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ON STRONG VARIATIONS OF WEYL TYPE THEOREMS

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Abstract. An operator T acting on a Banach space X satisfies the property (UW_E) if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E(T)$, where $\sigma_a(T)$ is the aproximate point spectrum of T, $\sigma_{SF_+^-}(T)$ is the upper semi-Weyl spectrum of T and $E(T)$ is the set of all eigenvalues of T that are isolated in the spectrum $\sigma(T)$ of T. In this paper, we introduce and study two new spectral properties, namely (V_E) and (V_{E_a}), in connection with Weyl type theorems. Among other results, we have that T satisfies property (V_E) if and only if T satisfies property (UW_E) and $\sigma(T) = \sigma_a(T)$.

1. Introduction and preliminaries

Throughout this paper, $L(X)$ denotes the algebra of all bounded linear operators acting on an infinite-dimensional complex Banach space X. We refer to [**25**] for details about notations and terminologies. However, we give the following notations that will be useful in the sequel:

- Browder spectrum: $\sigma_b(T)$
- Weyl spectrum: $\sigma_W(T)$
- Upper semi-Browder spectrum: $\sigma_{ub}(T)$
- Upper semi-Weyl spectrum: $\sigma_{SF_+^-}(T)$
- Drazin invertible spectrum: $\sigma_D(T)$
- B-Weyl spectrum: $\sigma_{BW}(T)$
- Left Drazin invertible spectrum: $\sigma_{LD}(T)$
- Upper semi-B-Weyl spectrum: $\sigma_{SBF_+^-}(T)$
- approximate point spectrum: $\sigma_a(T)$
- surjectivity spectrum: $\sigma_s(T)$

In this paper, we introduce two new spectral properties of type Weyl theorems, namely, the properties (V_E) and (V_{E_a}), respectively. In addition, we establish the precise relationships between these properties and other variants of Weyl's theorem recently introduced in [**8**], [**9**], [**24**], [**25**], [**26**], [**27**] and [**29**].

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Recall that an operator $T \in L(X)$ is said to have the *single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0) if for every open disc $\mathbb{D}_{\lambda_0} \subseteq \mathbb{C}$ centered at λ_0, the only analytic function $f : \mathbb{D}_{\lambda_0} \to X$ which satisfies the equation

$$(\lambda I - T)f(\lambda) = 0 \qquad \text{for all } \lambda \in \mathbb{D}_{\lambda_0},$$

is $f \equiv 0$ on $\mathbb{D}_{\lambda_0}$ (see [**17**]). The operator T is said to have SVEP if it has SVEP at every point $\lambda \in \mathbb{C}$. Evidently, every $T \in L(X)$ has SVEP at each point of the resolvent set $\rho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover, T has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T has SVEP at every isolated point of the spectrum. Note that (see [**1**, Theorem 3.8])

$$p(\lambda I - T) < \infty \quad \Longrightarrow \quad T \text{ has SVEP at} \lambda, \tag{1}$$

and dually,

$$q(\lambda I - T) < \infty \quad \Longrightarrow \quad T^* \text{ has SVEP at } \lambda. \tag{2}$$

It is easily seen from definition of localized SVEP that

$$\lambda \notin \operatorname{acc} \sigma_a(T) \quad \Longrightarrow \quad T \text{ has SVEP at } \lambda, \tag{3}$$

where $\operatorname{acc} K$ means the set of all accumulation points of $K \subseteq \mathbb{C}$, and

$$\lambda \notin \operatorname{acc} \sigma_s(T) \quad \Longrightarrow \quad T^* \text{ has SVEP at } \lambda. \tag{4}$$

Remark 1.1. If $\lambda I - T$ is a semi B-Fredholm operator, then the implications (1)–(4) are equivalences (see [**2**]).

Lemma 1.2. ([**3**, Lemma 2.4]) *Let $T \in L(X)$. Then*

(i) *T is upper semi B-Fredholm and $\alpha(T) < \infty$ if and only if $T \in \Phi_+(X)$.*
(ii) *T is lower semi B-Fredholm and $\beta(T) < \infty$ if and only if $T \in \Phi_-(X)$.*

Denote by $\operatorname{iso} K$, the set of all isolated points of $K \subseteq \mathbb{C}$. If $T \in L(X)$, define

$$\begin{aligned} E^0(T) &= \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(\lambda I - T) < \infty\}, \\ E^0_a(T) &= \{\lambda \in \operatorname{iso} \sigma_a(T) : 0 < \alpha(\lambda I - T) < \infty\}, \\ E(T) &= \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(\lambda I - T)\}, \\ E_a(T) &= \{\lambda \in \operatorname{iso} \sigma_a(T) : 0 < \alpha(\lambda I - T)\}. \end{aligned}$$

