (\tau - t_m)^n$ with $c_n = \frac{1}{n} (2\Delta)^n$, ODE (2.10) can be converted into the following equation

$$(2.12) \quad \begin{aligned} & [e^{2b+2\Delta t_m} \sum_{n=0}^{\infty} c_n (\tau - t_m)^n + 1] \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} (\tau - t_m)^n \\ & - [(i\rho\sigma\phi + m_j - 2\Delta) e^{2b+2\Delta t_m} \sum_{n=0}^{\infty} c_n (\tau - t_m)^n \\ & + (i\rho\sigma\phi + m_j + 2\Delta)] \sum_{n=0}^{\infty} (n+1) a_{n+1} (\tau - t_m)^n \\ & + [e^{2b+2\Delta t_m} \sum_{n=0}^{\infty} c_n (\tau - t_m)^n + 1] q_0 q_2 \sum_{n=0}^{\infty} a_n (\tau - t_m)^n = 0. \end{aligned}$$

It should be remarked that Equation (2.12) holds for any τ , which implies that the coefficients of $\{(\tau - t_m)^k, \text{ for all } k \geq 0\}$ equal to zero. Hence, we obtain the following equation

$$(2.13) \quad (k+1)(k+2) a_{k+2} + e^{2b+2\Delta t_m} \sum_{l=0}^k [(k-l+2)(k-l+1) a_{k-l+2} c_l] + I_2 - I_1 = 0$$

for any $k \geq 0$. Here, I_1 and I_2 are defined as

$$\begin{aligned} I_1 &= \left\{ (i\rho\sigma\phi + m_j - 2\Delta) e^{2b+2\Delta t_m} \right. \\ &\quad \times \sum_{l=0}^k [(k-l+1) a_{k-l+1} c_l] + (i\rho\sigma\phi + m_j + 2\Delta) (k+1) a_{k+1} \left. \right\} = 0, \\ I_2 &= e^{2b+2\Delta t_m} q_0 q_2 \sum_{l=0}^k (a_{k-l} c_l) + q_0 q_2 a_k. \end{aligned}$$

In order to reach the final solution, a_0 and a_1 need to be figured out. However, from the boundary condition $u'(0) = 0$, one can only obtain $a_1 = 0$ while the

value of a_0 keeps unknown. To solve this problem, both sides of Equation (2.13) are divided by a_0 and we define

$$(2.14) \quad \hat{a}_k = \frac{a_k}{a_0}, \quad k \geq 0.$$

In this case, $\{\hat{a}_k, k \geq 0\}$ can be evaluated with $\hat{a}_0 = 1$, $\hat{a}_1 = 0$, through

$$(2.15) \quad \hat{a}_{k+2} = \frac{\hat{I}_1 - \hat{I}_2 - e^{2b+2\Delta t_m} \sum_{l=1}^k [(k-l+1)(k-l+2)\hat{a}_{k-l+2}c_l]}{(e^{2b+2\Delta t_m} + 1)(k+2)(k+1)}, \quad k \geq 0,$$

where

$$\begin{aligned} \hat{I}_1 &= \{(\mathrm{i} \rho \sigma \phi + m_j - 2\Delta) e^{2b+2\Delta t_m} \\ &\quad \times \sum_{l=0}^k [(k-l+1)\hat{a}_{k-i+1}c_i] + (\mathrm{i} \rho \sigma \phi + m_j + 2\Delta)(k+1)\hat{a}_{k+1}\}, \\ \hat{I}_2 &= e^{2b+2\Delta t_m} q_0 q_2 \sum_{i=0}^k (\hat{a}_{k-i}c_i) + q_0 q_2 \hat{a}_k. \end{aligned}$$

Once $\{\hat{a}_k, k \geq 0\}$ is derived, D can be easily worked out by

$$(2.16) \quad D(\tau) = -\frac{1}{q_2} \frac{\sum_{n=0}^{\infty} (n+1)\hat{a}_{n+1}(\tau - t_m)^n}{\sum_{n=0}^{\infty} \hat{a}_n(\tau - t_m)^n},$$

and thus C can be calculated as

$$(2.17) \quad C(\tau) = \int_0^{\tau} k\theta D(t) \, dt.$$

By now, we have derived a set of pricing formulae based on different points at which the series is expanded. In the following, the convergence of these solutions in the considered region is verified. It is well-known that the radius of convergence of the series solution to a second order linear ODE near an ordinary point is at least as large as the distance from the ordinary point to the nearest singularity of the ODE [6]. Moreover, according to the results obtained in [5], all the singularities in ODE (2.10) can be specified as

$$(2.18) \quad \tau_k = -\frac{b}{\Delta} + \mathrm{i} \frac{(2k+1)\pi}{2\Delta}, \quad k = 0, 1, 2, \dots$$

