ON STRONG VARIATIONS OF WEYL TYPE THEOREMS

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Abstract. An operator $T$ acting on a Banach space $X$ satisfies the property $(U_W)$ if $\sigma(T) \setminus \sigma_{SF^-}(T) = E(T)$, where $\sigma_a(T)$ is the approximate point spectrum of $T$, $\sigma_{SF^-}(T)$ is the upper semi-Weyl spectrum of $T$ and $E(T)$ is the set of all eigenvalues of $T$ that are isolated in the spectrum $\sigma(T)$ of $T$. In this paper, we introduce and study two new spectral properties, namely $(V_E)$ and $(V_Ea)$, in connection with Weyl type theorems. Among other results, we have that $T$ satisfies property $(V_E)$ if and only if $T$ satisfies property $(U_W)$ and $\sigma(T) = \sigma_a(T)$.

1. Introduction and preliminaries

Throughout this paper, $L(X)$ denotes the algebra of all bounded linear operators acting on an infinite-dimensional complex Banach space $X$. We refer to [25] for details about notations and terminologies. However, we give the following notations that will be useful in the sequel:

- Browder spectrum: $\sigma_b(T)$
- Weyl spectrum: $\sigma_W(T)$
- Upper semi-Browder spectrum: $\sigma_{ub}(T)$
- Upper semi-Weyl spectrum: $\sigma_{SF^-}(T)$
- Drazin invertible spectrum: $\sigma_D(T)$
- B-Weyl spectrum: $\sigma_{BW}(T)$
- Left Drazin invertible spectrum: $\sigma_{LD}(T)$
- Upper semi-B-Weyl spectrum: $\sigma_{SBF^-}(T)$
- approximate point spectrum: $\sigma_a(T)$
- surjectivity spectrum: $\sigma_s(T)$

In this paper, we introduce two new spectral properties of type Weyl theorems, namely, the properties $(V_E)$ and $(V_{Ea})$, respectively. In addition, we establish the precise relationships between these properties and other variants of Weyl’s theorem recently introduced in [8], [9], [24], [25], [26], [27] and [29].
Recall that an operator \( T \in L(X) \) is said to have the single valued extension property at \( \lambda_0 \in \mathbb{C} \) (abbreviated SVEP at \( \lambda_0 \)) if for every open disc \( \mathbb{D}_{\lambda_0} \subseteq \mathbb{C} \) centered at \( \lambda_0 \), the only analytic function \( f : \mathbb{D}_{\lambda_0} \to X \) which satisfies the equation
\[
(\lambda I - T)f(\lambda) = 0 \quad \text{for all } \lambda \in \mathbb{D}_{\lambda_0},
\]
is \( f \equiv 0 \) on \( \mathbb{D}_{\lambda_0} \) (see [17]). The operator \( T \) is said to have SVEP if it has SVEP at every point \( \lambda \in \mathbb{C} \). Evidently, every \( T \in L(X) \) has SVEP at each point of the resolvent set \( \rho(T) := \mathbb{C} \setminus \sigma(T) \). Moreover, \( T \) has SVEP at every point of the boundary \( \partial \sigma(T) \) of the spectrum. In particular, \( T \) has SVEP at every isolated point of the spectrum. Note that (see [1, Theorem 3.8])
\[
(1) \quad p(\lambda I - T) < \infty \implies T \text{ has SVEP at } \lambda,
\]
and dually,
\[
(2) \quad q(\lambda I - T) < \infty \implies T^* \text{ has SVEP at } \lambda.
\]
It is easily seen from definition of localized SVEP that
\[
(3) \quad \lambda \notin \text{acc } \sigma_a(T) \implies T \text{ has SVEP at } \lambda,
\]
where \( \text{acc } K \) means the set of all accumulation points of \( K \subseteq \mathbb{C} \), and
\[
(4) \quad \lambda \notin \text{acc } \sigma_s(T) \implies T^* \text{ has SVEP at } \lambda.
\]

**Remark 1.1.** If \( \lambda I - T \) is a semi \( B \)-Fredholm operator, then the implications (1)–(4) are equivalences (see [2]).

**Lemma 1.2.** ([3, Lemma 2.4]) Let \( T \in L(X) \). Then
(i) \( T \) is upper semi \( B \)-Fredholm and \( \alpha(T) < \infty \) if and only if \( T \in \Phi_+(X) \).
(ii) \( T \) is lower semi \( B \)-Fredholm and \( \beta(T) < \infty \) if and only if \( T \in \Phi_-(X) \).