Also, define

$$\begin{aligned} \Pi^0(T) &= \sigma(T) \setminus \sigma_b(T), & \Pi^0_a(T) &= \sigma_a(T) \setminus \sigma_{ub}(T), \\ \Pi(T) &= \sigma(T) \setminus \sigma_D(T), & \Pi_a(T) &= \sigma_a(T) \setminus \sigma_{LD}(T). \end{aligned}$$

Let $T \in L(X)$. Following Coburn [**15**], T is said to satisfy *Weyl's theorem*, in symbols $(\mathcal{W})$, if $\sigma(T) \setminus \sigma_W(T) = E^0(T)$. Following Rakočević [**21**], T is said to satisfy *a-Weyl's theorem*, in symbols $(a\mathcal{W})$, if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E^0_a(T)$. According to Berkani and Koliha [**11**], T is said to satisfy *generalized Weyl's theorem*, in symbols $(g\mathcal{W})$, if $\sigma(T) \setminus \sigma_{BW}(T) = E(T)$. Similarly, T is said to satisfy *generalized a-Weyl's theorem*, in symbol $(ga\mathcal{W})$, if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$.

Now, we describe several spectral properties introduced recently in [**14**], [**24**], [**25**], [**26**] and [**27**].

Definition 1.3. An operator $T \in L(X)$ is said to have:

(i) *property* (gaw) [**14**] if $\sigma(T) \setminus \sigma_{BW}(T) = E_a(T)$.

(ii) *property* (z) [**27**] if $\sigma(T) \setminus \sigma_{SF_+^-}(T) = E_a^0(T)$.

(iii) *property* (gz) [**27**] if $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$.

(iv) *property* (v) [**25**] if $\sigma(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$.

(v) *property* (gv) [**25**] if $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$.

(vi) *property* (Sw) [**24**] if $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E^0(T)$.

(vii) *property* (Saw) [**26**] if $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E_a^0(T)$.

Property (gv) (resp., (v)) is also called property (gt) (resp., (t)) in [**22**], and property (gh) (resp., (h)) in [**28**]. It was proved in [**25**, Corollary 2.12], that property (gv) (resp., (v)) is equivalent to property (gz) (resp., (z)). Also, it was proved in [**26**, Corollary 2.9], that properties (Sw) and (Saw) are equivalent.

2. Properties (V_E) and (V_{E_a}).

According to [**8**], $T \in L(X)$ has *property* (W_E) (resp., *property* (UW_{E_a})) if $\sigma(T) \setminus \sigma_W(T) = E(T)$ (resp. $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_a(T)$). It was shown in [**8**, Theorem 2.3] (resp., [**8**, Theorem 3.5]) that property (W_E) (resp. (UW_{E_a})) implies generalized Weyl's theorem (resp., property (W_E)) but not conversely. Following to [**9**], an operator $T \in L(X)$ is said to have *property* (UW_E) if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E(T)$. It was shown in [**9**, Theorem 3.5] that property (UW_E) implies property (W_E) but not conversely. Also in [**9**], it is shown that properties (UW_{E_a}) and (UW_E) are independient. According to [**29**], $T \in L(X)$ has *property* (Z_{E_a}) if $\sigma(T) \setminus \sigma_W(T) = E_a(T)$. It was proved in [**29**, Corollary 2.5] that property (Z_{E_a}) also implies property (W_E). In this section, we introduce and study two equivalent spectral properties that are stronger than the properties (UW_{E_a}), (UW_E) and (Z_{E_a}).

Definition 2.1. An operator $T \in L(X)$ is said to have *property* (V_E) if $\sigma(T) \setminus \sigma_{SF_+^-}(T) = E(T)$.

Example 2.2. 1. Let L be the unilateral left shift operator on $\ell^2(\mathbb{N})$. It is well known that $\sigma(L) = \sigma_{SF_+^-}(L) = \mathbf{D}(0,1)$, the closed unit disc on $\mathbb{C}$ and $E(L) = \emptyset$. Therefore, $\sigma(L) \setminus \sigma_{SF_+^-}(L) = E(L)$, and so L satisfies property (V_E).

2. Consider the Volterra operator V on the Banach space $C[0,1]$ defined by $V(f)(x) = \int_0^x f(t)\mathrm{d}t$ for all $f \in C[0,1]$. Note that V is injective and quasinilpotent. Thus, $\sigma(V) = \{0\}$, $\alpha(V) = 0$ and hence $E(V) = \emptyset$. Since the range $R(V)$ is not closed, then $\sigma_{SF_+^-}(V) = \{0\}$. Therefore, $\sigma(V) \setminus \sigma_{SF_+^-}(V) = E(V)$, that means V has property (V_E).