Combining both of the two facts, it is not difficult to find that the nearest singularity to any expansion point is always $-\frac{b}{\Delta} + \mathrm{i} \frac{\pi}{2\Delta}$. As a result, the radius of convergence for the series solution expanded at t_m is at least

$$(2.19) \quad R_m = \sqrt{\left(\sum_{k=1}^m \frac{1}{\Delta} \sqrt{b^2 + \frac{\pi^2}{4} * k}\right)^2 + \frac{1}{\Delta^2} \frac{\pi^2}{4}} > \frac{1}{\Delta} \sqrt{b^2 + \frac{\pi^2}{4} * (m+1)},$$

which shows that the solution expanded at $\tau = t_m$ converges in the region $t_m \leq \tau \leq t_{m+1}$. It should also be noticed that

$$(2.20) \quad \lim_{m \rightarrow +\infty} t_m = +\infty,$$

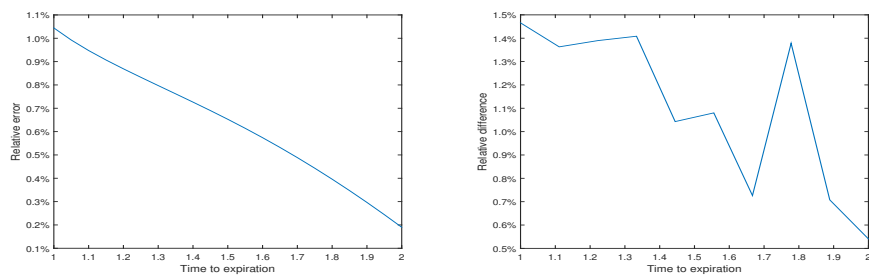
which implies that when $\tau > t_1$, there always exists $m > 1$ such that $\tau < t_m$.

Another issue that should also be mentioned is whether all these solutions including the formula obtained in [5] as well as those derived in this paper, have already covered the whole region $[0, +\infty]$. Given that the converged region for the solution presented in [5] is $[0, \frac{1}{\Delta}\sqrt{b^2 + \frac{\pi^2}{4}}]$, it is not difficult to find that when $b > 0$, $\frac{1}{\Delta}\sqrt{b^2 + \frac{\pi^2}{4}} > t_1$ holds. In this case, we have already finished our task. On the other hand, when $b \leq 0$, our solution expanded at t_1 can be used if τ falls in the gap $[\frac{1}{\Delta}\sqrt{b^2 + \frac{\pi^2}{4}}, t_1]$ since the radius of convergence for this solution is at least $\frac{1}{\Delta}\sqrt{b^2 + \frac{\pi^2}{4}} * 2$, which is obviously larger than the length of the interval $-\frac{b}{\Delta}$. Therefore, we can certainly reach the conclusion that for any $\tau \in [0, +\infty]$, we can always find a converged solution.

Once a new pricing formula is derived, it is natural to numerically check its accuracy, the results of which are presented in next subsection.

2.2. Numerical experiments

In this subsection, the accuracy of the newly derived formulae is numerically shown by comparing the results calculated with our formula and those obtained through the finite difference method and Monte-Carlo simulation. The parameters we use are listed as follows. The drift term u is 0.8 and the correlation ρ is 0.5. The mean-reverting speed k and mean-reverting level θ take the value of 5 and 0.01, respectively. The volatility of volatility β is 0.05 while the current volatility v_0 is 0.2. Both of the underlying price S and strike price K are set to be 100. It should be remarked here that the numerical results provided here are obtained with the series solution being expanded at $\tau = t_1$, which is different from those in [5], where the series solution is expanded at $\tau = 0$.



(a) Our price vs finite difference method price. (b) Our price vs Monte Carlo price.

Figure 1. Comparison of our price with those obtained by other numerical method..

The comparison results of option prices calculated through our formula and those obtained by solving the PDE with the finite difference method as well as by directly simulating the pricing dynamics are depicted in Figure 1. In specific,

Figure 1(a) exhibits the relative difference between our prices and finite difference prices, and it is clear that our pricing formula is quite accurate with the maximum relative difference being no more than 1.1 %. Figure 2(b) further verifies our formula by showing that the maximum relative difference between our prices and Monte-Carlo prices is less than 1.5 %, which is certainly acceptable in real markets.

3. CONCLUSION

In this paper, we have presented a slightly modified formula for European options under the Heston model with minimal entropy martingale measure. As a supplement to [5], in which a series solution is presented and the radius of convergence for such a solution is shown, the new solution constructed with a set of different formulae for different time range is accompanied by a set of complete convergence proofs for the solution that covers the entire time horizon of a European option contract.

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