Denote by iso \( K \), the set of all isolated points of \( K \subseteq \mathbb{C} \). If \( T \in L(X) \), define
\[
E^0(T) = \{ \lambda \in \text{iso } \sigma(T) : 0 < \alpha(\lambda I - T) < \infty \},
\]
\[
E^0_a(T) = \{ \lambda \in \text{iso } \sigma_a(T) : 0 < \alpha(\lambda I - T) < \infty \},
\]
\[
E_a(T) = \{ \lambda \in \text{iso } \sigma_a(T) : 0 < \alpha(\lambda I - T) \},
\]
\[
E_a(T) = \{ \lambda \in \text{iso } \sigma_a(T) : 0 < \alpha(\lambda I - T) \}.
\]

Also, define
\[
\Pi^0(T) = \sigma(T) \setminus \sigma_b(T), \quad \Pi^0_a(T) = \sigma_a(T) \setminus \sigma_ab(T),
\]
\[
\Pi(T) = \sigma(T) \setminus \sigma_D(T), \quad \Pi_a(T) = \sigma_a(T) \setminus \sigma_{LD}(T).
\]

Let \( T \in L(X) \). Following Coburn [15], \( T \) is said to satisfy Weyl’s theorem, in symbols \((W)\), if \( \sigma(T) \setminus \sigma_W(T) = E^0(T) \). Following Rakočević [21], \( T \) is said to satisfy a-Weyl’s theorem, in symbols \((aW)\), if \( \sigma_a(T) \setminus \sigma_{aW}(T) = E^0_a(T) \). According to Berkani and Koliha [11], \( T \) is said to satisfy generalized Weyl’s theorem, in symbols \((gW)\), if \( \sigma(T) \setminus \sigma_{BW}(T) = E(T) \). Similarly, \( T \) is said to satisfy generalized a-Weyl’s theorem, in symbol \((gaW)\), if \( \sigma_a(T) \setminus \sigma_{SBFW}(T) = E_a(T) \).

Now, we describe several spectral properties introduced recently in [14], [24], [25], [26] and [27].
Definition 1.3. An operator \( T \in L(X) \) is said to have:

(i) property \((gaw)\) [14] if \( \sigma(T) \setminus \sigma_{BW}(T) = E_\alpha(T) \).

(ii) property \((z)\) [27] if \( \sigma(T) \setminus \sigma_{SF^-}(T) = E_0(T) \).

(iii) property \((gz)\) [27] if \( \sigma(T) \setminus \sigma_{SBF^-}(T) = E_a(T) \).

(iv) property \((v)\) [25] if \( \sigma(T) \setminus \sigma_{SBF^-}(T) = E^0(T) \).

(v) property \((gv)\) [25] if \( \sigma(T) \setminus \sigma_{SBF^-}(T) = E(T) \).

(vi) property \((Sw)\) [24] if \( \sigma(T) \setminus \sigma_{SBF^-}(T) = E^0(T) \).

(vii) property \((Saw)\) [26] if \( \sigma(T) \setminus \sigma_{SBF^-}(T) = E_0(T) \).

Property \((gv)\) (resp., \((v)\)) is also called property \((gt)\) (resp., \((t)\)) in [22], and property \((gh)\) (resp., \((hi)\)) in [28]. It was proved in [25, Corollary 2.12], that property \((gv)\) (resp., \((v)\)) is equivalent to property \((gz)\) (resp., \((z)\)). Also, it was proved in [26, Corollary 2.9], that properties \((Sw)\) and \((Saw)\) are equivalent.

2. Properties \((V_E)\) and \((V_{E_\alpha})\).

According to [8], \( T \in L(X) \) has property \((W_E)\) (resp., property \((UW_{E_\alpha})\)) if \( \sigma(T) \setminus \sigma_W(T) = E(T) \) (resp. \( \sigma_a(T) \setminus \sigma_{SF^-}(T) = E_a(T) \)). It was shown in [8, Theorem 2.3] (resp., [8, Theorem 3.5]) that property \((W_E)\) (resp., \((UW_{E_\alpha})\)) implies generalized Weyl’s theorem (resp., property \((W_E)\)) but not conversely. Following to [9], an operator \( T \in L(X) \) is said to have property \((UW_E)\) if \( \sigma_a(T) \setminus \sigma_{SF^-}(T) = E(T) \). It was shown in [9, Theorem 3.5] that property \((UW_E)\) implies property \((W_E)\) but not conversely. Also in [9], it is shown that properties \((UW_{E_\alpha})\) and \((UW_E)\) are independent. According to [29], \( T \in L(X) \) has property \((Z_{E_\alpha})\) if \( \sigma(T) \setminus \sigma_W(T) = E_0(T) \). It was proved in [29, Corollary 2.5] that property \((Z_{E_\alpha})\) also implies property \((W_E)\). In this section, we introduce and study two equivalent spectral properties that are stronger than the properties \((UW_{E_\alpha})\), \((UW_E)\) and \((Z_{E_\alpha})\).