Theorem 2.3. *For $T \in L(X)$, the following statements are equivalent:*

(i) *T has property (V_E),*

(ii) *T has property (UW_E) and $\sigma(T) = \sigma_a(T)$,*

(iii) *T has property (UW_{E_a}) and $\sigma(T) = \sigma_a(T)$.*

Proof. (i) $\Rightarrow$ (ii). Suppose that T satisfies property (V_E) and let $\lambda \in \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$. Since $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) \subseteq \sigma(T) \setminus \sigma_{SF_+^-}(T) = E(T)$, we have $\lambda \in E(T)$ and so, $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) \subseteq E(T)$.

To show the opposite inclusion $E(T) \subseteq \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$, let $\lambda \in E(T)$. Then, $\lambda \in \operatorname{iso}\sigma(\mathrm{T})$ and $\alpha(\lambda I - T) > 0$, so $\lambda I - T$ is not bounded below and hence, $\lambda \in \sigma_a(T)$. As T satisfies property (V_E) and $\lambda \in E(T)$, it follows that $\lambda I - T$ is upper semi-Weyl. Therefore, $\lambda \in \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$. Thus, $E(T) \subseteq \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$ and T satisfies property (UW_E). Consequently, $\sigma(T) \setminus \sigma_{SF_+^-}(T) = E(T)$ and $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E(T)$. Therefore, $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$ and $\sigma(T) = \sigma_a(T)$.

(ii) $\Rightarrow$ (i). Suppose that T satisfies property (UW_E) and $\sigma(T) = \sigma_a(T)$. Then, $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E(T)$. Thus, $\sigma(T) \setminus \sigma_{SF_+^-}(T) = E(T)$ and T satisfies property (V_E).

(ii) $\Leftrightarrow$ (iii). Obvious. □

The next example shows that, in general, property (UW_{E_a}) does not imply property (V_E).

Example 2.4. Let R be the unilateral right shift operator on $\ell^2(\mathbb{N})$ and $U \in L(\ell^2(\mathbb{N}))$ be defined by

$$U(x_1, x_2, x_3, \cdots) = (0, x_2, x_3, \cdots).$$

Define an operator T on $X = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $T = R \oplus U$. Then, $\sigma(T) = \mathbf{D}(0,1)$, the closed unit disc on $\mathbb{C}$, $\sigma_a(T) = \Gamma \cup \{0\}$, where Γ denotes the unit circle of $\mathbb{C}$ and $\sigma_{SF_+^-}(T) = \Gamma$. Moreover, $E_a(T) = \{0\}$ and $E(T) = \emptyset$. Therefore, $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_a(T)$ and $\sigma(T) \setminus \sigma_{SF_+^-}(T) \neq E(T)$. Thus, T satisfies properties (UW_{E_a}), but T does not satisfy property (V_E).

The next example shows that, in general, property (UW_E) does not imply property (V_E).

Example 2.5. Let R be the unilateral right shift operator on $\ell^2(\mathbb{N})$. Define an operator T on $X = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $T = R \oplus 0$. Then, $\sigma(T) = \mathbf{D}(0,1)$, $\sigma_a(T) = \sigma_{SF_+^-}(T) = \Gamma \cup \{0\}$ and $E(T) = \emptyset$. Therefore, $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E(T)$ and $\sigma(T) \setminus \sigma_{SF_+^-}(T) \neq E(T)$. Thus, T satisfies property (UW_E), but T does not satisfy property (V_E).

The next result gives the relationship between the properties (V_E) and (W_E).

Theorem 2.6. *Let $T \in L(X)$. Then T has property (V_E) if and only if T has property (W_E) and $\sigma_{SF_+^-}(T) = \sigma_W(T)$.*

Proof. Sufficiency: Suppose that T satisfies property (V_E), then by Theorem 2.3, T satisfies property (UW_E). Property (UW_E) implies by [**9**, Theorem 3.2] that

T satisfies property (W_E). Consequently, $\sigma(T) \setminus \sigma_{SF_+^-}(T) = E(T)$ and $\sigma(T) \setminus \sigma_W(T) = E(T)$. Therefore, $\sigma_{SF_+^-}(T) = \sigma_W(T)$.

Necessity: Suppose that T satisfies property (W_E) and $\sigma_{SF_+^-}(T) = \sigma_W(T)$. Then, $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \sigma(T) \setminus \sigma_W(T) = E(T)$, and so T satisfies property (V_E). □

The next example shows that, in general, property (W_E) does not imply property (V_E).

