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STRONGLY n-GORENSTEIN PROJECTIVE, INJECTIVE, AND FLAT MODULES

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Abstract. This paper generalizes the idea of the authors in [**3**]. Namely, we define and study a particular case of modules with Gorenstein projective, injective, and flat, respectively, dimension less or equal $n \geq 0$, which we call strongly n-Gorenstein projective, injective and flat modules. These three classes of modules give us a new characterization of the first modules, and they are a generalization of the notions of strongly Gorenstein projective, injective, and flat modules, respectively.

1. Introduction

Throughout the paper, all rings are commutative with identity and all modules are unitary.
Let R be a ring, and let M be an R-module. As usual, we use $\mathrm{pd}_R(M)$, $\mathrm{id}_R(M)$, and $\mathrm{fd}_R(M)$, respectively, to denote the classical projective dimension, injective dimension, and flat dimension of M.

For a two-sided Noetherian ring R, Auslander and Bridger [**1**] introduced the G-dimension, $\mathrm{Gdim}_R(M)$, for every finitely generated R-module M. They showed that $\mathrm{Gdim}_R(M) \leq \mathrm{pd}_R(M)$ for all finitely generated R-modules M, and equality holds if $\mathrm{pd}_R(M)$ is finite.

Several decades later, Enochs and Jenda [**11, 12**] introduced the notion of Gorenstein projective dimension (G-projective dimension for short) as an extension of G-dimension to modules that are not necessarily finitely generated, and the Gorenstein injective dimension (G-injective dimension for short) as a dual notion of Gorenstein projective dimension. Then, to complete the analogy with the classical homological dimension, Enochs, Jenda, and Torrecillas [**13**] introduced the Gorenstein flat dimension. Some references are [**4, 8, 9, 11, 12, 13, 15**].

Recall that an R-module M is called Gorenstein projective if there exists an exact sequence of projective R-modules

$$\mathbf{P} : \ldots \to P_1 \to P_0 \to P^0 \to P^1 \to \ldots$$

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such that $M \cong \mathrm{Im}(P_0 \to P^0)$ and the functor $\mathrm{Hom}_R(-,Q)$ leaves $\mathbf{P}$ exact whenever Q is a projective R-module. The complex $\mathbf{P}$ is called a complete projective resolution.

The Gorenstein injective R-modules are defined dually.

An R-module M is called Gorenstein flat if there exists an exact sequence of flat R-modules

$$\mathbf{F} : \ldots \to F_1 \to F_0 \to F^0 \to F^1 \to \ldots$$

such that $M \cong \mathrm{Im}(F_0 \to F^0)$ and the functor $I \otimes_R -$ leaves $\mathbf{F}$ exact whenever I is an injective R-module. The complex $\mathbf{F}$ is called a complete flat resolution.

The Gorenstein projective, injective, and flat dimensions are defined in terms of resolutions and denoted by $\mathrm{Gpd}(-)$, $\mathrm{Gid}(-)$, and $\mathrm{Gfd}(-)$, respectively ([**8, 15**]).

In [**4**], the authors proved the equality for any associative ring R.

$$\begin{aligned} &\sup\{\mathrm{Gpd}_R(M) \mid \text{M is a (left) R-module}\} \\ &= \sup\{\mathrm{Gid}_R(M) \mid \text{M is a (left) R-module}\}. \end{aligned}$$

They called the common value of the above quantities the left Gorenstein global dimension of R and denoted it by $l.\mathrm{Ggldim}(R)$. Similarly, they set

$$l \cdot \mathrm{wGgldim}(R) = \sup\{\mathrm{Gfd}_R(M) \mid \text{M is a (left) R-module}\}$$

which called the left Gorenstein weak dimension of R. Since in this paper, all rings are commutative, we drop the letter l.

In [**3**], the authors introduced a particular case of Gorenstein projective, injective, and flat modules, respectively, which are defined as follows:

Definition 1.1.

(1) A module M is said to be strongly Gorenstein projective (SG-projective for short) if there exists an exact sequence of projective modules of the form

$$\mathbf{P} = \ \ldots \to P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \to \ldots$$

such that $M \cong \mathrm{Im}(f)$ and $\mathrm{Hom}(-,Q)$ leaves $\mathbf{P}$ exact whenever Q is a projective module.

The exact sequence $\mathbf{P}$ is called a strongly complete projective resolution and denoted by (P,f).

(2) The strongly Gorenstein injective module is defined dually.

(3) A module M is said to be strongly Gorenstein flat (SG-flat for short) if there exists an exact sequence of flat modules of the form

$$\mathbf{F} = \ \cdots \to F \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} F \to \ldots$$

such that $M \cong \mathrm{Im}(f)$ and $I\otimes$-leaves $\mathbf{F}$ exact whenever I is an injective module. The exact sequence $\mathbf{F}$ is called a strongly complete flat resolution and denoted by (F,f).

The principal role of the strongly Gorenstein projective and injective modules is to give a simple characterization of Gorenstein projective and injective modules, respectively, as follows

Theorem 1.2. ([**3**, Theorem 2.7]) *A module is Gorenstein projective (resp., injective) if and only if it is a direct summand of a strongly Gorenstein projective (resp., injective) module.*

Using [**3**, Theorem 3.5] together with [**15**, Theorem 3.7], we have the next result.

Proposition 1.3. *Let R be a coherent ring. A module is Gorenstein flat if and only if it is a direct summand of a strongly Gorenstein flat module.*

This result allows us to show that the strongly Gorenstein projective, injective and flat modules have simplier characterizations than their Gorenstein correspondent modules.

Theorem 1.4. ([**3**, Propositions 2.9 and 3.6])

(1) *A module M is strongly Gorenstein projective if and only if there exists a short exact sequence of modules: $0 \to M \to P \to M \to 0$ where P is projective and $Ext(M, Q) = 0$ for any projective module Q.*

(2) *A module M is strongly Gorenstein injective if, and only if, there exists a short exact sequence of modules: $0 \to M \to I \to M \to 0$, where I is injective and $\mathrm{Ext}(E, M) = 0$ for any injective module E.*

(3) *A module M is strongly Gorenstein flat if and only if there exists a short exact sequence of modules: $0 \to M \to F \to M \to 0$, where F is flat and $\mathrm{Tor}(M, I) = 0$ for any injective module I.*

Along this paper, we need the following Lemmas:

Lemma 1.5. *Let $0 \to N \to N' \to N'' \to 0$ be an exact sequence of R-modules. Then:*

(1) $\mathrm{Gpd}_R(N) \leq \max\{\mathrm{Gpd}_R(N'), \mathrm{Gpd}_R(N'') - 1\}$
with equality if $\mathrm{Gpd}_R(N') \neq \mathrm{Gpd}_R(N'')$.

(2) $\mathrm{Gpd}_R(N') \leq \max\{\mathrm{Gpd}_R(N), \mathrm{Gpd}_R(N")\}$
with equality if $\mathrm{Gpd}_R(N'') \neq \mathrm{Gpd}_R(N) + 1$.

(3) $\mathrm{Gpd}_R(N'') \leq \max\{\mathrm{Gpd}_R(N'), \mathrm{Gpd}_R(N) + 1\}$
with equality if $\mathrm{Gpd}_R(N') \neq \mathrm{Gpd}_R(N)$.

