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ON m-PROJECTIVELY FLAT ALMOST PSEUDO RICCI SYMMETRIC MANIFOLDS

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Abstract. In the present paper, we study m-projectively flat and almost pseudo Ricci symmetric manifolds.

1. Introduction

As an extended class of pseudo Ricci symmetric manifolds, M. C. Chaki and T. Kawaguchi [**4**] introduced the notion of almost pseudo Ricci symmetric manifolds. An n-dimensional Riemannian manifold $(M^n; g)$ is called an almost pseudo Ricci symmetric manifold if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = [A(X)+B(X)]S(Y,Z)+A(Y)S(X,Z)+A(Z)S(Y,X), \tag{1.1}$$

where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g and A, B are nowhere vanishing associated 1-forms such that $g(X,\rho_1) = A(X)$ and $g(X,\rho_2) = B(X)$ for all vector fields X and ρ_1, ρ_2 are called the basic vector fields of the manifold. An n-dimensional manifold of this kind is denoted by $A(PRS)_n$. In particular if, $B = A$, then (1.1) reduces to

$$(\nabla_X S)(Y,Z) = 2A(X)S(Y,Z)+A(Y)S(X,Z)+A(Z)S(Y,X), \tag{1.2}$$

which represents a pseudo Ricci symmetric manifold [**3**]. In [**4**], Chaki and Kawaguchi also studied conformally flat $A(PRS)_n$. In 1971, Pokhariyal and Mishra [**8**] established a new curvature tensor known as an m-projective curvature tensor on Riemannian manifolds. Many geometers such as Ojha ([**6**], [**7**]), Singh [**9**], Chaubey and Ojha [**5**] studied properties of an m-projective curvature tensor in different manifolds. A Riemannian manifold is flat if its curvature tensor vanishes at each point. Following this sense, Ojha [**7**] and Zengin [**12**] considered the m-projectively flat curvature tensor in the Sasakian and LP-Sasakian, manifolds, respectively.

Motivated by the above study the author considers the m-projectively flat $A(PRS)_n$ and established some geometrical properties. The paper is organized as follows: Section 2 concerns with preliminaries. Section 3 is devoted to the study

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of an m-projectively flat $A(PRS)_n$ and it is proved that such a manifold is of quasi constant curvature. It is shown that in an m-projectively flat $A(PRS)_n$, the vector field defined by $g(X, \rho) = H(X)$ is a unit proper concircular vector field. The notion of special conformally flat manifold which generalizes the notion of sub projective manifold was introduced by Chen and Yang [**2**]. In the same paper, the authors also introduced the notion of K-special conformally flat manifold which generalizes the notion of special conformally flat manifold as well as sub projective manifold. In this section, it is shown that an m-projectively flat $A(PRS)_n$ with non constant and non negative scalar curvature is a K-special conformally flat manifold. It is also proved that such a simply connected manifold with non constant and non negative scalar curvature can be isometrically immersed in an Euclidean manifold E^{n+1} as a hypersurface.

2. Preliminaries

Let Q be the symmetric endomorphism of the tangent bundle of the manifold corresponding to the Ricci tensor S, i.e., $S(X, Y) = g(QX, Y)$ for all vector fields X, Y. Let $\{e_i : i = 1, 2, \ldots, n\}$ be an orthonormal basis of the tangent apace at any point of the manifold. Putting $Y = Z = e_i$ in (1.1) and then taking summation over $i, 1 \leq i \leq n$, we obtain

$$dr(X) = r[A(X) + B(X)] + 2A(QX), \tag{2.1}$$

where r is the scalar curvature of the manifold.
Again from (1.1), we get

$$(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = B(X)S(Y, Z) - B(Y)S(X, Z). \tag{2.2}$$

Putting $Y = Z = e_i$ in (2.2) and then taking summation $\{e_i : i = 1, 2, \ldots, n\}$, we obtain

$$\mathrm{dr}(X) = 2rB(X) - 2B(QX). \tag{2.3}$$

If the scalar curvature r is constant, then

$$\mathrm{dr}(X) = 0 \tag{2.4}$$

for all X. Using (2.4) in (2.3), we obtain

$$B(QX) = rB(X), \tag{2.5}$$

i.e.,

$$S(X, \rho_2) = rg(X, \rho_2). \tag{2.6}$$

Thus we can state the following theorem.