Example 2.7. Let Q be defined on $\ell^1(\mathbb{N})$ by

$$Q(x_1, x_2, x_3, \dots, x_k, \dots) = (0, \alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_{k-1} x_{k-1}, \dots),$$

where (α_i) is a sequence of complex numbers such that $0 < |\alpha_i| \leq 1$ and $\sum_{i=1}^{\infty} \alpha_i < \infty$. It follows from [**11**, Example 3.12], that

$$\overline{R(Q^n)} \neq R(Q^n), \qquad n = 1, 2, \dots.$$

Define the operator T on $X = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N}) \oplus \ell^1(\mathbb{N})$ by $T = R \oplus 0 \oplus Q$, where R is the unilateral right shift operator. Then, $\sigma(T) = \sigma_W(T) = \mathbf{D}(0,1)$, $\sigma_{SF_+^-}(T) = \Gamma \cup \{0\}$ and $E(T) = \emptyset$. We then have,

$$\sigma(T) \setminus \sigma_W(T) = E(T), \qquad \sigma(T) \setminus \sigma_{SF_+^-}(T) \neq E(T).$$

Hence, T satisfies property (W_E), but T does not satisfy property (V_E).

The next result gives the relationship between the property (V_E) and generalized Weyl's theorem.

Theorem 2.8. *Let $T \in L(X)$. Then T has property (V_E) if and only if T satisfies generalized Weyl's theorem and $\sigma_{SF_+^-}(T) = \sigma_{BW}(T)$.*

Proof. Sufficiency: Property (V_E) implies by Theorem 2.6, that T satisfies property (W_E), and property (W_E) implies by [**8**, Theorem 2.3], that T satisfies generalized Weyl's theorem. Consequently, $\sigma(T) \setminus \sigma_{SF_+^-}(T) = E(T)$ and $\sigma(T) \setminus \sigma_{BW}(T) = E(T)$. Therefore, $\sigma_{SF_+^-}(T) = \sigma_{BW}(T)$.

Necessity: Assume that T satisfies generalized Weyl's theorem and $\sigma_{SF_+^-}(T) = \sigma_{BW}(T)$. Then, $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \sigma(T) \setminus \sigma_{BW}(T) = E(T)$, that means T satisfies property (V_E). □

Remark 2.9. From Theorem 2.8, property (V_E) implies generalized Weyl's theorem. However, the converse is not true in general. Consider the operator T in Example 2.7, since T satisfies property (W_E), then it also satisfies generalized Weyl's theorem, but does not satisfy property (V_E).

Theorem 2.10. *Suppose that $T \in L(X)$ has property (V_E). Then:*

(i) *T has property (Z_{E_a}),*
(ii) *$E_a(T) = E_a^0(T) = \Pi_a^0(T) = \Pi_a(T) = \Pi^0(T) = \Pi(T) = E^0(T) = E(T)$.*

Proof. (i) Property (V_E) implies by Theorem 2.3, that $\sigma(T) = \sigma_a(T)$, and also implies by Theorem 2.6 that $\sigma_{SF_+^-}(T) = \sigma_W(T)$. Hence, $\sigma(T) \setminus \sigma_W(T) = \sigma(T) \setminus \sigma_{SF_+^-}(T) = E(T) = E_a(T)$ and so T satisfies property (Z_{E_a}).

(ii) Follows from (i) and [**29**, Lemma 2.3]. □

Example 2.11. Let R be the unilateral right shift operator defined on $\ell^2(\mathbb{N})$. Since $\sigma(R) = \sigma_W(R) = \mathbf{D}(0,1)$, $E(R) = E_a(R) = \emptyset$ and $\sigma_{SF_+^-}(R) = \Gamma$, then R satisfies property (Z_{E_a}), but does not satisfy property (V_E).

Theorem 2.12. *For $T \in L(X)$, the following statements are equivalent:*
(i) *T has property (V_E),*
(ii) *T has property (v) and $E^0(T) = E(T)$,*
(iii) *T has property (z) and $E^0(T) = E(T)$,*
(iv) *T has property (gv) and $\sigma_{SF_+^-}(T) = \sigma_{SBF_+^-}(T)$.*
(v) *T has property (gz) and $\sigma_{SF_+^-}(T) = \sigma_{SBF_+^-}(T)$.*

Proof. (i) ⇒ (ii). Suppose that T satisfies property (V_E). Then by Theorem 2.10, $E^0(T) = E(T)$, and hence $\sigma(T) \setminus \sigma_{SF_+^-}(T) = E(T) = E^0(T)$, that means T has property (v).

(ii) ⇒ (i). If T satisfies property (v) and $E^0(T) = E(T)$, then $\sigma(T) \setminus \sigma_{SF_+^-}(T) = E^0(T) = E(T)$ and T satisfies property (V_E).

(ii) ⇔ (iii). The equivalence between the properties (z) and (v) have been proved in [**25**, Corollary 2.12].

(i) ⇒ (iv). Assume that T satisfies property (V_E). By Theorem 2.3, T satisfies property (UW_{E_a}). Property (UW_{E_a}) implies by [**8**, Theorem 3.2] that T satisfies generalized a-Weyl's theorem and $\sigma_{SF_+^-}(T) = \sigma_{SBF_+^-}(T)$. Consequently, $E(T) = \sigma(T) \setminus \sigma_{SF_+^-}(T) = \sigma(T) \setminus \sigma_{SBF_+^-}(T)$, and hence T satisfies property (gv).