Proof. Using [**15**, Theorems 2.20 and 2.24], the argument is analogous to the one of [**7**, Corollary 2, p. 135]. □

Dually we have:

Lemma 1.6. *Let $0 \to N \to N' \to N'' \to 0$ be an exact sequence of R-modules. Then:*

(1) $\mathrm{Gid}_R(N) \leq \max\{\mathrm{Gid}_R(N'), \mathrm{Gid}_R(N'') + 1\}$
with equality if $\mathrm{Gid}_R(N') \neq \mathrm{Gid}_R(N'')$.

(2) $\mathrm{Gid}_R(N') \leq \max\{\mathrm{Gid}_R(N), \mathrm{Gid}_R(N'')\}$
with equality if $\mathrm{Gid}_R(N'') + 1 \neq \mathrm{Gid}_R(N)$.

(3) $\mathrm{Gid}_R(N'') \leq \max\{\mathrm{Gid}_R(N'), \mathrm{Gid}_R(N) - 1\}$
with equality if $\mathrm{Gid}_R(N') \neq \mathrm{Gid}_R(N)$.

And using [**15**, Proposition 3.11] and Lemma 1.6, we get the following lemma.

Lemma 1.7. *Let* $0 \to N \to N' \to N'' \to 0$ *be an exact sequence of modules over a coherent ring* R. *Then:*

(1) $\mathrm{Gfd}_R(N) \leq \max\{\mathrm{Gfd}_R(N'), \mathrm{Gfd}_R(N'') - 1\}$
with equality if $\mathrm{Gfd}_R(N') \neq \mathrm{Gfd}_R(N'')$.

(2) $\mathrm{Gfd}_R(N') \leq \max\{\mathrm{Gfd}_R(N), \mathrm{Gfd}_R(N'')\}$
with equality if $\mathrm{Gfd}_R(N'') \neq \mathrm{Gfd}_R(N) + 1$.

(3) $\mathrm{Gfd}_R(N'') \leq \max\{\mathrm{Gfd}_R(N'), \mathrm{Gfd}_R(N) + 1\}$
with equality if $\mathrm{Gfd}_R(N') \neq \mathrm{Gfd}_R(N)$.

In [**15**], Holm gave a characterization of modules with finite Gorenstein projective, injective and flat dimensions ([**15**, Theorems 2.20, 2.22 and 3.14]). In this three characterizations, Holm imposed the finitness of theses dimensions. Almost by definition, one has the inclusion

$$\{M|\ \mathrm{pd}(M) \leq n\} \subseteq \{M|\ \mathrm{Gpd}(M) \leq n\}.$$

The main idea of this paper is to introduce and study an intermediate class of modules called strongly n-Gorenstein projective modules. Similarly, we define the strongly n-Gorenstein injective and flat modules.

The simplicity of these modules manifests in the fact that they have simplier characterizations than their corresponding Gorenstein modules. Moreover, with such modules, we are able to give nice new characterizations of modules with Gorenstein projective, injective, and flat dimensions equal to n.

2. Strongly n-Gorenstein projective and injective modules

In this section, we introduce and study strongly n-Gorenstein projective and injective modules which are defined as follows:

Definition 2.1. Let n be a positive integer.

(1) An R-module M is said to be strongly n-Gorenstein projective if there exists a short exact sequence

$$0 \to M \to P \to M \to 0,$$

where $\mathrm{pd}(P) \leq n$ and $\mathrm{Ext}^{n+1}(M, Q) = 0$ whenever Q is projective.

(2) An R-module M is said to be strongly n-Gorenstein injective if there exists a short exact sequence

$$0 \to M \to I \to M \to 0,$$

where $\mathrm{id}(I) \leq n$ and $\mathrm{Ext}^{n+1}(E, M) = 0$ whenever E is injective.

A direct consequence of the above definition is such that, the strongly 0-Gorenstein projective modules are just the strongly Gorenstein projective modules (by [**3**, Proposition 2.9]). Also every module with finite projective dimension less than or equal to n is a strongly n-Gorenstein projective module.

In [**6**], the authors introduced n-Strongly Gorenstein projective modules as follows: Let $n \geq$ be a positive integer. An R-module M is called n-strongly Gorenstein projective if there exists an exact sequence

$$0 \to M \to P_{n-1} \to \cdots \to P_0 \to M \to 0,$$

where each P_i is projective such that $\mathrm{Hom}_R(\cdot, Q)$ leaves the sequence exact whenever Q is a projective R-module. It is clear that every n-strongly Gorenstein projective module is Gorenstein projective ([**6**, Proposition 2.5]). The class of strongly 0-Gorenstein projective modules and the class of 1-strongly Gorenstein projective module, (in the sens of [**6**]) coincide with the class of strongly Gorenstein projective modules. However, in general case, the notion of strongly n-Gorenstein projective modules and that of m-strongly Gorenstein projective modules are different.

Example 2.2.

(1) Let $n \geq 1$ be an integer and let R be a ring with $\mathrm{gldim}(R) = n$. There exists an R-module M such that $\mathrm{pd}_R(M) = n$. Then, M is a strongly n-Gorenstein projective module which is not m-strongly Gorenstein projective for any positive integer m.

(2) Consider the local ring $R := k[[X, Y]]/(XY)$, where k is a field. Set $\overline{X}$ the residue class of X in R. Then, the ideal $(\overline{X})$ is a 2-strongly Gorenstein projective R-module which is not a strongly n-Gorenstein projective module for any positive integer n.

Proof. (1) We have the exact sequence

$$0 \to M \to M \oplus M \to M \to 0$$

with $\mathrm{pd}_R(M \oplus M) = n$ and $\mathrm{Ext}_R^{n+1}(M, Q) = 0$ for each module (in particular, projective module) Q. Hence, M is a strongly n-projective module. However, $\mathrm{Gpd}_R(M) = \mathrm{pd}_R(M) = n$. Then, M cannot be an m-strongly Gorenstein projective module for some positive integer m since every m-strongly Gorenstein projective module is Gorenstein projective ([**6**, Proposition 2.5]).

(2) The ideal $(\overline{X})$ is a 2-strongly Gorenstein projective R-module which is not strongly Gorenstein projective (by [**6**, Example 2.6]). If $(\overline{X})$ is a strongly n-Gorenstein projective module for some positive integer n, then there exists an exact sequence

$$0 \to (\overline{X}) \to P \to (\overline{X}) \to 0$$

with $\mathrm{pd}_R(P) \leq n$. The module $(\overline{X})$ is Gorenstein projective (since it is a 2-strongly Gorenstein projective). Then, P is a Gorenstein projective module with finite projective dimension (by [**15**, Theorem 2.5]), and so a projective module (by [**15**, Proposition 2.27]). Thus, $(\overline{X})$ is a strongly Gorenstein projective module, which is impossible. □

The main difference between the notion of strongly n-Gorenstein projective modules and that of m-strongly Gorenstein projective module is that all n-strongly Gorenstein projective module are Gorenstein projective but strongly n-Gorenstein projective modules can have Gorenstein projective dimension > 0.

Proposition 2.3. *Let n be a positive integer and M be a strongly n-Gorenstein projective module. Then, the following holds:*

(1) $0 \to N \to P_n \to \cdots \to P_1 \to M \to 0$ *is an exact sequence, where all P_i are projective, then N is a strongly Gorenstein projective module and consequently,* $\mathrm{Gpd}(M) \leq n$.