Definition 2.1. An operator \( T \in L(X) \) is said to have property \((V_E)\) if \( \sigma(T) \setminus \sigma_{SF^-}(T) = E(T) \).

Example 2.2. 1. Let \( L \) be the unilateral left shift operator on \( \ell^2(\mathbb{N}) \). It is well known that \( \sigma(L) = \sigma_{SF^-}(L) = D(0, 1) \), the closed unit disc on \( \mathbb{C} \) and \( E(L) = \emptyset \). Therefore, \( \sigma(L) \setminus \sigma_{SF^-}(L) = E(L) \), and so \( L \) satisfies property \((V_E)\).

2. Consider the Volterra operator \( V \) on the Banach space \( C[0, 1] \) defined by \( V(f)(x) = \int_0^x f(t)dt \) for all \( f \in C[0, 1] \). Note that \( V \) is injective and quasinilpotent. Thus, \( \sigma(V) = \{0\} \), \( \alpha(V) = 0 \) and hence \( E(V) = \emptyset \). Since the range \( R(V) \) is not closed, then \( \sigma_{SF^-}(V) = \{0\} \). Therefore, \( \sigma(V) \setminus \sigma_{SF^-}(V) = E(V) \), that means \( V \) has property \((V_E)\).

Theorem 2.3. For \( T \in L(X) \), the following statements are equivalent:

(i) \( T \) has property \((V_E)\),
(ii) \( T \) has property \((UW_E)\) and \( \sigma(T) = \sigma_a(T) \),
(iii) $T$ has property $(UW_{E_a})$ and $\sigma(T) = \sigma_a(T)$.

Proof. (i) $\Rightarrow$ (ii). Suppose that $T$ satisfies property $(V_E)$ and let $\lambda \in \sigma_a(T) \setminus \sigma_{SF_+}(T)$. Since $\sigma_a(T) \setminus \sigma_{SF_+}(T) \subseteq \sigma(T) \setminus \sigma_{SF_+}(T) = E(T)$, we have $\lambda \in E(T)$ and so, $\sigma_a(T) \setminus \sigma_{SF_+}(T) \subseteq E(T)$.

To show the opposite inclusion $E(T) \subseteq \sigma_a(T) \setminus \sigma_{SF_+}(T)$, let $\lambda \in E(T)$. Then, $\lambda \in \sigma_a(T)$ and $\alpha(\lambda I - T) > 0$, so $\lambda I - T$ is not bounded below and hence, $\lambda \in \sigma_a(T)$. As $T$ satisfies property $(V_E)$ and $\lambda \in E(T)$, it follows that $\lambda I - T$ is upper semi-Weyl. Therefore, $\lambda \in \sigma_a(T) \setminus \sigma_{SF_+}(T)$. Thus, $E(T) \subseteq \sigma_a(T) \setminus \sigma_{SF_+}(T)$ and $T$ satisfies property $(UW_E)$. Consequently, $\sigma(T) \setminus \sigma_{SF_+}(T) = E(T)$ and $\sigma_a(T) \setminus \sigma_{SF_+}(T) = E(T)$. Therefore, $\sigma(T) \setminus \sigma_{SF_+}(T) = \sigma_a(T) \setminus \sigma_{SF_+}(T)$ and $\sigma(T) = \sigma_a(T)$.

(ii) $\Rightarrow$ (i). Suppose that $T$ satisfies property $(UW_E)$ and $\sigma(T) = \sigma_a(T)$. Then, $\sigma(T) \setminus \sigma_{SF_+}(T) = \sigma_a(T) \setminus \sigma_{SF_+}(T) = E(T)$. Thus, $\sigma(T) \setminus \sigma_{SF_+}(T) = E(T)$ and $T$ satisfies property $(V_E)$.

(ii) $\Leftrightarrow$ (iii). Obvious. $\square$

The next example shows that, in general, property $(UW_{E_a})$ does not imply property $(V_E)$.

**Example 2.4.** Let $R$ be the unilateral right shift operator on $\ell^2(\mathbb{N})$ and $U \in L(\ell^2(\mathbb{N}))$ be defined by

$$U(x_1, x_2, x_3, \cdots) = (0, x_2, x_3, \cdots).$$

Define an operator $T$ on $X = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $T = R \oplus U$. Then, $\sigma(T) = D(0, 1)$, the closed unit disc on $\mathbb{C}$, $\sigma_a(T) = \Gamma \cup \{0\}$, where $\Gamma$ denotes the unit circle of $\mathbb{C}$ and $\sigma_{SF_+}(T) = \Gamma$. Moreover, $E_a(T) = \{0\}$ and $E(T) = \emptyset$. Therefore, $\sigma_a(T) \setminus \sigma_{SF_+}(T) = E_a(T)$ and $\sigma(T) \setminus \sigma_{SF_+}(T) \neq E(T)$. Thus, $T$ satisfies properties $(UW_{E_a})$, but $T$ does not satisfy property $(V_E)$.