(iv) ⇒ (i). Suppose that T satisfies property (gv) and $\sigma_{SF_+^-}(T) = \sigma_{SBF_+^-}(T)$. Then $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \sigma(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$, and hence T satisfies property (V_E).

(iv) ⇔ (v). The equivalence between the properties (gz) and (gv) have been proved in [**25**, Corollary 2.12]. □

The following example shows that, in general, property (gv) (resp. (v)) does not imply property (V_E).

Example 2.13. Consider the operator $T = 0$ defined on the Hilbert space $\ell^2(\mathbb{N})$. Then, $\sigma(T) = \sigma_{SF_+^-}(T) = \{0\}$, $\sigma_{SBF_+^-}(T) = \emptyset$ and $E(T) = \{0\}$. Therefore, $\sigma(T) \setminus \sigma_{SF_+^-}(T) \neq E(T)$ and T does not satisfy property (V_E). On the other hand, $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$, that means T satisfies property (gv), in consequence T also satisfies property (v).

The next result gives the relationship between the properties (V_E) and (Sw).

Theorem 2.14. *For $T \in L(X)$, the following statements are equivalent:*

(i) *T has property (V_E),*
(ii) *T has property (Sw) and $E^0(T) = E(T)$,*
(iii) *T has property (Saw) and $E^0(T) = E(T)$.*

Proof. (i) $\Rightarrow$ (ii). Assume that T satisfies property (V_E). Then by Theorem 2.10, $E(T) = E^0(T)$, and by Theorem 2.12, $\sigma_{SF_+^-}(T) = \sigma_{SBF_+^-}(T)$. Therefore $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \sigma(T) \setminus \sigma_{SF_+^-}(T) = E(T) = E^0(T)$, that means T satisfies property (Sw).

(ii) $\Rightarrow$ (i). If T satisfies property (Sw) and $E(T) = E^0(T)$, then $\sigma(T) \setminus \sigma_{SF_+^-}(T) \subseteq \sigma(T) \setminus \sigma_{SBF_+^-}(T) = E^0(T) = E(T)$. This shows $\sigma(T) \setminus \sigma_{SF_+^-}(T) \subseteq E(T)$.

To show the opposite inclusion $E(T) \subseteq \sigma(T) \setminus \sigma_{SF_+^-}(T)$, let $\lambda \in E(T)$. Since $E(T) = E^0(T)$, then $\lambda \in E^0(T) = \sigma(T) \setminus \sigma_{SBF_+^-}(T)$. Thus $\lambda I - T$ is an upper semi B-Fredholm operator and $\alpha(\lambda I - T) < \infty$. By Lemma 1.2, $\lambda I - T$ is upper semi-Fredholm, and hence upper semi-Weyl. Therefore, $\lambda \in \sigma(T) \setminus \sigma_{SF_+^-}(T)$ and consequently $\sigma(T) \setminus \sigma_{SF_+^-}(T) = E(T)$.

(ii) $\Leftrightarrow$ (iii). The equivalence between the properties (z) and (v) have been proved in [**26**, Corollary 2.9]. □

The following example shows that, in general, property (Sw) does not imply property (V_E).

Example 2.15. Consider the operator Q defined in Example 2.7 and define an operator T on $X = \ell^1(\mathbb{N}) \oplus \ell^1(\mathbb{N})$ by $T = Q \oplus 0$. Then, $N(T) = \{0\} \oplus \ell^1(\mathbb{N})$, $\sigma(T) = \{0\}$, $E(T) = \{0\}$, $E^0(T) = \emptyset$. Since $R(T^n) = R(Q^n) \oplus \{0\}$, $R(T^n)$ is not closed for any $n \in \mathbb{N}$; in consequence T is not an upper semi B-Weyl (resp. upper semi-Weyl) operator and $\sigma_{SBF_+^-}(T) = \{0\}$ (resp. $\sigma_{SF_+^-}(T) = \{0\}$). Then, we have

$$\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E^0(T), \qquad \sigma(T) \setminus \sigma_{SF_+^-}(T) \neq E(T).$$

Hence, T satisfies property (Sw), but T does not satisfy property (V_E).

The next result gives the relationship between property (V_E) and Weyl's theorem.