Theorem 2.1. *In an $A(PRS)_n$ of constant curvature, the scalar curvature r is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ_2.*

The $m-$projective curvature tensor W^* of type $(1, 3)$ is defined by [**8**]

$$\begin{aligned} W^*(X, Y, Z) = R(X, Y, Z) + \big[&(S(Y, Z)X - S(X, Z)Y \\ &+ g(Y, Z)QX - g(X, Z)QY\big]. \end{aligned} \tag{2.7}$$

Differentiating (2.7) covariantly with respect to V, we get

$$(2.8)\qquad \begin{aligned}(D_V W^*)(X,Y,Z) = (D_V R)(X,Y,Z) - \frac{1}{2(n-1)}\big\{&(D_V S)(Y,Z)X \\ &- (D_V S)(X,Z)Y + g(Y,Z)(D_V Q)(X) \\ &- g(X,Z)(D_V Q)(Y)\big\}.\end{aligned}$$

Contracting the above with respect to V, we get

$$(2.9)\qquad \begin{aligned}(\operatorname{div} W^*)(X,Y,Z) = (\operatorname{div} R)(X,Y,Z) - \frac{1}{2(n-1)}\big\{&(D_X S)(Y,Z) \\ &- (D_Y S)(X,Z) + g(Y,Z)(\operatorname{div} Q)(X) \\ &- g(X,Z)(\operatorname{div} Q)(Y)\big\},\end{aligned}$$

where div denotes the divergence.
We know that in a Riemannian manifold

$$(2.10)\qquad (\operatorname{div} R)(X,Y,Z) = (D_X S)(Y,Z) - (D_Y S)(X,Z).$$

Using (2.10) in (2.9), we get

$$(2.11)\qquad \begin{aligned}(\operatorname{div} W^*)(X,Y,Z) = \frac{(2n-3)}{2(n-1)}\big[&(D_X S)(Y,Z) \\ &- (D_Y S)(X,Z)\big] - \frac{1}{4(n-1)}\big\{\operatorname{dr}(X)g(Y,Z) \\ &- \operatorname{dr}(Y)g(X,Z)\big\}.\end{aligned}$$

3. $m-$projectively flat $A(PRS)_n$

Let us consider an m-projectively flat $A(PRS)_n$. Then

$$(\operatorname{div} W^*)(X,Y,Z) = 0,$$

and hence, (2.11) yields

$$(3.1)\qquad 2(2n-3)\big[(D_X S)(Y,Z)-(D_Y S)(X,Z)\big] = \big\{\operatorname{dr}(X)g(Y,Z)-\operatorname{dr}(Y)g(X,Z)\big\}.$$

Making use of (2.2) and (2.3) in (3.1), we obtain

$$(3.2)\qquad \begin{aligned}B(X)S(Y,Z) - B(Y)S(X,Z) = \frac{1}{(2n-3)}\Big[&r\big\{B(X)g(Y,Z) - B(Y)g(X,Z)\big\} \\ &- \big\{B(QX)g(Y,Z) - B(QY)g(X,Z)\big\}\Big].\end{aligned}$$

Now putting $Z = \rho_2$ in (3.2), we get

$$(3.3)\qquad B(X)B(QY) - B(Y)B(QX) = 0,$$

provided $n > 2$.

Let $B(QX) = g(QX, \rho_2) = P(X) = g(X, \xi)$ for all X. Then (3.3) reduces to

$$(3.4)\qquad B(X)P(Y) = B(Y)P(X),$$

which implies that the vector fields ρ_2 and ξ are co-directional. This leads to the following theorem.

Theorem 3.1. *In an m-projectively flat $A(PRS)_n$ with $(n > 2)$, the vector fields ρ_2 and ξ are co-directional.*

Again setting $Y = Z = e_i$ in (3.2) and then taking summation $\{e_i : i = 1, 2, \ldots, n\}$, we obtain

$$B(QX) = r\, B(X), \tag{3.5}$$

provided that $n > 2$, i.e.,

$$S(X, \rho_2) = r\, g(X, \rho_2). \tag{3.6}$$

Hence, we can state the following theorem.