(2) *Moreover, if* $0 \to M \to P \to M \to 0$ *is a short exact sequence, where* $\mathrm{pd}(P) < \infty$*, then* $\mathrm{Gpd}(M) = \mathrm{pd}(P)$ *and consequently M is a strongly k-Gorenstein projective module with* $k := \mathrm{pd}(P)$.

Proof. (1) If $n = 0$, the result holds from [**3**, Proposition 2.9]. Otherwise, since M is a strongly n-Gorenstein projective module, there is a short exact sequence

$$0 \to M \to P \to M \to 0,$$

where $\mathrm{pd}(P) \leq n$. Consider the following n-step projective resolution of M

$$0 \to N \to P_n \to \cdots \to P_1 \to M \to 0.$$

Hence, there is a module Q such that the following diagram is commutative.

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & & 0 & & 0 & \\
 & & \downarrow & & \downarrow & & & \downarrow & & \downarrow & \\
0 & \to & N & \to & P_n & \to \cdots \to & & P_1 & \to & M & \to 0 \\
 & & \downarrow & & \downarrow & & & \downarrow & & \downarrow & \\
0 & \to & Q & \to & P_n \oplus P_n & \to \cdots \to & & P_1 \oplus P_1 & \to & P & \to 0 \\
 & & \downarrow & & \downarrow & & & \downarrow & & \downarrow & \\
0 & \to & N & \to & P_n & \to \cdots \to & & P_1 & \to & M & \to 0 \\
 & & \downarrow & & \downarrow & & & \downarrow & & \downarrow & \\
 & & 0 & & 0 & & & 0 & & 0 &
\end{array}$$

Clearly, Q is projective since $\mathrm{pd}(P) \leq n$ and for every projective module K, $\mathrm{Ext}(N,K) = \mathrm{Ext}^{n+1}(M,K) = 0$. Thus, by [**3**, Proposition 2.9], N is a strongly Gorenstein projective module (then, Gorenstein projective). So, $\mathrm{Gpd}(M) \leq n$.

(2) From the short exact sequence $0 \to M \to P \to M \to 0$ and [**15**, Proposition 2.27], Lemma 1.5, and since $\mathrm{Gpd}(M)$ is finite by (1) above, we have

$$k := \mathrm{pd}(P) = \mathrm{Gpd}(P) = \max\{\mathrm{Gpd}(M), \mathrm{Gpd}(M)\} = \mathrm{Gpd}(M).$$

Thus, $\mathrm{Gpd}(M) = \mathrm{pd}(P)$. By [**15**, Theorem 2.20], $\mathrm{Ext}^{k+1}(M,K) = 0$ whenever K is projective. Consequently, M is a strongly k-Gorenstein projective module. □

Using [**15**, Theorem 2.20], a direct consequence of Proposition 2.3, is that every strongly n-Gorenstein projective module is a strongly m-Gorenstein projective module whenever $n \leq m$.

Proposition 2.4.

(1) *If $(M_i)_{i\in I}$ is a family of strongly n-Gorenstein projective modules, then $\bigoplus_{i\in I} M_i$ is strongly n-Gorenstein projective.*

(2) *If $(M_i)_{i\in I}$ is a family of strongly n-Gorenstein injective modules, then $\prod_{i\in I} M_i$ is strongly n-Gorenstein injective.*

Proof. It is clear since $\mathrm{pd}(\oplus M_i) = \sup\{\mathrm{pd}(M_i)\}$ and $\mathrm{id}(\prod M_i) = \sup\{\mathrm{id}(M_i)\}$, and also since

$$\mathrm{Ext}^i(\oplus M_i, N) \cong \oplus \mathrm{Ext}^i(M_i, N) \quad \text{and} \quad \mathrm{Ext}^i(M, \prod N_i) \cong \prod \mathrm{Ext}^i(M, N_i)$$

for every modules M, N, M_i, N_i and all $i \geq 0$. □

It is clear that for a positive integer n and an R-module M.

"$\mathrm{pd}(M) \leq n$" $\Longrightarrow$ "M is strongly n-Gorenstein; projective" $\Longrightarrow$ "$\mathrm{Gpd}(M) \leq n$"

The converse is false as the following two examples shows

Example 2.5. Consider the quasi-Frobenius local ring $R := K[X]/(X^2)$, where K is a field and we by $\overline{X}$, denote the residue class in R of X. Let S be a Noetherian ring such that $\mathrm{gldim}(S) = n$. Consider a finitely generated S-module M of S such that $\mathrm{pd}_S(M) = n$. Set $T = R \times S$ and set $E := (\overline{X}) \times M$. Then:
(1) E is a strongly n-Gorenstein projective T-module and $\mathrm{Gpd}_T(E) = n$.
(2) However, $\mathrm{pd}_T(E) = \infty$.

Proof. (1) Consider the short exact sequence of R-modules

$$0 \to (\overline{X}) \xrightarrow{\kappa} R \xrightarrow{\phi} (\overline{X}) \to 0,$$

where κ is the injection and ϕ is the multiplication by $\overline{X}$. And consider also the short exact sequence of S-module:

$$0 \to M \xrightarrow{\iota} M \oplus M \xrightarrow{\pi} M \to 0$$

where ι and π, respectively, the canonical injection and projection. Hence, we have the short exact sequence of $R \times S$-module

$$(\star) \qquad 0 \to E \to R \times (M \oplus M) \to E \to 0.$$

By [**17**, Lemma 2.5(2)], $\mathrm{pd}_T(R \times (M \oplus M)) = \mathrm{pd}_S(M \oplus M) = n$. On the other hand, by [**5**, Theorem 3.1] and [**4**, Propositions 2.8 and 2.12], we have

$$\mathrm{Ggldim}(T) = \max\{\mathrm{Ggldim}(R), \mathrm{Ggldim}(S)\} = \mathrm{gldim}(S) = n < \infty.$$

Then, $\mathrm{Gpd}_T(E) < \infty$. Therefore, applying Lemma 1.5 to $(\star)$,

$$\mathrm{Gpd}_T(E) \leq \max\{\mathrm{Gpd}_T(R \times (M \oplus M)), \mathrm{Gpd}_T(E) - 1\}.$$

Thus, $\mathrm{Gpd}_T(E) \leq \mathrm{Gpd}_T(R \times (M \oplus M))$. Using Lemma 1.5 again to $(\star)$, we have

$$\mathrm{Gpd}_T(R \times (M \oplus M)) \leq \max\{\mathrm{Gpd}_T(E), \mathrm{Gpd}_T(E)\} = \mathrm{Gpd}_T(E)$$

So, $\mathrm{Gpd}_T(E) = \mathrm{Gpd}_T(R \times (M \oplus M))$. On the other hand, by [**15**, Propostion 2.27], $\mathrm{Gpd}_T(R \times (M \oplus M)) = \mathrm{pd}_T(R \times (M \oplus M)) = n$. Consequently, $\mathrm{Gpd}_T(E) = n$ and by $(\star)$ and [**15**, Theorem 2.20], E is a strongly n-Gorenstein projective T-module, as desired.