Theorem 2.16. *Let $T \in L(X)$. Then T has property (V_E) if and only if T satisfies Weyl's theorem and $\sigma_W(T) \setminus \sigma_{SF_+^-}(T) = E(T) \setminus E^0(T)$.*

Proof. Sufficiency: Suppose that T satisfies property (V_E). It follows by Theorem 2.8, T satisfies generalized Weyl's theorem. Since generalized Weyl's theorem implies Weyl's theorem, it is enough to show that $\sigma_W(T) \setminus \sigma_{SF_+^-}(T) = E(T) \setminus E^0(T)$. By Theorems 2.6 and 2.10, $\sigma_{SF_+^-}(T) = \sigma_W(T)$ and $E(T) = E^0(T)$, respectively. Thus, we conclude that $\sigma_W(T) \setminus \sigma_{SF_+^-}(T) = \emptyset = E(T) \setminus E^0(T)$.

Necessity: Assume that T satisfies Weyl's theorem and $\sigma_W(T) \setminus \sigma_{SF_+^-}(T) = E(T) \setminus E^0(T)$. Since $\sigma(T) \setminus \sigma_W(T) = E^0(T)$, $\sigma(T) = E^0(T) \cup \sigma_W(T)$ and $E^0(T) \cap \sigma_W(T) = \emptyset$. Thus,

$$\begin{aligned}\sigma(T) \setminus \sigma_{SF_+^-}(T) &= [E^0(T) \cup \sigma_W(T)] \setminus \sigma_{SF_+^-}(T) \\ &= E^0(T) \cup [\sigma_W(T) \setminus \sigma_{SF_+^-}(T)] \\ &= E^0(T) \cup [E(T) \setminus E^0(T)] = E(T)\end{aligned}$$

and hence T satisfies property (V_E). $\square$

Remark 2.17. By Theorem 2.16, property (V_E) implies Weyl's theorem. However, the converse is not true in general. Consider the operator T in Remark 2.9, since T satisfies generalized Weyl's theorem, then it also satisfies Weyl's theorem, but does not satisfy property (V_E).

Definition 2.18. An operator $T \in L(X)$ is said to have *property* (V_{E_a}) if $\sigma(T) \setminus \sigma_{SF_+^-}(T) = E_a(T)$.

Theorem 2.19. *Let $T \in L(X)$. Then T has property (V_{E_a}) if and only if T has property (UW_{E_a}) and $\sigma(T) = \sigma_a(T)$.*

Proof. Sufficiency: Assume that T satisfies property (V_{E_a}). Then

$$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) \subseteq \sigma(T) \setminus \sigma_{SF_+^-}(T) = E_a(T) \text{ and so } \sigma_a(T) \setminus \sigma_{SF_+^-}(T) \subseteq E_a(T).$$

To show the opposite inclusion $E_a(T) \subseteq \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$, let $\lambda \in E_a(T)$. Then, $\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(\mathrm{T})$ and hence $\lambda \in \sigma_a(T)$. As T satisfies property (V_{E_a}) and $\lambda \in E_a(T)$, it follows that $\lambda I - T$ is upper semi-Weyl. Therefore, $\lambda \in \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$. Thus, $E_a(T) \subseteq \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$ and T satisfies property (UW_{E_a}). Consequently, $\sigma(T) \setminus \sigma_{SF_+^-}(T) = E_a(T)$ and $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_a(T)$. Therefore, $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$ and $\sigma(T) = \sigma_a(T)$.

Necessity: Suppose that T satisfies property (UW_{E_a}) and $\sigma(T) = \sigma_a(T)$. Then, $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_a(T)$, in consequence T satisfies property (V_{E_a}). $\square$

Corollary 2.20. *Let $T \in L(X)$. Then T has property (V_{E_a}) if and only if T has property (V_E).*

Proof. Sufficiency: Suppose that T satisfies property (V_{E_a}). By Theorem 2.19, $\sigma(T) = \sigma_a(T)$, it follows that $\sigma(T) \setminus \sigma_{SF_+^-}(T) = E_a(T) = E(T)$, hence T satisfies property (V_E).

Necessity: Assume that T satisfies property (V_E). By Theorem 2.3, $\sigma(T) = \sigma_a(T)$ and so, $\sigma(T) \setminus \sigma_{SF_+^-}(T) = E(T) = E_a(T)$. Therefore, T satisfies property (V_{E_a}). $\square$

The next result gives the relationship between property (V_{E_a}) (or equivalently (V_E)) and property (Z_{E_a}).

Theorem 2.21. *Let $T \in L(X)$. Then T has property (V_{E_a}) if and only if T has property (Z_{E_a}) and $\sigma_{SF_+^-}(T) = \sigma_W(T)$.*

Proof. Sufficiency: Assume that T satisfies property (V_{E_a}). By Corollary 2.20, property (V_{E_a}) is equivalent to property (V_E), and by Theorem 2.6, property (V_E) implies that $\sigma_{SF_+^-}(T) = \sigma_W(T)$. Consequently, $\sigma(T) \setminus \sigma_W(T) = \sigma(T) \setminus \sigma_{SF_+^-}(T) = E_a(T)$. Therefore, T satisfies property (Z_{E_a}).