Theorem 3.2. *In an m-projectively flat $A(PRS)_n$ with $n > 2$, the scalar curvature r is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ_2.*

In view of (3.5), the relation (3.2) yields

$$B(X)S(Y, Z) = B(Y)S(X, Z). \tag{3.7}$$

Setting $X = \rho_2$ in (3.7), we get

$$S(Y, Z) = \frac{1}{B(\rho_2)} B(Y)B(QZ). \tag{3.8}$$

Again using (3.5) in (3.8), we obtain

$$S(Y, Z) = rH(Y)H(Z), \tag{3.9}$$

where $H(Y) = g(Y, \rho) = \frac{B(Y)}{\sqrt{B(\rho_2)}}$, ρ is a unit vector field.

From (3.9), it follows that if $r = 0$, then $S(Y, Z) = 0$, which is inadmissible by the definition of $A(PRS)_n$. Hence, we can state the following theorem.

Theorem 3.3. *In a Pseudo projectively flat $A(PRS)_n$ with $n > 2$, the scalar curvature can not vanish and the Ricci tensor is given by* (3.9).

As a generalization of the manifold of constant curvature, the notion of the manifold of quasi-constant curvature arose during the study of conformally flat hypersurfaces by Chen and Yano [**1**]. A Riemannian manifold (M^n, g) is said to be of quasi constant curvature [**1**] if it is conformally flat and its curvature tensor R of type $(0, 4)$ is of the form

$$\begin{aligned} {}'R(X, Y, Z, U) = a_1 &\{g(Y, Z)g(X, U) - g(X, Z)g(Y, U)\} \\ &+ a_2\{g(Y, Z)A(X)A(U) - g(X, Z)A(Y)A(U) \\ &+ g(X, U)A(Y)A(Z) - g(Y, U)A(X)A(Z)\}, \end{aligned} \tag{3.10}$$

where A is a nowhere vanishing 1-form and a_1, a_2 are scalars of which $a_2 \neq 0$. Now from (2.7), it follows that in an m-projectively flat $A(PRS)_n$, the curvature tensor R of type $(0,4)$ is of the following form

$$\begin{aligned} {}'R(X,Y,Z,U) = &\frac{1}{2(n-1)}\big\{S(Y,Z)g(X,U) - S(X,Z)g(Y,U) \\ &+ S(X,U)g(Y,Z) - S(Y,U)g(X,Z)\big\}. \end{aligned} \tag{3.11}$$

Using (3.9) in the above relation, we have

$$\begin{aligned} {}'R(X,Y,Z,U) = &a_1\big\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\big\} \\ &+ a_2\big\{g(Y,Z)H(X)H(U) - g(X,Z)H(Y)H(U) \\ &+ g(X,U)H(Y)H(Z) - g(Y,U)H(X)H(Z)\big\}, \end{aligned} \tag{3.12}$$

where $a_1 = 0$ and $a_2 = \frac{r}{2(n-1)}$ are scalars. By virtue of (3.11), it follows from (3.12) that an m-projectively flat $A(PRS)_n$ is a manifold of quasi constant curvature. This leads to the following theorem.

Theorem 3.4. *Every m-projectively flat $A(PRS)_n$ is a manifold of quasi constant curvature.*

In consequence of (3.9), we have

$$\begin{aligned} (D_X S)(Y,Z) = &\ \mathrm{dr}(X)H(Y)H(Z) + r[(D_X H)(Y)H(Z) \\ &+ (D_X H)(Z)H(Y)]. \end{aligned} \tag{3.13}$$

Using (3.13) in (3.1), we obtain

$$\begin{aligned} &2(2n-3)\Big[\mathrm{dr}(X)H(Y)H(Z) - \mathrm{dr}(Y)H(X)H(Z) \\ &+ r\big\{(D_X H)(Y)H(Z) + (D_X H)(Z)H(Y) \\ &- (D_Y H)(X)H(Z) - (D_Y H)(Z)H(X)\big\}\Big] \\ =&\big\{\mathrm{dr}(X)g(Y,Z) - \mathrm{dr}(Y)g(X,Z)\big\}. \end{aligned} \tag{3.14}$$

Putting $Y = Z = e_i$ in (3.14) and then taking summation $\{e_i : i = 1, 2, \ldots, n\}$, we obtain

$$\big[\mathrm{dr}(\rho)H(X) + r\{(D_\rho H)(X) + H(X)\sum_{i=1}^{n}(D_{e_i}H)(e_i)\}\big] = \Big\{\frac{3n-5}{2(2n-3)}\Big\}\mathrm{dr}(X). \tag{3.15}$$