(2) Using [**17**, Lemma 2.5(2)], $\mathrm{pd}_T(E) = \sup\{\mathrm{pd}_R(\overline{X}), \mathrm{pd}_S(M)\}$. Now, suppose that $\mathrm{pd}_R(\overline{X}) < \infty$. Thus, by [**4**, Proposition 2.8 and Corollary 2.10], $\overline{X}$ is projective, and then free since R is local. Absurd, since $\overline{X}^2 = 0$. Consequently, $\mathrm{pd}_T(E) = \infty$. □

Example 2.6. Consider the Noetherian local ring $R := K[[X,Y]]/(XY)$, where K is a field, and we denote by $\overline{X}$ the residue class in R of X. Let S be a Noetherian ring such that $\mathrm{gldim}(S) = n$. Let M be a finitely generated S-module such that $\mathrm{pd}_S(M) = n$. Set $T = R \times S$, and set $E := (\overline{X}) \times M$. Then:

(1) $\mathrm{Gpd}_T(E) = n$.

(2) There is no positive integer k for such E which is a strongly k-Gorenstein T-module.

Proof. (1) By [**5**, Lemma 3.2] and [**15**, Theorem 2.27],

$$n = \mathrm{pd}_S(M) = \mathrm{Gpd}_S(M) = \mathrm{Gpd}_S(E \otimes_T S) \leq \mathrm{Gpd}_T(E)$$

On the other hand, see [**4**, Propostions 2.8 and 2.10 and 2.12], the conditions of [**5**, Lemma 3.3] are satisfied. Hence, we have

$$\mathrm{Gpd}_T(E) \leq \max\{\mathrm{Gpd}_R(\overline{X}), \mathrm{Gpd}_S(M)\} = \mathrm{pd}_S(M) = n.$$

Consequently, $\mathrm{Gpd}_T(E) = n$, as desired.

(2) Suppose the existence of a positive integer k such that E is strongly k-Gorenstein projective T-module. Then, there exists a short exact sequence of T-modules $0 \to E \to P \to E \to 0$, where $\mathrm{pd}_T(P) < \infty$. Since R is a projective T-module, and since $(\overline{X}) \cong_R E \otimes_T R$, we have a short exact sequence of R-modules

$$0 \to (\overline{X}) \to P \otimes_T R \to (\overline{X}) \to 0.$$

Notice that $\mathrm{pd}_R(P \otimes_T R) < \infty$ since R is a projective T-module. Using [**4**, Propositions 2.8 and 2.10], we get that $P \otimes_T R$ is a projective R-module and that $(\overline{X})$, is a Gorenstein projective R-module. So, by [**15**, Theorem 2.20], $(\overline{X})$ is strongly Gorenstein projective module. That is absurd due to (by [**3**, Example 2.13(2)]). □

Now, we give our main result of this paper.

Theorem 2.7. *Let M be an R-module and n a positive a integer. Then, $\mathrm{Gpd}_R(M) \leq n$ if and only if M is a direct summand of a strongly n-Gorenstein projective module.*

Proof. If $n = 0$, the result holds from [**3**, Theorem 2.7]. So, assume that $0 < \mathrm{Gpd}(M) \leq n$. From [**15**, Theorem 2.10], there is an exact sequence of R-module $0 \to K \to G \to M$ where G is Gorenstein projective and $\mathrm{pd}(K) \leq n-1$. By the definition of Gorenstein projective module, there is a short exact sequence

$$0 \to G \to P \to G^0 \to 0,$$

where P is projective and G' is Gorenstein projective. Hence, consider the following pushout diagram

$$\begin{array}{ccccccc}
 & 0 & & 0 & & & \\
 & \downarrow & & \downarrow & & & \\
 & K & = & K & & & \\
 & \downarrow & & \downarrow & & & \\
0\to & G & \to & P & \to & G^0 & \to 0 \\
 & \downarrow & & \downarrow & & \| & \\
0\to & M & \to & D & \to & G^0 & \to 0 \\
 & \downarrow & & \downarrow & & & \\
 & 0 & & 0 & & &
\end{array}$$

From the vertical middle short exact sequence, $\mathrm{pd}(D) \leq \mathrm{pd}(K) + 1 \leq n$. Now, consider the Gorenstein projective resolution of M

$$0 \to G_n \to P_n \to \cdots \to P_1 \to M \to 0,$$

where all P_i are projective and G_n is Gorenstein projective. Devise this sequence on short exact sequence as

$$\begin{array}{cccccccc}
0\to & G_1 & \to & P_1 & \to & M & \to 0 \\
0\to & G_2 & \to & P_2 & \to & G_1 & \to 0 \\
 & \vdots & & \vdots & & \vdots & \\
0\to & G_n & \to & P_n & \to & G_{n-1} & \to 0.
\end{array}$$

Clearly, by Lemma 1.5, for all $1 \leq i \leq n$, $\mathrm{Gpd}(G_i) \leq n - i \leq n$.

Consider also the following projective resolution of G_n

$$\cdots \to P_{n+2} \to P_{n+1} \to G_n \to 0$$

and devise this long sequence on short exact sequences as $0 \to G_{i+1} \to P_{i+1} \to G_i \to 0$ for all $i \geq n$. It is clear that for all $i \geq n$, G_i is a Gorenstein projective module (by [**15**, Theorem 2.5]).

On the other hand, since G^0 is Gorenstein projective, there is a co-proper right projective resolution of G^0

$$0 \to G^0 \to P^1 \to P^2 \to P^3 \to \ldots$$

such that for every $i \geq 1$, $G^i = \mathrm{Im}(P^i \to P^{i+1})$ is Gorenstein projective. If we devise this sequence on short exact sequence, we get $0 \to G^i \to P^{i+1} \to G^{i+1} \to 0$ for all $i \geq 0$.

Briefly, we have

$$\begin{array}{cccccccc}
 & \vdots & & \vdots & & \vdots & \\
0\to & G^1 & \to & P^2 & \to & G^2 & \to 0 \\
0\to & G^0 & \to & P^1 & \to & G^1 & \to 0 \\
0\to & M & \to & D & \to & G^0 & \to 0 \\
0\to & G_1 & \to & P_1 & \to & M & \to 0 \\
0\to & G_2 & \to & P_2 & \to & G_1 & \to 0 \\
 & \vdots & & \vdots & & \vdots &
\end{array}$$

Thus, we have a sum short exact sequence $0 \to N \to Q \to N \to 0$, where $N = \oplus_{i\geq 1}G_i \oplus M \oplus_{i\geq 0} G^i$ and $Q = \oplus_{i\geq 1}P_i \oplus D \oplus_{i\geq 1} P^i$. And clearly $\mathrm{pd}(Q) = \mathrm{pd}(D) \leq n$ and $\mathrm{Gpd}(N) = \max\{\mathrm{Gpd}(G_i), \mathrm{Gpd}(G^i), \mathrm{Gpd}(M)\} \leq n$ (by [**15**, Proposition 2.19]). Thus, by [**15**, Theorem 2.20], N is a strongly n-Gorenstein projective module and M is a direct summand of N.

The condition "if" follows from [**15**, Propostion 2.19] and Proposition 2.3. □

Dually, we have the following theorem.

Theorem 2.8. *Let M be an R-module and n be a positive integer. Then, $\mathrm{Gid}_R(M) \leq n$ if and only if M is a direct summand of a strongly n-Gorenstein injective module.*

Proof. The proof is similar to the one of Theorem 2.7 by replacing the direct sum by the direct product and by using [**15**, Theorem 2.15], the dual of [**15**, Propostion 2.19] and [**3**, Theorem 2.7]. □

Remark 2.9. From the proof of Theorem 2.7, if $\mathrm{Gpd}(M) = n$, then there exists a strongly n-Gorenstein projective module N such that $\mathrm{Gpd}(N) = n$ and M is a direct summand of N.