The next example shows that, in general, property $(UW_E)$ does not imply property $(V_E)$.

**Example 2.5.** Let $R$ be the unilateral right shift operator on $\ell^2(\mathbb{N})$. Define an operator $T$ on $X = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $T = R \oplus 0$. Then, $\sigma(T) = D(0, 1)$, $\sigma_a(T) = \sigma_{SF_+}(T) = \Gamma \cup \{0\}$ and $E(T) = \emptyset$. Therefore, $\sigma_a(T) \setminus \sigma_{SF_+}(T) = E(T)$ and $\sigma(T) \setminus \sigma_{SF_+}(T) \neq E(T)$. Thus, $T$ satisfies property $(UW_E)$, but $T$ does not satisfy property $(V_E)$.

The next result gives the relationship between the properties $(V_E)$ and $(W_E)$.

**Theorem 2.6.** Let $T \in L(X)$. Then $T$ has property $(V_E)$ if and only if $T$ has property $(W_E)$ and $\sigma_{SF_+}(T) = \sigma_W(T)$.

Proof. Sufficiency: Suppose that $T$ satisfies property $(V_E)$, then by Theorem 2.3, $T$ satisfies property $(UW_E)$. Property $(UW_E)$ implies by [9, Theorem 3.2] that
$T$ satisfies property $(W_E)$. Consequently, $\sigma(T) \smallsetminus \sigma_{SF^+}(T) = E(T)$ and $\sigma(T) \smallsetminus \sigma_W(T) = E(T)$. Therefore, $\sigma_{SF^+}(T) = \sigma_W(T)$.

Necessity: Suppose that $T$ satisfies property $(W_E)$ and $\sigma_{SF^+}(T) = \sigma_W(T)$. Then, $\sigma(T) \smallsetminus \sigma_{SF^+}(T) = \sigma(T) \smallsetminus \sigma_W(T) = E(T)$, and so $T$ satisfies property $(V_E)$. \hfill \square

The next example shows that, in general, property $(W_E)$ does not imply property $(V_E)$.

**Example 2.7.** Let $Q$ be defined on $\ell^1(\mathbb{N})$ by

$$Q(x_1, x_2, x_3, \ldots, x_k, \ldots) = (0, \alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_k x_k, \ldots),$$

where $(\alpha_i)$ is a sequence of complex numbers such that $0 < |\alpha_i| \leq 1$ and $\sum_{i=1}^{\infty} \alpha_i < \infty$. It follows from [11, Example 3.12], that

$$R(Q^n) \neq R(Q^n), \quad n = 1, 2, \ldots.$$ Define the operator $T$ on $X = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N}) \oplus \ell^1(\mathbb{N})$ by $T = R \oplus 0 \oplus Q$, where $R$ is the unilateral right shift operator. Then, $\sigma(T) = \sigma_W(T) = D(0, 1)$, $\sigma_{SF^+}(T) = \Gamma \cup \{0\}$ and $E(T) = \emptyset$. We then have,

$$\sigma(T) \smallsetminus \sigma_W(T) = E(T), \quad \sigma(T) \smallsetminus \sigma_{SF^+}(T) \neq E(T).$$

Hence, $T$ satisfies property $(W_E)$, but $T$ does not satisfy property $(V_E)$.

The next result gives the relationship between the property $(V_E)$ and generalized Weyl’s theorem.

**Theorem 2.8.** Let $T \in L(X)$. Then $T$ has property $(V_E)$ if and only if $T$ satisfies generalized Weyl’s theorem and $\sigma_{SF^+}(T) = \sigma_{BW}(T)$.

**Proof.** Sufficiency: Property $(V_E)$ implies by Theorem 2.6, that $T$ satisfies property $(W_E)$, and property $(W_E)$ implies by [8, Theorem 2.3], that $T$ satisfies generalized Weyl’s theorem. Consequently, $\sigma(T) \smallsetminus \sigma_{SF^+}(T) = E(T)$ and $\sigma(T) \smallsetminus \sigma_{BW}(T) = E(T)$. Therefore, $\sigma_{SF^+}(T) = \sigma_{BW}(T)$.