Necessity: Assume that T satisfies property (Z_{E_a}) and $\sigma_{SF_+^-}(T) = \sigma_W(T)$. Then, $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \sigma(T) \setminus \sigma_W(T) = E_a(T)$, that means T satisfies property (V_{E_a}). □

Similar to Theorem 2.21, we have the following result.

Theorem 2.22. *Let $T \in L(X)$. Then T has property (V_{E_a}) if and only if T has property (gaw) and $\sigma_{SF_+^-}(T) = \sigma_{BW}(T)$.*

Recall that $T \in L(X)$ is said to satisfy *a-Browder's theorem* (resp., *generalized a-Browder's theorem*) if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi_a^0(T)$ (resp., $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a(T)$). From [**7**, Theorem 2.2] (see also [**4**, Theorem 3.2(ii)]), a-Browder's theorem and generalized a-Browder's theorem are equivalent. It is well known that a-Browder's theorem for T implies Browder's theorem for T, i.e., $\sigma(T) \setminus \sigma_W(T) = \Pi^0(T)$. Also by [**7**, Theorem 2.1], Browder's theorem for T is equivalent to generalized Browder's theorem for T, i.e. $\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T)$.

For $T \in L(X)$, define $\Pi_+^0(T) = \sigma(T) \setminus \sigma_{ub}(T)$. The following theorem describes the relationship between a-Browder's theorem and property (V_E).

Theorem 2.23. *For $T \in L(X)$, the following statements are equivalent:*

(i) *T has property (V_E),*

(ii) *T satisfies a-Browder's theorem and $\Pi_+^0(T) = E(T)$.*

Proof. (i) ⇒ (ii) Assume that T satisfies property (V_E). Then $E(T) = E^0(T)$ and T satisfies property (v) by Theorems 2.10 and 2.12, respectively. Property (v) implies by [**25**, Theorem 2.17] that T satisfies a-Browder's theorem and $\Pi_+^0(T) = E^0(T)$. Consequently, T satisfies a-Browder's theorem and $\Pi_+^0(T) = E^0(T) = E(T)$.

(ii) ⇒ (i) If T satisfies a-Browder's theorem and $\Pi_+^0(T) = E(T)$, then $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \sigma(T) \setminus \sigma_{ub}(T) = \Pi_+^0(T) = E(T)$. Therefore, T satisfies property (V_E). □

Remark 2.24. By Theorem 2.23, property (V_E) implies a-Browder's theorem. However, the converse is not true in general. Indeed, the operator T defined in Example 2.15 does not satisfy property (V_E), but $\sigma_a(T) = \sigma_{SF_+^-}(T) = \{0\}$ and $\Pi_a^0(T) = \emptyset$, it follows that T satisfies a-Browder's theorem.

Corollary 2.25. *If $T \in L(X)$ has SVEP at each $\lambda \notin \sigma_{SF_+^-}(T)$, then T has property (V_E) if and only if $E(T) = \Pi_+^0(T)$.*

Proof. By Theorem [**5**, Teorema 2.3], the hypothesis T has SVEP at each $\lambda \notin \sigma_{SF_+^-}(T)$ is equivalent to T satisfies a-Browder's theorem. Therefore, if $E(T) = \Pi_+^0(T)$, then $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \sigma(T) \setminus \sigma_{ub}(T) = \Pi_+^0(T) = E(T)$. ☐

Remark 2.26. It was proved in [**12**, Lemma 2.1], that if T^* has SVEP at every $\lambda \notin \sigma_{SF_+^-}(T)$ (resp., T has SVEP at every $\lambda \notin \sigma_{SF_-^+}(T)$), then $\sigma_W(T) = \sigma_{SF_+^-}(T)$ and $\sigma_a(T) = \sigma(T)$ (resp., $\sigma_W(T^*) = \sigma_{SF_+^-}(T^*)$ and $\sigma_a(T^*) = \sigma(T^*)$). Under the above results, clearly we have that if T^* has SVEP at every $\lambda \notin \sigma_{SF_+^-}(T)$ (resp. T has SVEP at every $\lambda \notin \sigma_{SF_-^+}(T)$), then the properties (W_E), (UW_E), (UW_{E_a}), (Z_{E_a}), (V_E) and (V_{E_a}) are equivalent for T (resp. for T^*).

In the following table summarizes the meaning of various theorems and properties that are related with property (V_E).