Proposition 2.10. *For any module M and any positive integer n, the following statements are equivalent:*

1. *M is strongly n-Gorenstein projective.*
2. *There is an exact sequence $0 \to M \to Q \to M \to 0$, where $\mathrm{pd}(Q) \leq n$ and $\mathrm{Ext}^i(M, P) = 0$ for every module P with finite projective dimension and all $i > n$.*
3. *There is an exact sequence $0 \to M \to Q \to M \to 0$, where $\mathrm{pd}(Q) < \infty$ and $\mathrm{Ext}^i(M, P) = 0$ for every projective module P and all $i > n$.*

Proof. $1 \Rightarrow 2$. By definition of strongly n-Gorenstein projective modules, we have just to prove that for every $i > n$ and all module P with finite projective dimension, we have $\mathrm{Ext}^i(M, P) = 0$. That is clear from [**15**, Theorem 2.20] since $\mathrm{Gpd}(M) \leq n$ (by Proposition 2.3).
$2 \Rightarrow 3$. Obvious.
$3 \Rightarrow 1$. Since $\mathrm{Ext}^i(M, P) = 0$ for every projective module P and all $i > n$, from the short exact sequence $0 \to M \to Q \to M \to 0$, we have for all $i > n$,

$$\cdots \to 0 = \mathrm{Ext}^i(M, P) \to \mathrm{Ext}^i(Q, P) \to \mathrm{Ext}^i(Q, P) = 0 \to \dots$$

Thus $\mathrm{Ext}^i(Q, P) = 0$. On the other hand, $\mathrm{Gpd}(Q) = \mathrm{pd}(Q) < \infty$ (by [**15**, Proposition 2.27]). Then, from [**15**, Theorem 2.20], $\mathrm{pd}(Q) = \mathrm{Gpd}(Q) \leq n$. Consequently, M is strongly n-Gorenstein projective. □

Proposition 2.11. *If $\mathrm{Ggldim}(R) < \infty$ then:*

1. *M is strongly n-Gorenstein projective if and only if there exists an exact sequence $0 \to M \to Q \to M \to 0$, where $\mathrm{pd}(Q) \leq n$.*
2. *M is strongly n-Gorenstein injective if and only if there exists an exact sequence $0 \to M \to E \to M \to 0$, where $\mathrm{id}(E) \leq n$.*

Proof. (1) The condition "only if" is clear by definition of the strongly n-Gorenstein projective module. So, we claim the "if" condition. Since $\mathrm{Ggldim}(R) < \infty$, $\mathrm{Gpd}(M) < \infty$. Thus, there is an integer k such that $\mathrm{Ext}^i(M,P) = 0$ for all $i > k$ and for all projective module P. Thus, using the long exact sequence

$$\cdots \to \mathrm{Ext}^i(Q,P) \to \mathrm{Ext}^i(M,P) \to \mathrm{Ext}^{i+1}(M,P) \to \mathrm{Ext}^{i+1}(Q,P) \to \dots,$$

we deduce that $\mathrm{Ext}^{n+1}(M,P) = \mathrm{Ext}^{n+j}(M,P)$ for all $j > 0$ (since $\mathrm{pd}(Q) \leq n$). Thus, if $j > k$, we conclude that for every projective module P, $\mathrm{Ext}^{n+1}(M,P) = 0$. Consequently, M is strongly n-Gorenstein projective.

(2) The proof is dual to (1). □

Proposition 2.12. *Let M be a strongly n-Gorenstein projective R-module $(n \geq 1)$. Then, there is an epimorphism $\varphi N \twoheadrightarrow M$, where N is strongly Gorenstein projective and $K = Ker(\varphi)$ satisfies* $\mathrm{pd}(K) = \mathrm{Gpd}(M) - 1 \leq n - 1$.

Proof. Assume that M is a strongly n-Gorenstein projective module.

The proof will be similar to the one of [**15**, Theorem 2.10]. For completness we include the proof here. Let $0 \to N \to P_n \to \cdots \to P_1 \to M \to 0$ be an exact sequence, where all P_i are projective and N is strongly Gorenstein projective (the existence of this sequence is guaranted by Proposition 2.3). By definition of a strongly Gorenstein projective module, there is an exact sequence $0 \to N \to Q \to \cdots \to Q \to N \to 0$, where Q is projective and such that the functor $\mathrm{Hom}(-,P)$ leaves this sequence exact whenever P is projective. Thus, there exist homomorphism, $Q \to P_i$ for $i = 1,\dots,n$ and $N \to M$ such that the following diagram is commutative.

$$\begin{array}{ccccccccc} 0 \to & N & \to & Q & \to \cdots \to & Q & \to & N & \to 0 \\ & \| & & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & N & \to & P_n & \to \cdots \to & P_1 & \to & M & \to 0 \end{array}$$

This diagram gives a chain map between complexes

$$\begin{array}{ccccccc} 0 \to & Q & \to \cdots \to & Q & \to & N & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & P_n & \to \cdots \to & P_1 & \to & M & \to 0 \end{array}$$

which induces an isomorphism in homology. Its mapping cone is exact, and all the modules in it, except for $P_1 \oplus N$, which is strongly Gorenstein projective, are projective. Hence the kernel K of $\varphi\ P_1 \oplus N \twoheadrightarrow M$ satisfies $\mathrm{pd}(K) \leq n - 1$, as desired. □

Proposition 2.13.

(A) *Let $0 \to N \overset{\alpha}{\to} P \overset{\beta}{\to} N' \to 0$ be an exact sequence of R-modules.*

<u>*Case 1*</u> *"P projective and* $\mathrm{Gpd}(N') = n < \infty$*"*

(1) *If N' is strongly Gorenstein projective, then so is N.*

(2) *If $n \geq 1$ and N' is strongly n-Gorenstein projective, then N is strongly $(n-1)$-Gorenstein projective and* $\mathrm{Gpd}(N) = n - 1$.

Case 2 "$\mathrm{pd}(P) = n < \infty$"

If N is a strongly Gorenstein projective module, then N' is strongly $(n+1)$-Gorenstein projective.

(B) *Let* $0 \to N \overset{\mu}{\to} N' \overset{\nu}{\to} Q \to 0$ *be an exact sequence, where* $\mathrm{pd}(Q) = n < \infty$.

(1) *If $n > 0$ and N' is strongly Gorenstein projective, then N is strongly $(n-1)$-Gorenstein projective.*

(2) *If Q is projective then N is strongly Gorenstein projective if and only if N' is strongly Gorenstein projective.*

Proof. (A) *Case 1*

(1) Clear.

(2) If N' is strongly n-Gorenstein projective module, there is a short exact sequence $0 \to N' \to Q \to N' \to 0$ where $\mathrm{pd}(Q) \leq n$. Since $\mathrm{Gpd}(N') = n$ we deduce that $\mathrm{pd}(Q) = n$ (by Proposition 2.3). On the other hand, we have the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & & N & \to & P & \to & N' & \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & & Q' & \to & P \oplus P & \to & Q & \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & & N & \to & P & \to & N' & \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 & & 0 & & 0 & & 0 &
\end{array}$$

Since P is projective, we get $\mathrm{pd}(Q') = n - 1$, and since $\mathrm{Gpd}(N') = n$, we deduce that $\mathrm{Gpd}(N) = n - 1$ (by Lemma 1.5). Thus, N' is strongly $(n-1)$-Gorenstein projective (by [**15**, Theorem 2.20]).