Now, putting $Y = Z = \rho$ in (3.14), we get

$$2(2n-3)r(D_\rho H)(X) = (4n-7)\{\mathrm{dr}(X) - \mathrm{dr}(\rho)H(X)\}. \tag{3.16}$$

By the virtue of (3.16), the equation (3.15) yields

$$(n-2)\,\mathrm{dr}(X) - \mathrm{dr}(\rho)H(X) + 2(2n-3)r\,H(X)\sum_{n}^{i=1}(D_{e_i}H)(e_i) = 0. \tag{3.17}$$

Now putting $X = \rho$ in the relation (3.17), we have

$$2(2n-3)r\sum_{n}^{i=1}(D_{e_i}H)(e_i) = -(n-3)\,\mathrm{dr}(\rho). \tag{3.18}$$

From (3.18) and (3.17), we get

$$\mathrm{dr}(X) = \mathrm{dr}(\rho)H(X). \tag{3.19}$$

Now setting $Z = \rho$ in (3.14) and then using (3.19), we have

$$2(2n-3)r[(D_X H)(Y) - (D_Y H)(X)], \tag{3.20}$$

which implies that

$$(D_X H)(Y) - (D_Y H)(X) = 0 \tag{3.21}$$

for $n > 2$. The above equation shows that a 1-form H is closed.

By virtue of (3.19), the equation (3.16) gives

$$(\nabla_\rho H)(X) = 0, \tag{3.22}$$

provided $n > 2$, which implies that $D_\rho \rho = 0$ and hence, we can state the following theorem

Theorem 3.5. *In an m-projectively flat $A(PRS)_n$ with $n > 2$, the integral curves of the generator ρ are geodesic.*

Again replacing Y for $= \rho$ in (3.14) and then using (3.19) and (3.22), we get

$$(D_X H)(Z) = \frac{\mathrm{dr}(\rho)}{2r(2n-3)}\{H(X)H(Z) - g(X,Z)\}. \tag{3.23}$$

Now, let us consider a non zero scalar function $f = \frac{\mathrm{dr}(\rho)}{2r(2n-3)}$, where r is non zero scalar curvature tensor. We have

$$\nabla_X f = -\frac{1}{2r^2(2n-3)}\,\mathrm{dr}(\rho)(X) + \frac{1}{2r(2n-3)}d^2r(\rho,X). \tag{3.24}$$

Again from (3.19), we get

$$d^2r(X,Y) = d^2r(\rho,Y)H(X) + \mathrm{dr}(\rho)(\nabla_Y H)(X). \tag{3.25}$$

We know that in a Riemannian manifold, the second covariant differential of any function $h \in C^\infty(M^n)$ is defined by

$$d^2h(X,Y) = X(Yh) - (\nabla_X Y)h$$

for all vector fields X, Y, which shows that

$$d^2h(X,Y) = d^2h(Y,X).$$

Taking account of (3.21), the equation (3.25) gives

$$d^2h(\rho,Y)H(X) = d^2h(\rho,X)H(Y). \tag{3.26}$$

Again setting $Y = \rho$ in (3.26), we get

$$d^2h(\rho,X) = d^2h(\rho,\rho)H(X) = -\phi H(X), \tag{3.27}$$

where $\phi = d^2r(\rho,\rho)$ is a scalar function.

Now in consequence of (3.19) and (3.27), (3.24) assumes the form

$$\nabla_X f = \delta\ H(X), \tag{3.28}$$

where $\delta = -\frac{1}{2r^2(2n-3)}[r\phi + \{\mathrm{dr}(\rho)\}^2]$ is a non zero scalar.

Now we consider a 1-form λ given by

$$\lambda(X) = \frac{1}{2r(2n-3)}\,\mathrm{dr}(\rho)H(X) = f\,H(X). \tag{3.29}$$

In view of (3.21) and (3.28), equation (3.29) becomes

$$d\lambda(X,Y) = 0, \tag{3.30}$$

i.e., the 1-form λ is closed. Therefore, (3.23) can be rewritten as

$$(\nabla_X H)(Y) = \lambda(X)H(Y) - f\,g(X,Y), \tag{3.31}$$

where λ is closed. But this means that the vector field ρ corresponding to the 1-form H defined by $g(X,\rho) = H(X)$ is a proper concircular vector field ([**10**], [**11**]). Hence we can state the following theorem.