(W_E) [**8**]	$\sigma(T) \setminus \sigma_W(T) = E(T)$	(W_Π) [**9**]	$\sigma(T) \setminus \sigma_W(T) = \Pi(T)$
$\mathcal{W}$ [**15**]	$\sigma(T) \setminus \sigma_W(T) = E^0(T)$	$\mathcal{B}$ [**19**]	$\sigma(T) \setminus \sigma_W(T) = \Pi^0(T)$
(Z_{E_a}) [**29**]	$\sigma(T) \setminus \sigma_W(T) = E_a(T)$	(Z_{Π_a}) [**29**]	$\sigma(T) \setminus \sigma_W(T) = \Pi_a(T)$
(aw) [**14**]	$\sigma(T) \setminus \sigma_W(T) = E_a^0(T)$	(ab) [**14**]	$\sigma(T) \setminus \sigma_W(T) = \Pi_a^0(T)$
$g\mathcal{W}$ [**11**]	$\sigma(T) \setminus \sigma_{BW}(T) = E(T)$	$g\mathcal{B}$ [**11**]	$\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T)$
(Bw) [**18**]	$\sigma(T) \setminus \sigma_{BW}(T) = E^0(T)$	(Bb) [**23**]	$\sigma(T) \setminus \sigma_{BW}(T) = \Pi^0(T)$
(gaw) [**14**]	$\sigma(T) \setminus \sigma_{BW}(T) = E_a(T)$	(gab) [**14**]	$\sigma(T) \setminus \sigma_{BW}(T) = \Pi_a(T)$
(Baw) [**30**]	$\sigma(T) \setminus \sigma_{BW}(T) = E_a^0(T)$	(Bab) [**30**]	$\sigma(T) \setminus \sigma_{BW}(T) = \Pi_a^0(T)$
(v) [**25**]	$\sigma(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$	(ah) [**28**]	$\sigma(T) \setminus \sigma_{SF_+^-}(T) = \Pi^0(T)$
(z) [**27**]	$\sigma(T) \setminus \sigma_{SF_+^-}(T) = E_a^0(T)$	(az) [**27**]	$\sigma(T) \setminus \sigma_{SF_+^-}(T) = \Pi_a^0(T)$
(gv) [**25**]	$\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$	(gah) [**28**]	$\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \Pi(T)$
(Sw) [**24**]	$\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E^0(T)$	(Sb) [**24**]	$\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \Pi^0(T)$
(gz) [**27**]	$\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$	(gaz) [**27**]	$\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a(T)$
(Saw) [**26**]	$\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E_a^0(T)$	(Sab) [**26**]	$\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a^0(T)$
(UW_E) [**9**]	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E(T)$	(UW_Π) [**9**]	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi(T)$
(w) [**20**]	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$	(b) [**13**]	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi^0(T)$
(UW_{E_a}) [**8**]	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_a(T)$	(UW_{Π_a}) [**9**]	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi_a(T)$
$a\mathcal{W}$ [**21**]	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_a^0(T)$	$a\mathcal{B}$ [**16**]	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi_a^0(T)$
(gw) [**6**]	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$	(gb) [**13**]	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi(T)$
(Bgw) [**23**]	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^0(T)$	(Bgb) [**23**]	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi^0(T)$
$ga\mathcal{W}$ [**11**]	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$	$ga\mathcal{B}$ [**11**]	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a(T)$
$(SBaw)$ [**10**]	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a^0(T)$	$(SBab)$ [**10**]	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a^0(T)$

Table 1.

Theorem 2.27. *Suppose that $T \in L(X)$ has property (V_E). Then:*

(i) $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T) = \sigma_{SF_+^-}(T) = \sigma_W(T) = \sigma_{LD}(T) = \sigma_D(T) = \sigma_{ub}(T) = \sigma_b(T)$ *and* $\sigma(T) = \sigma_a(T)$.

(ii) *All properties given in Table 1 are equivalent, and T satisfies each of these properties.*

Proof. (i) By Theorem 2.3, the equality $\sigma(T) = \sigma_a(T)$ holds. The equalities $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T) = \sigma_{SF_+^-}(T) = \sigma_W(T)$ follows from Theorems 2.6, 2.8 and 2.12. Since the inclusions $\sigma_{SBF_+^-}(T) \subseteq \sigma_{LD}(T) \subseteq \sigma_{ub}(T) \subseteq \sigma_b(T)$ and $\sigma_{SBF_+^-}(T) \subseteq \sigma_{LD}(T) \subseteq \sigma_D(T) \subseteq \sigma_b(T)$ hold, it is sufficient to prove $\sigma_{SBF_+^-}(T) = \sigma_b(T)$. Indeed, since T has property (V_E), by Theorem 2.23, T satisfies generalized a-Browder's theorem or equivalently a-Browder's theorem. As a-Browder's theorem implies Browder's theorem, it follows that $\sigma_{SBF_+^-}(T) = \sigma_W(T) = \sigma_b(T)$,

(ii) By Theorem 2.3, T satisfies property (UW_E), and the equivalence between all properties follows from (i) and Theorem 2.10. □

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