Case 2

Since N is a strongly Gorenstein projective module, there is an exact sequence $0 \to N \overset{u}{\to} Q \overset{v}{\to} N \to 0$ where Q is projective and $\mathrm{Ext}(N, K) = 0$ for every module K with finite projective dimension. Thus, since $\mathrm{pd}(P) < \infty$, the short sequence

$$0 \to \mathrm{Hom}(N, P) \overset{\circ v}{\to} \mathrm{Hom}(Q, P) \overset{\circ u}{\to} \mathrm{Hom}(N, P) \to 0$$

is exact. Hence, for $\alpha : N \to P$, there is a morphism $\lambda : Q \to P$ such that $\alpha = \lambda \circ u$. Thus, the following diagram is commutative

$$\begin{array}{ccccccc}
0 \to & N & \overset{u}{\to} & Q & \overset{v}{\to} & N & \to 0 \\
 & \alpha \downarrow & & \phi \downarrow & & \downarrow \alpha & \\
0 \to & P & \overset{i}{\to} & P \oplus P & \overset{j}{\to} & P & \to 0,
\end{array}$$

where $\phi Q \to P \oplus P$ is defined by $\phi(q) = (\lambda(q), \alpha \circ v(q))$, and i and j, respectively, the canonical injection and projection. Thus, applying the Snake Lemma, we deduce an exact sequence of the form:

$$0 \to N' \to (P \oplus P)/\phi(Q) \to N' \to 0$$

and clearly $\mathrm{pd}(P \oplus P/\phi(Q)) \leq n+1$ and $\mathrm{Gpd}(N') \leq n+1$. Thus, by [**15**, Theorem 2.20], N' is strongly $(n+1)$-Gorenstein projective, as desired.

(B)
Suppose that N' is strongly Gorenstein projective. Thus, there is an exact sequence $0 \to N' \overset{u}{\to} P \overset{u}{\to} N' \to 0$, where P is projective and $\mathrm{Ext}(N,K)=0$ for every module K with finite projective dimension. Then, similar as in (A) *Case 2*, there is a morphism $\phi\, P \to Q \oplus Q$ such that the following diagram is commutative

$$\begin{array}{ccccccc} 0 \to & N' & \overset{u}{\to} & P & \overset{v}{\to} & N' & \to 0 \\ & \nu \downarrow & & \phi \downarrow & & \downarrow \nu & \\ 0 \to & Q & \overset{i}{\to} & Q \oplus Q & \overset{j}{\to} & Q & \to 0 \end{array}$$

Hence, applying Snake Lemma, we get an exact sequence of the form $0 \to N \to \mathrm{Ker}(\phi) \to N \to 0$.

(1) If $n>0$, then $\mathrm{pd}(\mathrm{Ker}(\phi)) = n-1$ and also $\mathrm{Gpd}(N) = n-1$ (by Lemma 1.5). Therefore, N is strongly $(n-1)$-Gorenstein projective (by [**15**, Theorem 2.20]).
(2) If Q is projective, then $\mathrm{Ker}(\phi)$ is projective and N is Gorenstein projective. Thus, N is strongly Gorenstein projective. Conversely, if N is strongly Gorenstein projective, it is clear that $N' \cong N \oplus P$ is strongly Gorenstein projective, as desired.

□

Dually, we have the following proposition.

Proposition 2.14.
(A) *Let* $0 \to N \overset{\alpha}{\to} I \overset{\beta}{\to} N' \leftrightarrow 0$ *be an exact sequence of* R*-modules.*
Case 1 *"I is injective and* $\mathrm{Gid}(N) = n < \infty$*"*
(1) *If* N *is strongly Gorenstein injective, then so is* N'.
(2) *If* $n \geq 1$ *and* N *is strongly* n*-Gorenstein injective then* N' *is strongly* $(n-1)$*-Gorenstein injective and* $\mathrm{Gid}(N') = n-1$.

Case 2 "$\mathrm{id}(P) = n < \infty$"
If N' *is a strongly Gorenstein injective module, then* N *is strongly* $(n+1)$*-Gorenstein injective.*

(B) *Let* $0 \to E \overset{\mu}{\to} N' \overset{\nu}{\to} N \to 0$ *be an exact sequence, where* $\mathrm{id}(E) = n < \infty$.
(1) *If* $n>0$ *and* N' *is strongly Gorenstein injective, then* N *is strongly* $(n-1)$*-Gorenstein injective.*
(2) *If* E *is injective, then* N *is strongly Gorenstein injective if and only if* N' *is strongly Gorenstein injective.*

Corollary 2.15. *Let* R *be a ring. The following statements are equivalent:*

(1) *Every Gorenstein projective module is strongly Gorenstein projective.*
(2) *Every module such that* $\mathrm{Gpd}(M) \leq 1$ *is strongly* 1*-Gorenstein projective.*

Proof. Assume that every Gorenstein projective module is strongly Gorenstein projective and consider a module M such that $\mathrm{Gpd}(M) \leq 1$. Consider a short exact sequence $0 \to N \to P \to M \to 0$, where P is projective, and so N is Gorenstein projective. Hence, by the hypothesis, condition, N is a strongly Gorenstein projective module. Thus, by Proposition 2.13 (Case 2), M is strongly 1-Gorenstein projective module, as desired.

Conversely, assume that every module such $\mathrm{Gpd}(M) \leq 1$ is strongly 1-Gorenstein projective. Let M be a Gorenstein projective module. Thus, by the hypothesis condition M is strongly 1-Gorenstein projective. Then, there is an exact sequence $0 \to M \to Q \to M \to 0$, where $\mathrm{pd}(Q) \leq 1$. Since M is Gorenstein projective, so is Q and then it is projective (by [**15**, Theorem 2.5 and Proposition 2.27]). Consequently, M is a strongly Gorenstein projective module. □

Proposition 2.16. *Let R be a ring. The following statements are equivalent:*

(1) *Every module is strongly n-Gorenstein projective.*

(2) *Every module is strongly n-Gorenstein injective.*

Proof. We prove only one implication and the other is similar.
Assume that every module is strongly n-Gorenstein projective. Thus $\mathrm{Ggldim}(R) \leq n$ (by Proposition 2.3 and the hypothesis condition). Now, consider an arbitrary module M. Clearly $\mathrm{Gid}(M) \leq n$ (since $\mathrm{Ggldim}(R) \leq n$). Then, for every injective module I, $\mathrm{Ext}^{n+1}(I, M) = 0$ ([**15**, Theorem 2.22]). On the other hand, there is an exact sequence $0 \to M \to P \to M \to 0$, where $\mathrm{pd}(P) \leq n$. By [**4**, Corollary 2.10], $\mathrm{id}(P) \leq n$. Consequently, M is strongly n-Gorenstein injective, as desired. □

Proposition 2.17. *Let R be a ring with finite Gorenstein global dimension and n be a positive integer. The following statements are equivalent:*

(1) $\mathrm{Ggldim}(R) \leq n$.