Necessity: Assume that $T$ satisfies generalized Weyl’s theorem and $\sigma_{SF^+}(T) = \sigma_{BW}(T)$. Then, $\sigma(T) \smallsetminus \sigma_{SF^+}(T) = \sigma(T) \smallsetminus \sigma_{BW}(T) = E(T)$, that means $T$ satisfies property $(V_E)$. \hfill \square

**Remark 2.9.** From Theorem 2.8, property $(V_E)$ implies generalized Weyl’s theorem. However, the converse is not true in general. Consider the operator $T$ in Example 2.7, since $T$ satisfies property $(W_E)$, then it also satisfies generalized Weyl’s theorem, but does not satisfy property $(V_E)$.

**Theorem 2.10.** Suppose that $T \in L(X)$ has property $(V_E)$. Then:

(i) $T$ has property $(Z_{E_a})$,
(ii) $E_a(T) = E_a^0(T) = \Pi_a^0(T) = \Pi_a(T) = \Pi(T) = E^0(T) = E(T)$. 

Since \( \sigma \) satisfies property (a), then \( \sigma(T) = \sigma_a(T) \) and also implies by Theorem 2.6 that \( \sigma_{SF^-}(T) = \sigma_W(T) \). Hence, \( \sigma(T) \setminus \sigma_W(T) = \sigma(T) \setminus \sigma_{SF^-}(T) = E(T) = E_a(T) \) and so \( T \) satisfies property \((Z_{E_a})\).

(ii) Follows from (i) and [29, Lemma 2.3]. \( \square \)

**Example 2.11.** Let \( R \) be the unilateral right shift operator defined on \( \ell^2(\mathbb{N}) \). Since \( \sigma(R) = \sigma_W(R) = D(0,1) \), \( E(R) = E_a(R) = \emptyset \) and \( \sigma_{SF^-}(R) = \Gamma \), then \( R \) satisfies property \((Z_{E_a})\), but does not satisfy property \((V_E)\).

**Theorem 2.12.** For \( T \in L(X) \), the following statements are equivalent:

(i) \( T \) has property \((V_E)\).

(ii) \( T \) has property \((v)\) and \( E^0(T) = E(T) \).

(iii) \( T \) has property \((z)\) and \( E^0(T) = E(T) \).

(iv) \( T \) has property \((gz)\) and \( \sigma_{SF^-}(T) = \sigma_{SBF^-}(T) \).

(v) \( T \) has property \((gz)\) and \( \sigma_{SF^-}(T) = \sigma_{SBF^-}(T) \).

**Proof.** (i) \( \Rightarrow \) (ii). Suppose that \( T \) satisfies property \((V_E)\). Then by Theorem 2.10, \( E^0(T) = E(T) \), and hence \( \sigma(T) \setminus \sigma_{SF^-}(T) = E(T) = E^0(T) \), that means \( T \) has property \((v)\).

(ii) \( \Rightarrow \) (i). If \( T \) satisfies property \((v)\) and \( E^0(T) = E(T) \), then \( \sigma(T) \setminus \sigma_{SF^-}(T) = E^0(T) = E(T) \) and \( T \) satisfies property \((V_E)\).

(ii) \( \Leftrightarrow \) (iii). The equivalence between the properties \((z)\) and \((v)\) have been proved in [25, Corollary 2.12].

(i) \( \Rightarrow \) (iv). Assume that \( T \) satisfies property \((V_E)\). By Theorem 2.3, \( T \) satisfies property \((UWE_a)\). Property \((UWE_a)\) implies by [8, Theorem 3.2] that \( T \) satisfies generalized \( \alpha \)-Weyl’s theorem and \( \sigma_{SF^-}(T) = \sigma_{SBF^-}(T) \). Consequently, \( E(T) = \sigma(T) \setminus \sigma_{SF^-}(T) = \sigma(T) \setminus \sigma_{SBF^-}(T) \), and hence \( T \) satisfies property \((gz)\).

(iv) \( \Rightarrow \) (i). Suppose that \( T \) satisfies property \((gz)\) and \( \sigma_{SF^-}(T) = \sigma_{SBF^-}(T) \). Then \( \sigma(T) \setminus \sigma_{SF^-}(T) = \sigma(T) \setminus \sigma_{SBF^-}(T) = E(T) \), and hence \( T \) satisfies property \((V_E)\).

(iv) \( \Leftrightarrow \) (v). The equivalence between the properties \((gz)\) and \((gz)\) have been proved in [25, Corollary 2.12]. \( \square \)

The following example shows that, in general, property \((gz)\) (resp. \((v)\)) does not imply property \((V_E)\).

