ON m-PROJECTIVELY FLAT ALMOST PSEUDO RICCI SYMMETRIC MANIFOLDS

J. P. SINGH

Abstract. In the present paper, we study m-projectively flat and almost pseudo Ricci symmetric manifolds.

1. Introduction

As an extended class of pseudo Ricci symmetric manifolds, M. C. Chaki and T. Kawaguchi \[4\] introduced the notion of almost pseudo Ricci symmetric manifolds. An n-dimensional Riemannian manifold \((M^n; g)\) is called an almost pseudo Ricci symmetric manifold if its Ricci tensor \(S\) of type \((0, 2)\) is not identically zero and satisfies the condition

\[(\nabla_X S)(Y, Z) = [A(X) + B(X)]S(Y, Z) + A(Y)S(X, Z) + A(Z)S(Y, X),\]

where \(\nabla\) denotes the operator of covariant differentiation with respect to the metric tensor \(g\) and \(A, B\) are nowhere vanishing associated 1-forms such that \(g(X, \rho_1) = A(X)\) and \(g(X, \rho_2) = B(X)\) for all vector fields \(X\) and \(\rho_1, \rho_2\) are called the basic vector fields of the manifold. An n-dimensional manifold of this kind is denoted by \(A(PRS)_n\). In particular if, \(B = A\), then (1.1) reduces to

\[(\nabla_X S)(Y, Z) = 2A(X)S(Y, Z) + A(Y)S(X, Z) + A(Z)S(Y, X),\]

which represents a pseudo Ricci symmetric manifold \[3\]. In \[4\], Chaki and Kawaguchi also studied conformally flat \(A(PRS)_n\). In 1971, Pokhariyal and Mishra \[8\] established a new curvature tensor known as an m-projective curvature tensor on Riemannian manifolds. Many geometers such as Ojha \([6, 7]\), Singh \[9\], Chaubey and Ojha \[5\] studied properties of an m-projective curvature tensor in different manifolds. A Riemannian manifold is flat if its curvature tensor vanishes at each point. Following this sense, Ojha \[7\] and Zengin \[12\] considered the m-projectively flat curvature tensor in the Sasakian and LP-Sasakian, manifolds, respectively.

Motivated by the above study the author considers the m-projectively flat \(A(PRS)_n\) and established some geometrical properties. The paper is organized as follows: Section 2 concerns with preliminaries. Section 3 is devoted to the study

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of an \( m \)-projectively flat \( A(PRS)_n \) and it is proved that such a manifold is of quasi constant curvature. It is shown that in an \( m \)-projectively flat \( A(PRS)_n \), the vector field defined by \( g(X, \rho) = H(X) \) is a unit proper concircular vector field. The notion of special conformally flat manifold which generalizes the notion of sub projective manifold was introduced by Chen and Yang \([2]\). In the same paper, the authors also introduced the notion of \( K \)-special conformally flat manifold which generalizes the notion of special conformally flat manifold as well as sub projective manifold. In this section, it is shown that an \( m \)-projectively flat \( A(PRS)_n \) with non constant and non negative scalar curvature is a \( K \)-special conformally flat manifold. It is also proved that such a simply connected manifold with non constant and non negative scalar curvature can be isometrically immersed in an Euclidean manifold \( E^{n+1} \) as a hypersurface.

2. Preliminaries

Let \( Q \) be the symmetric endomorphism of the tangent bundle of the manifold corresponding to the Ricci tensor \( S \), i.e., \( S(X,Y) = g(QX,Y) \) for all vector fields \( X,Y \). Let \( \{ e_i : i = 1, 2, \ldots, n \} \) be an orthonormal basis of the tangent space at any point of the manifold. Putting \( Y = Z = e_i \) in (1.1) and then taking summation over \( i, 1 \leq i \leq n \), we obtain

\[
(2.1) \quad dr(X) = r[A(X) + B(X)] + 2A(QX),
\]

where \( r \) is the scalar curvature of the manifold.

Again from (1.1), we get

\[
(2.2) \quad (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = B(X)S(Y,Z) - B(Y)S(X,Z).
\]

Putting \( Y = Z = e_i \) in (2.2) and then taking summation \( \{ e_i : i = 1, 2, \ldots, n \} \), we obtain

\[
(2.3) \quad dr(X) = 2rB(X) - 2B(QX).
\]

If the scalar curvature \( r \) is constant, then

\[
(2.4) \quad dr(X) = 0
\]

for all \( X \). Using (2.4) in (2.3), we obtain

\[
(2.5) \quad B(QX) = rB(X),
\]

i.e.,

\[
(2.6) \quad S(X, \rho_2) = rg(X, \rho_2).
\]

Thus we can state the following theorem.