Theorem 3.6. *In an m-projectively flat $A(PRS)_n$ of non constant scalar curvature, the vector field ρ defined by $g(X,\rho) = H(X)$ is a unit proper concircular vector field.*

In 1973, Chen and Yano [**2**] introduced the notion of special conformally flat manifolds which generalizes the notion of sub projective manifolds. According to them, a conformally flat Riemannian manifold is said to be a special conformally flat manifold if the tensor field H of type (0,2) defined by

$$H(X,Y) = -\frac{1}{n-2}S(X,Y) + \frac{r}{2(n-1)(n-2)}g(X,Y) \tag{3.32}$$

is expressible in the form

$$H(X,Y) = -\frac{\alpha^2}{2}g(X,Y) + \beta(X\alpha)(Y\alpha), \tag{3.33}$$

where α, β are two scalars such that α is positive.

In view of (3.9), the expression (3.32) can be written as

$$H(X,Y) = -\frac{r}{n-2}H(X)H(Y) + \frac{r}{2(n-1)(n-2)}g(X,Y). \tag{3.34}$$

Putting

$$\alpha^2 = -\frac{r}{2(n-1)(n-2)} > 0, \qquad \text{provided } r < 0, \tag{3.35}$$

then

$$2\alpha(X\alpha) = -\frac{\mathrm{dr}(X)}{(n-1)(n-2)}, \tag{3.36}$$

which implies by virtue of (3.19) that

$$2\alpha(X\alpha) = -\frac{\mathrm{dr}(\rho)H(X)}{(n-1)(n-2)}. \tag{3.37}$$

Hence

$$H(X)H(Y) = -\frac{4(n-1)(n-2)r(X\alpha)(Y\alpha)}{\Omega^2}, \tag{3.38}$$

where $\Omega = \mathrm{dr}(\rho)$. Thus in consequence of (3.35), the expression (3.34) can be written as

$$H(X,Y) = -\frac{\alpha^2}{2}g(X,Y) + \beta(X\alpha)(Y\alpha), \tag{3.39}$$

where $\frac{4(n-1)r^2}{\Omega^2}$. Hence the manifold under consideration is a special conformally flat manifold. Since an Einstein m-projectively flat manifold is conformally flat [**5**], we can state the following theorem.

Theorem 3.7. *An Einstein m-projectively flat* $A(PRS)_n$ *with non constant negative scalar curvature tensor is a special conformally flat manifold.*

Also in [**2**], the authors proved that every simply connected special conformally flat manifold can be isometrically immersed in an Euclidean manifold E^{n+1} as a hypersurface. Therefore, by virtue of Theorem 3.7, we can state the following theorem

Theorem 3.8. *Every simply connected m-projectively flat* $(APRS)_n$ *with non constant negative scalar curvature tensor can be isometrically immersed in an Euclidean manifold* E^{n+1} *as a hypersurface.*

The notion of K-special conformally flat manifold which generalizes the notion of special conformally flat manifold as well as sub projective manifold was introduced by Chen and Yano [**2**]. According to them, a conformally flat manifold is said to be K-special conformally flat manifold if the tensor H of type (0,2) defined in (3.32) is expressible in the form

$$H(X,Y) = -\frac{K+\alpha^2}{2}g(X,Y) + \beta\gamma\pi(X)\pi(Y), \tag{3.40}$$

where $(X\alpha) = \beta\pi(X)$ on G, G is an open set on M^n defined by

$$G = \{p^n : \beta \neq 0\} \tag{3.41}$$

and π is a 1-form on G, α, β, γ are scalar functions and K is a constant. We consider an Einstein m-projectively flat $A(PRS)_n$. Then such a manifold is conformally flat. Using (3.9) in (3.32), we obtain (3.34). Let put

$$K + \alpha^2 = -\frac{r}{2(n-1)(n-2)} > 0, \qquad \text{provided } r < 0, \tag{3.42}$$

where K is a constant. Then proceeding similarly as before, it can be easily shown that

$$H(X,Y) = -\frac{K+\alpha^2}{2}g(X,Y) + \beta\gamma\pi(X)\pi(Y), \tag{3.43}$$

where $\beta = \frac{4(n-1)r^2}{\Omega^2}$, $\gamma = \frac{16r^3(n-1)^2\{r+K(n-1)(n-2)\}}{\Omega^4}$. Thus we can state the following theorem.

Theorem 3.9. *An Einstein m-projectively flat* $A(PRS)_n$ *with non constant negative scalar curvature tensor is a K-special conformally flat manifold.*

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