(2) *Every strongly Gorenstein projective module is a strongly n-Gorenstein injective module.*

(3) *Every strongly Gorenstein injective module is a strongly n-Gorenstein projective module.*

Proof. We claim that $\mathrm{Ggldim}(R) \leq n$ if and only if every strongly Gorenstein projective module is a strongly n-Gorenstein injective module. The proof of the other equivalence is analogous. So, suppose that $\mathrm{Ggldim}(R) \leq n$, and consider a strongly Gorenstein projective module M. For such module, there is an exact sequence $0 \to M \to P \to M \to 0$, where P is projective. From [**4**, Corollary 2.10], $\mathrm{id}(P) \leq n$. Hence, from Proposition 2.11, M is strongly n-Gorenstein injective.

Conversely, suppose that every strongly Gorenstein projective module is strongly n-Gorenstein injective module and let P be a projective module (then strongly Gorenstein projective). By the hypothesis condition, P is strongly n-Gorenstein injective. Thus, there is an exact sequence $0 \to P \to E \to P \to 0$, where $\mathrm{id}(E) \leq n$. Hence, $P \oplus P \cong E$. Consequently, $\mathrm{id}(P) \leq n$. Then, from [**4**, Theorem 2.1 and Lemma 2.2], $\mathrm{Ggldim}(R) \leq n$, as desired. □

3. Strongly n-Gorenstein flat modules

In this section, we introduce and study the strongly n-Gorenstein flat modules which are defined as follows.

Definition 3.1. An R-module M is said to be strongly n-Gorenstein flat, if there exists a short exact sequence

$$0 \to M \to F \to M \to 0,$$

where $\mathrm{fd}_R(P) \leq n$ and $\mathrm{Tor}_R^{n+1}(M, I) = 0$ whenever I is injective.

A direct consequence of the above definition is such that the strongly 0-Gorenstein flat modules are just the strongly Gorenstein flat modules (by [**3**, Proposition 3.6]). Also every module with finite flat dimension less than or equal to n is a strongly n-Gorenstein flat module.

In [**21**], the authors introduced n-Strongly Gorenstein flat modules[1] as follows: Let $n \geq$ be a positive integer. An R-module M is called n-strongly Gorenstein flat if there exists an exact sequence of R-modules.

$$0 \to M \to F_{n-1} \to \cdots \to F_0 \to M \to 0,$$

where each F_i is flat such that $. \times_R I$ leaves the sequence exact whenever I is an injective R-module. It is clear that every n-strongly Gorenstein flat module is Gorenstein flat. The class of strongly 0-Gorenstein flat modules and the class of 1-strongly Gorenstein flat modules in the sense of [**21**]) coincide with the class of strongly Gorenstein flat modules. However, in general case, the notion of strongly n-Gorenstein flat modules and that of m-strongly Gorenstein flat modules are different.

Example 3.2.

(1) Let $n \geq 1$ be an integer and let R be a ring with $\mathrm{wdim}(R) = n$. There exists an R-module M such that $\mathrm{fd}_R(M) = n$. Then, M is a strongly n-Gorenstein flat module which is not an m-strongly Gorenstein flat module for each positive integer m.

(2) Consider a Noetherian local ring $R := k[[X, Y]]/(XY)$, where k is a field. Set $\overline{X}$ the residue class of X in R. Then, the ideal $(\overline{X})$ is a 2-strongly Gorenstein projective R-module which is not a strongly n-Gorenstein projective module for each positive integer n.

Proof. (1) We have the exact sequence

$$0 \to M \to M \oplus M \to M \to 0$$

with $\mathrm{fd}_R(M \oplus M) = n$ and $\mathrm{Tor}_R^{n+1}(M, I) = 0$ for each module (in particular, injective module) I. Hence, M is a strongly n-flat module. However, $\mathrm{Gfd}_R(M) = \mathrm{fd}_R(M) = n \geq 1$ (by [**2**, Theorem 2.2]). Then, M cannot be an m-strongly Gorenstein flat module for each positive integer m since every m-strongly Gorenstein flat module is Gorenstein flat.

[1]In [**21**], the definition of n-Strongly Gorenstein flat modules is given in the associative context. Here, we use the commutative version of this definition.

(2) The ideal $(\overline{X})$ is a 2-strongly Gorenstein projective S-module which is not strongly Gorenstein projective (by [**21**, Example 4.8]). If $(\overline{X})$ is a strongly n-Goren- stein flat module for some positive integer n, then there exists an exact sequence

$$0 \to (\overline{X}) \to F \to (\overline{X}) \to 0$$

with $\mathrm{fd}_R(P) \leq n$. The module $(\overline{X})$ is Gorenstein flat (since it is 2-strongly Gorenstein flat). Then, by [**15**, Theorem 3.7], F is a Gorenstein projective module with finite flat dimension, and so a flat module by [**2**, Theorem 2.2]. Thus, $(\overline{X})$ is a strongly Gorenstein projective flat module, which is impossible. □

The main difference between the notion of strongly n-Gorenstein flat modules and that of m-strongly Gorenstein flat module is that all n-strongly Gorenstein flat module are Gorenstein flat but strongly n-Gorenstein projective modules can have Gorenstein flat dimension> 0.

Proposition 3.3. *Let n be a positive integer and M be a strongly n-Gorenstein flat R-module. Then, the following statements hold.*

(1) *If $0 \to N \to P_n \to \cdots \to P_1 \to M \to 0$ is an exact sequence, where all P_i are projective, then N is a strongly Gorenstein flat module, and consequently, $\mathrm{Gfd}(M) \leq n$.*

(2) *Moreover, if $0 \to M \to F \to M \to 0$ is a short exact sequence, where $\mathrm{fd}(F) < \infty$, then $\mathrm{Gfd}(M) = \mathrm{fd}(F)$, and consequently, M is a strongly k-Gorenstein flat module with $k := \mathrm{pd}(P)$.*

Proof. (1) Using an n-step projective resolution of M and [**3**, Proposition 3.6], the proof is analogous to Proposition 2.3.

(2) Consider an exact short sequence $(\mp)$ $\quad 0 \to M \to F \to M \to 0$, where $\mathrm{fd}(F) < \infty$. We claim $\mathrm{Gfd}(M) = \mathrm{fd}(F)$.

Consider an n step projective resolution

$$0 \to N \to P_n \to \cdots \to P_1 \to M \to 0$$

From (1) above, N is a strongly Gorenstein flat module. Thus, there is a short exact sequence $(\star)$ $\quad 0 \to N \to P \to N \to 0$, where P is flat and $\mathrm{Tor}(N, I) = 0$ whenever I is injective. Hence, from $(\star)$, for all $i > 0$, $\mathrm{Tor}^i(N, I) = 0$. So, we have $\mathrm{Tor}^{n+i}(M, I) = \mathrm{Tor}^i(N, I) = 0$.

Now, suppose that $\mathrm{fd}(F) := k$ and let I be an arbitrary injective. From the short exact sequence $(\mp)$, we have the long exact sequence

$$\ldots \mathrm{Tor}^{i+1}(F, I) \to \mathrm{Tor}^{i+1}(M, I) \to \mathrm{Tor}^i(M, I) \to \mathrm{Tor}^i(F, I) \to \ldots$$

Hence, for all $i > k$, $\mathrm{Tor}^i(M, I) = \mathrm{Tor}^{i+1}(M, I) = \cdots = \mathrm{Tor}^{i+n}(M, I) = 0$. In particular, $\mathrm{Tor}^{k+1}(M, I) = 0$. Consequently, M is a strongly k-Gorenstein flat module. Then, from (1) above, $\mathrm{Gfd}(M) \leq k = \mathrm{fd}(F)$.