**Theorem 2.1.** In an \( A(PRS)_n \) of constant curvature, the scalar curvature \( r \) is an eigenvalue of the Ricci tensor \( S \) corresponding to the eigenvector \( \rho_2 \).

The \( m \)-projective curvature tensor \( W^* \) of type \((1,3)\) is defined by \([8]\)

\[
W^*(X,Y,Z) = R(X,Y,Z) + [(S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY)].
\]
Differentiating (2.7) covariantly with respect to $V$, we get
\[
(D_V W^*)(X, Y, Z) = (D_V R)(X, Y, Z) - \frac{1}{2(n-1)} \left\{ (D_V S)(Y, Z) X \\
- (D_V S)(Y, Z) Y + g(Y, Z) (D_V Q)(X) \\
- g(X, Z) (D_V Q)(Y) \right\}. 
\] (2.8)
Contracting the above with respect to $V$, we get
\[
(D_V W^*)(X, Y, Z) = (D_V R)(X, Y, Z) - \frac{1}{2(n-1)} \left\{ (D_V S)(Y, Z) X \\
- (D_V S)(Y, Z) Y + g(Y, Z) (D_V Q)(X) \\
- g(X, Z) (D_V Q)(Y) \right\}. 
\] (2.9)
where $\text{div}$ denotes the divergence.

We know that in a Riemannian manifold
\[
(D_V W^*)(X, Y, Z) = (D_V R)(X, Y, Z) - \frac{1}{2(n-1)} \left\{ (D_V S)(Y, Z) X \\
- (D_V S)(Y, Z) Y + g(Y, Z) (D_V Q)(X) \\
- g(X, Z) (D_V Q)(Y) \right\}. 
\] (2.10)
Using (2.10) in (2.9), we get
\[
(D_V W^*)(X, Y, Z) = (2n-3) \left\{ (D_V S)(Y, Z) X \\
- (D_V S)(Y, Z) Y + g(Y, Z) (D_V Q)(X) \\
- g(X, Z) (D_V Q)(Y) \right\}. 
\] (3.1)

Let us consider an $m$-projectively flat $A(PRS)_n$. Then
\[
(D_V W^*)(X, Y, Z) = 0, 
\] and hence, (2.11) yields
\[
(D_V W^*)(X, Y, Z) = (2n-3) \left\{ (D_V S)(Y, Z) X \\
- (D_V S)(Y, Z) Y + g(Y, Z) (D_V Q)(X) \\
- g(X, Z) (D_V Q)(Y) \right\}. 
\] (3.2)

Making use of (2.2) and (2.3) in (3.1), we obtain
\[
B(X)S(Y, Z) - B(Y)S(X, Z) = \frac{1}{2(n-3)} \left\{ (B(X)g(Y, Z) - B(Y)g(X, Z)) \\
- B(QX)g(Y, Z) - B(QY)g(X, Z) \right\}. 
\] (3.3)
Now putting $Z = \rho_2$ in (3.2), we get
\[
B(X)B(QY) - B(Y)B(QX) = 0, 
\] provided $n > 2$.
Let $B(QX) = g(QX, \rho_2) = P(X) = g(X, \xi)$ for all $X$. Then (3.3) reduces to
\[
B(X)P(Y) = B(Y)P(X), 
\] (3.4)
which implies that the vector fields \( \rho_2 \) and \( \xi \) are co-directional. This leads to the following theorem.

**Theorem 3.1.** In an \( m \)-projectively flat \( A(PRS)_n \) with \( n > 2 \), the vector fields \( \rho_2 \) and \( \xi \) are co-directional.

Again setting \( Y = Z = e_i \) in (3.2) and then taking summation \( \{e_i : i = 1, 2, \ldots, n\} \), we obtain

\[
B(QX) = r B(X),
\]

provided that \( n > 2 \), i.e.,

\[
S(X, \rho_2) = r g(X, \rho_2).
\]

Hence, we can state the following theorem.

**Theorem 3.2.** In an \( m \)-projectively flat \( A(PRS)_n \) with \( n > 2 \), the scalar curvature \( r \) is an eigenvalue of the Ricci tensor \( S \) corresponding to the eigenvector \( \rho_2 \).