**Example 2.13.** Consider the operator \( T = 0 \) defined on the Hilbert space \( \ell^2(\mathbb{N}) \). Then, \( \sigma(T) = \sigma_{SF^-}(T) = \{0\} \), \( \sigma_{SBF^-}(T) = \emptyset \) and \( E(T) = \{0\} \). Therefore, \( \sigma(T) \setminus \sigma_{SF^-}(T) \neq E(T) \) and \( T \) does not satisfy property \((V_E)\). On the other hand, \( \sigma(T) \setminus \sigma_{SBF^-}(T) = E(T) \), that means \( T \) satisfies property \((gz)\), in consequence \( T \) also satisfies property \((v)\).

The next result gives the relationship between the properties \((V_E)\) and \((Sw)\).
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2.10, \( E \) in \( \text{property (} \sigma \text{)} \) consequently \( E \)

\( \text{semi-Weyl} \) operator and \( \text{closed for any} \ n \ \sigma \ \lambda \text{semi-Fredholm, and hence upper semi-Weyl. Therefore,} \)

\( \text{Theorem 2.8}, \) respectively. Thus, we conclude that \( T \)

\( \text{Theorem 2.16.} \) \( \text{The next result gives the relationship between property (} \sigma \text{)} \) and \( \text{Weyl’s theorem.} \)

\( \text{Example 2.15.} \) \( \text{Consider the operator} \)

\( \text{This shows} \ \sigma(T) \ \text{and} \ \sigma_{\text{SBF}}(T) \ \text{is upper} \)

\( \text{The following example shows that, in general, property (} \sigma \text{) does not imply} \)

\( \text{Therefore,} \ \lambda \ \text{is an upper} \)

\( \text{Sufficiency: Suppose that} \ T \text{satisfies property (} \sigma \text{)}. \text{It follows by} \)

\( \text{Theorem 2.8}, \ T \text{satisfies generalized Weyl’s theorem. Since generalized Weyl’s theo-

Necessity: Assume that $T$ satisfies Weyl’s theorem and $\sigma W(T) \setminus \sigma_{SF^-}(T) = E(T) \setminus E^0(T)$. Since $\sigma(T) \setminus \sigma W(T) = E^0(T)$, $\sigma(T) = E^0(T) \cup \sigma W(T)$ and $E^0(T) \cap \sigma W(T) = \emptyset$. Thus,

$$\sigma(T) \setminus \sigma_{SF^-}(T) = [E^0(T) \cup \sigma W(T)] \setminus \sigma_{SF^-}(T)$$

$$= E^0(T) \cup [\sigma W(T) \setminus \sigma_{SF^-}(T)]$$

$$= E^0(T) \cup [E(T) \setminus E^0(T)] = E(T)$$

and hence $T$ satisfies property $(V_E)$. □

**Remark 2.17.** By Theorem 2.16, property $(V_E)$ implies Weyl’s theorem. However, the converse is not true in general. Consider the operator $T$ in Remark 2.9, since $T$ satisfies generalized Weyl’s theorem, then it also satisfies Weyl’s theorem, but does not satisfy property $(V_E)$.

**Definition 2.18.** An operator $T \in L(X)$ is said to have property $(V_{E_a})$ if $\sigma(T) \setminus \sigma_{SF^-}(T) = E_a(T)$.

**Theorem 2.19.** Let $T \in L(X)$. Then $T$ has property $(V_{E_a})$ if and only if $T$ has property $(UW_{E_a})$ and $\sigma(T) = \sigma_a(T)$.

*Proof.* Sufficiency: Assume that $T$ satisfies property $(V_{E_a})$. Then $\sigma_a(T) \setminus \sigma_{SF^-}(T) \subseteq \sigma(T) \setminus \sigma_{SF^-}(T) = E_a(T)$ and so $\sigma_a(T) \setminus \sigma_{SF^-}(T) \subseteq E_a(T)$.

To show the opposite inclusion $E_a(T) \subseteq \sigma_a(T) \setminus \sigma_{SF^-}(T)$, let $\lambda \in E_a(T)$. Then, $\lambda \in \sigma_a(T)$ and hence $\lambda \in \sigma(T)$. As $T$ satisfies property $(V_{E_a})$ and $\lambda \in E_a(T)$, it follows that $E \setminus T$ is upper semi-Weyl. Therefore, $\lambda \in \sigma_a(T) \setminus \sigma_{SF^-}(T)$.

Thus, $E_a(T) \subseteq \sigma_a(T) \setminus \sigma_{SF^-}(T)$ and $T$ satisfies property $(UW_{E_a})$.