Conversely, we claim $\mathrm{fd}(F) \leq \mathrm{Gfd}(M)$. Applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ to the short exact sequence $0 \to M \to F \to M \to 0$ we get the exactness of $0 \to \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \to 0$. On the other hand, from [**15**, Proposition 3.11], $\mathrm{Gid}(\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) \leq \mathrm{Gfd}(M) \leq n$ and by [**19**, Lemma 3.51 and

Theorem 3.52], $\mathrm{id}(\mathrm{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})) = \mathrm{fd}(F) < \infty$. Hence, by [**15**, Theorem 2.22] and the injective counterpart of Proposition 2.10, $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is a strongly n-Gorenstein injective module. So, from the injective counterpart of Proposition 2.3 and by [**15**, Proposition 3.11],

$$\mathrm{fd}(F) = \mathrm{id}(\mathrm{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})) = \mathrm{Gid}(\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) \leq \mathrm{Gfd}(M).$$

Thus, we have the desired equality. □

Theorem 3.4. *Let R be a coherent ring, M be an R-module and n be a positive integer. Then, $Gf(M) \leq n$ if and only if M is a direct summand of a strongly n-Gorenstein flat module.*

Proof. Using [**3**, Theorem 3.5], [**15**, Proposition 3.13, Theorems 3.14 and 3.23], Lemma 1.7, and Proposition 3.3, the proof of this result is analogous to the one of Theorem 2.7. □

Proposition 3.5. *For a module M and a positive integer n, the following statements are equivalent:*

(1) *M is strongly n-Gorenstein flat.*
(2) *There is an exact sequence $0 \to M \to F \to M \to 0$, where $\mathrm{fd}(F) \leq n$ and $\mathrm{Tor}^i(M, I) = 0$ for every module I with finite injective dimension and all $i > n$.*
(3) *There is an exact sequence $0 \to M \to F \to M \to 0$, where $\mathrm{fd}(F) < \infty$ and $\mathrm{Tor}^i(M, I) = 0$ for every injective module P and all $i > n$.*

Proof. (1) $\Rightarrow$ (2) Assume that M is a strongly n-Gorenstein flat module. Then, there is an exact sequence $0 \to M \to F \to M \to 0$, where $\mathrm{fd}(F) \leq n$. On the other hand, if $0 \to N \to P_n \to \cdots \to P_1 \to M \to 0$ is an n-step projective resolution of M, by Proposition 3.3, N is a strongly Gorenstein flat module. Thus, there is a short exact sequence $(\star)$ $\;0 \to N \to P \to N \to 0$, where P is flat and $\mathrm{Tor}(N, I) = 0$ whenever $\mathrm{id}(I) < \infty$ (from [**3**, Proposition 3.6]). Hence, from $(\star)$, for all $i > 0$, $\mathrm{Tor}^i(N, I) = 0$. So, we have $\mathrm{Tor}^{n+i}(M, I) = \mathrm{Tor}^i(N, I) = 0$, as desired.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) As in the proof of Proposition 2.10(3 $\Rightarrow$ 1), we prove that for every injective module I and all $i > n$, we have $\mathrm{Tor}^i(F, I) = 0$. Suppose that $m := \mathrm{fd}(F) > n$ and let M be an arbitrary module. Pick a short exact sequence $0 \to M \to I \to I/M \to 0$, where I is injective. So, we have the long exact sequence

$$\cdots \to \mathrm{Tor}^{i+1}(F, I) \to \mathrm{Tor}^{i+1}(F, I/M) \to \mathrm{Tor}^i(F, M) \to \mathrm{Tor}^i(F, I) \to \ldots$$

Thus, for $i > n$, we have $\mathrm{Tor}^i(F, M) = \mathrm{Tor}^{i+1}(F, I/M)$. Hence, $\mathrm{Tor}^m(F, M) = \mathrm{Tor}^{m+1}(F, I/M) = 0$. Then, $\mathrm{fd}(F) \leq m - 1$. Absurd. Thus, $\mathrm{fd}(F) \leq n$. So, it is clear that M is a strongly n-Gorenstein flat module, as desired. □

Proposition 3.6. *Let M be a strongly n-Gorenstein flat module over a coherent ring R $(n \geq 1)$. Then, there is an epimorphism $\varphi\, N \twoheadrightarrow M$, where N is strongly Gorenstein flat and $K = \mathrm{Ker}(\varphi)$ satisfies $\mathrm{fd}(K) = \mathrm{Gfd}(M) - 1 \leq n - 1$.*

Proof. Using [**3**, Proposition 3.6] and [**15**, Lemma 3.17], the proof is analogous to that in [**15**, Lemma 3.17] and Proposition 2.12. □

Proposition 3.7. *Let M be an R-module and n be a positive integer. Then, following statements are equivalent:*

(1) *M is a strongly n-Gorenstein projective and M admits a finite n-presentation.*

(2) *M is a strongly n-Gorenstein flat module and M admits a finite $n+1$-presentation.*

Proof. Any way, in this Proposition M, admits a finite n-presentation. Thus, we can consider an n-step free resolution $0 \to N \to F_n \to \cdots \to F_1 \to M \to 0$, where F_i are finitely generated free and N is finitely generated.

(1) If M is strongly n-Gorenstein projective module, then N is a finitely generated strongly Gorenstein projective module. Thus, from [**3**, Proposition 3.9], N is a finitely presented strongly Gorenstein flat module. Then, M admits a finite $(n+1)$--presentation and for all injective module I, we have $\mathrm{Tor}^{n+1}(M,I)=\mathrm{Tor}(N,I)=0$. On the other hand, there is an exact sequence $0 \to M \to Q \to M \to 0$, where $\mathrm{fd}(Q) \leq \mathrm{pd}(Q) \leq n$. Consequently, M is a strongly n-Gorenstein flat module which admits a finite $n+1$-presentation.

(2) Now, if M is a strongly n-Gorenstein flat module which admits a finite $(n+1)$- -presentation. Then, N is a finitely presented strongly flat module. Thus, from [**3**, Proposition 3.9], N is a strongly Gorenstein projective module. Hence, for every projective module P, $\mathrm{Ext}^{n+1}(M,P) = \mathrm{Ext}(N,P) = 0$. On the other hand, there is an exact sequence $0 \to M \to F \to M \to 0$, where $\mathrm{fd}(F) \leq n$. But, from this short exact sequence we see that F also admits a finite $(n+1)$-presentation. Thus, $\mathrm{pd}(F) = \mathrm{fd}(F) \leq n$. Consequently, M is a strongly n-Gorenstein projective module, as desired. □

Corollary 3.8. *If R is a coherent ring and M a finitely presented module. Then, M is strongly n-Gorenstein projective if and only if M is strongly n-Gorenstein flat.*

Finally, it is clear that for a module M and a positive integer n, we have:

$$\text{“}\mathrm{fd}(M) \leq n\text{”} \Longrightarrow \text{“}M \text{ is strongly } n\text{-Gorenstein flat”} \Longrightarrow \text{“}Gf(M) \leq n\text{”}.$$

Also, the converse are false, in general, by the same Examples (2.5 and 2.6) in Section 2, since T is Noetherian and E is finitely presented (since E is finitely generated and T is Noetherian) (by Corollary 3.8).

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