In view of (3.5), the relation (3.2) yields

\[
B(X)S(Y, Z) = B(Y)S(X, Z).
\]

Setting \( X = \rho_2 \) in (3.7), we get

\[
S(Y, Z) = \frac{1}{B(\rho_2)} B(Y) B(QZ).
\]

Again using (3.5) in (3.8), we obtain

\[
S(Y, Z) = r H(Y) H(Z),
\]

where \( H(Y) = g(Y, \rho) = \frac{B(Y)}{\sqrt{B(\rho_2)}} \), \( \rho \) is a unit vector field.

From (3.9), it follows that if \( r = 0 \), then \( S(Y, Z) = 0 \), which is inadmissible by the definition of \( A(PRS)_n \). Hence, we can state the following theorem.

**Theorem 3.3.** In a Pseudo projectively flat \( A(PRS)_n \) with \( n > 2 \), the scalar curvature can not vanish and the Ricci tensor is given by (3.9).

As a generalization of the manifold of constant curvature, the notion of the manifold of quasi-constant curvature arose during the study of conformally flat hypersurfaces by Chen and Yano [1]. A Riemannian manifold \( (M^n, g) \) is said to be of quasi constant curvature [1] if it is conformally flat and its curvature tensor \( R \) of type \( (0, 4) \) is of the form

\[
' R(X, Y, Z, U) = a_1 \{g(Y, Z)g(X, U) - g(X, Z)g(Y, U)\}
\]

\[
+ a_2 \{g(Y, Z)A(U)A(U) - g(X, Z)A(Y)A(U)
\]

\[
+ g(X, U)A(Y)A(Z) - g(Y, U)A(X)A(Z)\},
\]

where \( A \) is a nowhere vanishing 1-form and \( a_1, a_2 \) are scalars of which \( a_2 \neq 0 \). Now from (2.7), it follows that in an \( m \)-projectively flat \( A(PRS)_n \), the curvature
tensor $R$ of type $(0,4)$ is of the following form

$$
R(X, Y, Z, U) = \frac{1}{2(n-1)} \{S(Y, Z)g(X, U) - S(X, Z)g(Y, U)
+ S(X, U)g(Y, Z) - S(Y, U)g(X, Z).\}
$$

(3.11)

Using (3.9) in the above relation, we have

$$
R(X, Y, Z, U) = a_1 \{g(Y, Z)g(X, U) - g(X, Z)g(Y, U)\}
+ a_2 \{g(Y, Z)H(X)H(U) - g(X, Z)H(Y)H(U)
+ g(X, U)H(Y)H(Z) - g(Y, U)H(X)H(Z)\},
$$

(3.12)

where $a_1 = 0$ and $a_2 = \frac{r}{2(n-1)}$ are scalars. By virtue of (3.11), it follows from (3.12) that an $m$-projectively flat $A(\text{PRS})_n$ is a manifold of quasi constant curvature. This leads to the following theorem.

**Theorem 3.4.** Every $m$-projectively flat $A(\text{PRS})_n$ is a manifold of quasi constant curvature.

In consequence of (3.9), we have

$$
(D_X S)(Y, Z) = dr(X)H(Y)H(Z) + r[(D_X H)(Y)H(Z)
- (D_X H)(Z)H(Y)]
$$

(3.13)

Using (3.13) in (3.1), we obtain

$$
2(2n - 3)\left[dr(X)H(Y)H(Z) - dr(Y)H(X)H(Z)
+ r \{(D_X H)(Y)H(Z) + (D_X H)(Z)H(Y)
- (D_Y H)(X)H(Z) - (D_Y H)(Z)H(X)\}\right]
= \{dr(X)g(Y, Z) - dr(Y)g(X, Z)\}.
$$

(3.14)

Putting $Y = Z = e_i$ in (3.14) and then taking summation $\{e_i : i = 1, 2, \ldots, n\},$ we obtain

$$
\left[dr(\rho)H(X) + r[(D_{e_i} H)(X) + H(X)\sum_{i=1}^{n}(D_{e_i} H)(e_i)]\right] = \left\{\frac{3n - 5}{2(2n - 3)}\right\} dr(X).
$$

(3.15)

Now, putting $Y = Z = \rho$ in (3.14), we get

$$
2(2n - 3)r(D_{\rho} H)(X) = (4n - 7)\{dr(X) - dr(\rho)H(X)\}.
$$

(3.16)