Consequently, $\sigma(T) \setminus \sigma_{SF^-}(T) = E_a(T)$ and $\sigma_a(T) \setminus \sigma_{SF^-}(T) = E_a(T)$. Therefore,

$$\sigma(T) \setminus \sigma_{SF^-}(T) = \sigma_a(T) \setminus \sigma_{SF^-}(T)$$

and $\sigma(T) = \sigma_a(T)$.

Necessity: Suppose that $T$ satisfies property $(UW_{E_a})$ and $\sigma(T) = \sigma_a(T)$. Then, $\sigma(T) \setminus \sigma_{SF^-}(T) = \sigma_a(T) \setminus \sigma_{SF^-}(T) = E_a(T)$, in consequence $T$ satisfies property $(V_{E_a})$. □

**Corollary 2.20.** Let $T \in L(X)$. Then $T$ has property $(V_{E_a})$ if and only if $T$ has property $(V_{E})$.

*Proof.* Sufficiency: Suppose that $T$ satisfies property $(V_{E_a})$. By Theorem 2.19, $\sigma(T) = \sigma_a(T)$, it follows that $\sigma(T) \setminus \sigma_{SF^-}(T) = E_a(T) = E(T)$, hence $T$ satisfies property $(V_{E})$.

Necessity: Assume that $T$ satisfies property $(V_{E})$. By Theorem 2.3, $\sigma(T) = \sigma_a(T)$ and so, $\sigma(T) \setminus \sigma_{SF^-}(T) = E(T) = E_a(T)$. Therefore, $T$ satisfies property $(V_{E_a})$. □

The next result gives the relationship between property $(V_{E_a})$ (or equivalently $(V_{E})$) and property $(Z_{E_a})$.


Theorem 2.21. Let $T \in L(X)$. Then $T$ has property $(V_{E_a})$ if and only if $T$ has property $(Z_{E_a})$ and $\sigma_{SF^+}(T) = \sigma_W(T)$.

Proof. Sufficiency: Assume that $T$ satisfies property $(V_{E_a})$. By Corollary 2.20, property $(V_{E_a})$ implies that $\sigma_{SF^+}(T) = \sigma_W(T)$. Consequently, $\sigma(T) \setminus \sigma_W(T) = \sigma(T) \setminus \sigma_{SF^+}(T) = E_a(T)$. Therefore, $T$ satisfies property $(Z_{E_a})$.

Necessity: Assume that $T$ satisfies property $(Z_{E_a})$ and $\sigma_{SF^+}(T) = \sigma_W(T)$. Then, $\sigma(T) \setminus \sigma_{SF^+}(T) = \sigma(T) \setminus \sigma_W(T) = E_a(T)$, that means $T$ satisfies property $(V_{E_a})$. $\square$

Similar to Theorem 2.21, we have the following result.

Theorem 2.22. Let $T \in L(X)$. Then $T$ has property $(V_{E_a})$ if and only if $T$ has property $(gau)$ and $\sigma_{SF^+}(T) = \sigma_{BW}(T)$.

Recall that $T \in L(X)$ is said to satisfy $a$-Browder’s theorem (resp., generalized $a$-Browder’s theorem) if $\sigma_a(T) \setminus \sigma_{SF^+}(T) = \Pi_a^0(T)$ (resp., $\sigma_a(T) \setminus \sigma_{SBF^+}(T) = \Pi_a(T)$). From [7, Theorem 2.2] (see also [4, Theorem 3.2(ii)]), $a$-Browder’s theorem and generalized $a$-Browder’s theorem are equivalent. It is well known that $a$-Browder’s theorem for $T$ implies Browder’s theorem for $T$, i.e., $\sigma(T) \setminus \sigma_W(T) = \Pi(0)(T)$. Also by [7, Theorem 2.1], Browder’s theorem for $T$ is equivalent to generalized Browder’s theorem for $T$, i.e, $\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T)$.

For $T \in L(X)$, define $\Pi_+(T) = \sigma(T) \setminus \sigma_{ub}(T)$. The following theorem describes the relationship between $a$-Browder’s theorem and property $(V_E)$.

Theorem 2.23. For $T \in L(X)$, the following statements are equivalent:

(i) $T$ has property $(V_E)$.

(ii) $T$ satisfies $a$-Browder’s theorem and $\Pi_+(T) = E(T)$.

Proof. (i) $\Rightarrow$ (ii) Assume that $T$ satisfies property $(V_E)$. Then $E(T) = E^0(T)$ and $T$ satisfies property $(v)$ by Theorems 2.10 and 2.12, respectively. Property $(v)$ implies by [25, Theorem 2.17] that $T$ satisfies $a$-Browder’s theorem and $\Pi_+(T) = E^0(T)$. Consequently, $T$ satisfies $a$-Browder’s theorem and $\Pi_+(T) = E^0(T) = E(T)$.