By the virtue of (3.16), the equation (3.15) yields

$$
(n - 2) dr(X) - dr(\rho)H(X) + 2(2n - 3)r H(X)\sum_{i=1}^{n}(D_{e_i} H)(e_i) = 0.
$$

(3.17)

Now putting $X = \rho$ in the relation (3.17), we have

$$
2(2n - 3)r \sum_{i=1}^{n}(D_{e_i} H)(e_i) = -(n - 3) dr(\rho).
$$

(3.18)
From (3.18) and (3.17), we get

\[ \text{dr}(X) = \text{dr}(\rho)H(X). \]  

(3.19)

Now setting \( Z = \rho \) in (3.14) and then using (3.19), we have

\[ 2(2n-3)r[(DXH)(Y) - (DYH)(X)], \]

which implies that

\[ (DXH)(Y) - (DYH)(X) = 0 \]

for \( n > 2 \). The above equation shows that a 1-form \( H \) is closed.

By virtue of (3.19), the equation (3.16) gives

\[ (\nabla_\rho H)(X) = 0, \]

(3.22)

provided \( n > 2 \), which implies that \( D_\rho \rho = 0 \) and hence, we can state the following theorem

**Theorem 3.5.** In an \( m \)-projectively flat \( A(PRS)_n \) with \( n > 2 \), the integral curves of the generator \( \rho \) are geodesic.

Again replacing \( Y \) for \( = \rho \) in (3.14) and then using (3.19) and (3.22), we get

\[ (DXH)(Z) = \frac{\text{dr}(\rho)}{2r(2n-3)}\{H(X)H(Z) - g(X, Z)\}. \]

(3.23)

Now, let us consider a non zero scalar function \( f = \frac{\text{dr}(\rho)}{2r(2n-3)} \), where \( r \) is non zero scalar curvature tensor. We have

\[ \nabla_X f = -\frac{1}{2r^2(2n-3)}\text{dr}(\rho)(X) + \frac{1}{2r(2n-3)}d^2 r(\rho, X). \]

(3.24)

Again from (3.19), we get

\[ d^2 r(X, Y) = d^2 r(\rho, Y)H(X) + \text{dr}(\rho)(\nabla_Y H)(X). \]

(3.25)

We know that in a Riemannian manifold, the second covariant differential of any function \( h \in C^\infty(M^n) \) is defined by

\[ d^2 h(X, Y) = X(Yh) - (\nabla_X Y)h \]

for all vector fields \( X, Y \), which shows that

\[ d^2 h(X, Y) = d^2 h(Y, X). \]

Taking account of (3.21), the equation (3.25) gives

\[ d^2 h(\rho, Y)H(X) = d^2 h(\rho, X)H(Y). \]

(3.26)

Again setting \( Y = \rho \) in (3.26), we get

\[ d^2 h(\rho, X) = d^2 h(\rho, \rho)H(X) = -\phi H(X), \]

where \( \phi = d^2 \rho(\rho, \rho) \) is a scalar function.

Now in consequence of (3.19) and (3.27), (3.24) assumes the form

\[ \nabla_X f = \delta H(X), \]

where \( \delta = -\frac{1}{2r^2(2n-3)}[r\phi + (\text{dr}(\rho))^2] \) is a non zero scalar.
Now we consider a 1-form $\lambda$ given by

$$\lambda(X) = \frac{1}{2r(2n-3)} \text{dr(}\rho\text{)H}(X) = f \, H(X).$$

In view of (3.21) and (3.28), equation (3.29) becomes

$$d\lambda(X, Y) = 0,$$

i.e., the 1-form $\lambda$ is closed. Therefore, (3.23) can be rewritten as

$$\nabla_X H(Y) = \lambda(X)H(Y) - f \, g(X, Y),$$

where $\lambda$ is closed. But this means that the vector field $\rho$ corresponding to the 1-form $H$ defined by $g(X, \rho) = H(X)$ is a proper concircular vector field ([10], [11]). Hence we can state the following theorem.

**Theorem 3.6.** In an $m$-projectively flat $\text{A(}PRS\text{)}_n$ of non constant scalar curvature, the vector field $\rho$ defined by $g(X, \rho) = H(X)$ is a unit proper concircular vector field.

In 1973, Chen and Yano [2] introduced the notion of special conformally flat manifolds which generalizes the notion of sub projective manifolds. According to them, a conformally flat Riemannian manifold is said to be a special conformally flat manifold if the tensor field $H$ of type $(0,2)$ defined by

$$H(X, Y) = \frac{1}{n-2} S(X, Y) + \frac{r}{2(n-1)(n-2)} g(X, Y)$$

is expressible in the form

$$H(X, Y) = -\frac{\alpha^2}{2} g(X, Y) + \beta(X\alpha)(Y\alpha),$$

where $\alpha, \beta$ are two scalars such that $\alpha$ is positive.

In view of (3.9), the expression (3.32) can be written as

$$H(X, Y) = -\frac{r}{n-2} H(X)H(Y) + \frac{r}{2(n-1)(n-2)} g(X, Y).$$

Putting

$$\alpha^2 = \frac{r}{2(n-1)(n-2)} > 0,$$

provided $r < 0$,

then

$$2\alpha(X\alpha) = -\frac{\text{dr(X)}}{(n-1)(n-2)},$$

which implies by virtue of (3.19) that

$$2\alpha(X\alpha) = -\frac{\text{dr(}\rho\text{)H(X)}}{(n-1)(n-2)}.$$

Hence

$$H(X)H(Y) = -\frac{4(n-1)(n-2)r(X\alpha)(Y\alpha)}{\Omega^2},$$
where \( \Omega = \text{dr}(\rho) \). Thus in consequence of (3.35), the expression (3.34) can be written as

\[
H(X,Y) = -\frac{\alpha^2}{2} g(X,Y) + \beta(X\alpha)(Y\alpha),
\]

where \( \frac{4(n-1)^2}{2r} \). Hence the manifold under consideration is a special conformally flat manifold. Since an Einstein \( m \)-projectively flat manifold is conformally flat \([5]\), we can state the following theorem.

**Theorem 3.7.** An Einstein \( m \)-projectively flat \( A(PRS)_n \) with non constant negative scalar curvature tensor is a special conformally flat manifold.

Also in \([2]\), the authors proved that every simply connected special conformally flat manifold can be isometrically immersed in an Euclidean manifold \( E^{n+1} \) as a hypersurface. Therefore, by virtue of Theorem 3.7, we can state the following theorem

**Theorem 3.8.** Every simply connected \( m \)-projectively flat \( A(PRS)_n \) with non constant negative scalar curvature tensor can be isometrically immersed in an Euclidean manifold \( E^{n+1} \) as a hypersurface.

The notion of \( K \)-special conformally flat manifold which generalizes the notion of special conformally flat manifold as well as sub projective manifold was introduced by Chen and Yano \([2]\). According to them, a conformally flat manifold is said to be \( K \)-special conformally flat manifold if the tensor \( H \) of type \((0,2)\) defined in (3.32) is expressible in the form

\[
H(X,Y) = -\frac{K + \alpha^2}{2} g(X,Y) + \beta \gamma \pi(X)\pi(Y),
\]

where \((X\alpha) = \beta \pi(X)\) on \( G \), \( G \) is an open set on \( M^n \) defined by

\[
G = \{ p^n : \beta \neq 0 \}
\]

and \( \pi \) is a 1-form on \( G \), \( \alpha, \beta, \gamma \) are scalar functions and \( K \) is a constant. We consider an Einstein \( m \)-projectively flat \( A(PRS)_n \). Then such a manifold is conformally flat. Using (3.9) in (3.32), we obtain (3.34). Let put

\[
K + \alpha^2 = -\frac{r}{2(n-1)(n-2)} > 0, \quad \text{provided} \quad r < 0,
\]

where \( K \) is a constant. Then proceeding similarly as before, it can be easily shown that

\[
H(X,Y) = -\frac{K + \alpha^2}{2} g(X,Y) + \beta \gamma \pi(X)\pi(Y),
\]

where \( \beta = \frac{4(n-1)^2}{16r} \), \( \gamma = \frac{16r^2(n-1)^2(r + K(n-1)(n-2))}{16} \). Thus we can state the following theorem.

**Theorem 3.9.** An Einstein \( m \)-projectively flat \( A(PRS)_n \) with non constant negative scalar curvature tensor is a \( K \)-special conformally flat manifold.
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J. P. Singh, Department of Mathematics, Mizoram University Tanhriil, Aizawl, India, e-mail: jpsmaths@gmail.com