(ii) $\Rightarrow$ (i) If $T$ satisfies $a$-Browder’s theorem and $\Pi_+(T) = E(T)$, then $\sigma(T) \setminus \sigma_{SF^+}(T) = \sigma(T) \setminus \sigma_{ub}(T) = \Pi_+(T) = E(T)$. Therefore, $T$ satisfies property $(V_E)$. $\square$

Remark 2.24. By Theorem 2.23, property $(V_E)$ implies $a$-Browder’s theorem. However, the converse is not true in general. Indeed, the operator $T$ defined in Example 2.15 does not satisfy property $(V_E)$, but $\sigma_a(T) = \sigma_{SF^+}(T) = \{0\}$ and $\Pi_+(T) = \emptyset$, it follows that $T$ satisfies $a$-Browder’s theorem.

Corollary 2.25. If $T \in L(X)$ has SVEP at each $\lambda \notin \sigma_{SF^+}(T)$, then $T$ has property $(V_E)$ if and only if $E(T) = \Pi_+(T)$. 
Proof. By Theorem [5, Teorema 2.3], the hypothesis T has SVEP at each \( \lambda \notin \sigma_{SF-}^-(T) \) is equivalent to T satisfies \( a \)-Browder’s theorem. Therefore, if \( E(T) = \Pi^0_\lambda(T) \), then \( \sigma(T) \setminus \sigma_{SF-}^-(T) = \sigma(T) \setminus \sigma_{ab}(T) = \Pi^0_\lambda(T) = E(T) \).

### Remark 2.26
It was proved in [12, Lemma 2.1], that if \( T^* \) has SVEP at every \( \lambda \notin \sigma_{SF-}^+(T) \) (resp., \( T \) has SVEP at every \( \lambda \notin \sigma_{SF-}^+(T) \)), then \( \sigma_W(T) = \sigma_{SF-}^+(T) \) and \( \sigma_a(T) = \sigma(T) \) (resp., \( \sigma_W(T^*) = \sigma_{SF-}^+(T^*) \) and \( \sigma_a(T^*) = \sigma(T^*) \)). Under the above results, clearly we have that if \( T^* \) has SVEP at every \( \lambda \notin \sigma_{SF-}^+(T) \) (resp., \( T \) has SVEP at every \( \lambda \notin \sigma_{SF-}^+(T) \)), then the properties \( (W_E), (UW_E), (UW_{E_a}), (Z_{E_b}), (V_E) \) and \( (V_{E_b}) \) are equivalent for \( T \) (resp. for \( T^* \)).

In the following table summarizes the meaning of various theorems and properties that are related with property \((V_E)\).
Theorem 2.27. Suppose that $T \in L(X)$ has property $(V_E)$. Then:

(i) $\sigma_{SBF}^+(T) = \sigma_{BW}^+(T) = \sigma_{SF}^-(T) = \sigma_{LD}(T) = \sigma_{D}(T) = \sigma_{ub}(T) = \sigma_b(T)$ and $\sigma(T) = \sigma_a(T)$.

(ii) All properties given in Table 1 are equivalent, and $T$ satisfies each of these properties.

Proof. (i) By Theorem 2.3, the equality $\sigma(T) = \sigma_a(T)$ holds. The equalities $\sigma_{SBF}^+(T) = \sigma_{BW}^+(T) = \sigma_{SF}^-(T) = \sigma_{W}(T)$ follows from Theorems 2.6, 2.8 and 2.12. Since the inclusions $\sigma_{SBF}^+(T) \subseteq \sigma_{LD}(T) \subseteq \sigma_{ub}(T) \subseteq \sigma_b(T)$ and $\sigma_{SBF}^+(T) \subseteq \sigma_{D}(T) \subseteq \sigma_{D}(T) \subseteq \sigma_b(T)$ hold, it is sufficient to prove $\sigma_{SBF}^+(T) = \sigma_b(T)$. Indeed, since $T$ has property $(V_E)$, by Theorem 2.23, $T$ satisfies generalized $a$-Browder’s theorem or equivalently $a$-Browder’s theorem. As $a$-Browder’s theorem implies Browder’s theorem, it follows that $\sigma_{SBF}^+(T) = \sigma_{W}(T) = \sigma_b(T)$.

(ii) By Theorem 2.3, $T$ satisfies property $(UW_E)$, and the equivalence between all properties follows from (i) and Theorem 2.10.

References

26. Sanabria J., Carpintero C., Rosas E. and García O., On property (Sw) and other spectral properties type Weyl-Browder theorems, Submitted.