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EXISTENCE AND ULAM STABILITY RESULTS FOR TWO-ORDERS FRACTIONAL DIFFERENTIAL EQUATION

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ABSTRACT. In this paper, we deal with the existence of a unique solution and some Ulam's type stability concepts for an initial value problem of a class of two-orders fractional differential equations involving Caputo's fractional derivative. We investigate two types of Ulam stability: Ulam-Hyers stability and Ulam-Hyers-Rassias stability for the considered problem of two fractional orders. We use the Banach fixed point theorem and fractional calculus. Finally, we give an example to illustrate the results.

1. INTRODUCTION

Fractional differential equations arise naturally in various fields of science and engineering (mathematical physics, finance, hydrology, biology, thermodynamics, control theory, mechanic and bioengineering). In fact, fractional derivatives have a nonlocal character which made them an excellent instrument for the description of memory and hereditary properties of some processes. Fractional calculus can be considered as a generalization of ordinary differentiation and integration to arbitrary order (real or complex). In recent years, there has been a significant development in fractional differential equations; see the books of Diethelm [**5**], Kilbas et al. [**13**] and Lakshmikantham et al. [**14**].

Some results on the existence of solutions of fractional differential initial or boundary value problems were widely discussed by many mathematicians, for example, we refer to [**1, 2, 3, 15, 19, 20**]. Recently, Ulam stability for fractional differential equations has attracted the attention of many mathematicians [**4, 11, 12, 17, 18**] and the references therein. The classical concept of Ulam-Hyers stability means that for a Ulam-Hyers stable system one does not seek the exact solution. It is required to find a function which satisfies a suitable approximation inequality. This approach can guarantee that there exists a close exact solution useful in many applications such as numerical analysis and optimization, where finding the exact solution is impossible. To know more about the problem of

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the so-called Ulam stability posed by Ulam in 1940 and its different types see [**7, 8, 9, 10, 16**].

For example in [**4**], Benchohra and Lazreg established four types of Ulam stability for the following initial value problem for implicit fractional-order differential equation

$$\begin{cases} {}^C D^{\alpha}_{0+} y(t) = f(t, y(t), {}^C D^{\alpha}_{0+} y(t)), & t \in [0,T]\,,\ 0 < \alpha \le 1 \\ y(0) = y_0, \end{cases}$$

where ${}^C D^{\alpha}_{0+}$ is the Caputo fractional derivative, $f\colon\ [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function $y_0 \in \mathbb{R}$.

In [**12**], Ibrahim studied the existence of the solution and the Ulam-Hyers stability for the boundary value differential problem for Lane-Emden equation involving two fractional orders of the form

$$\begin{cases} D^{\beta}_{0+}\left(D^{\alpha}_{0+} + \dfrac{a}{t}\right) u(t) + f(t, u(t)) = g(t), & t \in [0,1],\ 0 < \alpha, \beta \le 1 \\ u(0) = \mu, \quad u(1) = \nu, \end{cases}$$

where D^{α}_{0+} is the Riemann-Liouville fractional derivative, $f(t, u(t))$ is a continuous real valued function and $g(t) \in C[0,1]$.

Motivated by these works, we study the existence of the solution and the Ulam stability for the following nonlinear fractional differential equation with two orders

$$ {}^C D^{\beta}_{0+}\left(p(t)\,{}^C D^{\alpha}_{0+} u(t)\right) + h(t)u(t) = f(t, u(t)), \qquad t \in [0,T] \tag{1}$$

subject to the initial history condition

$$ u(t) = \phi(t), \qquad t \in\ [-r, 0], \tag{2}$$

where ${}^C D^{\alpha}_{0+}$ is the Caputo fractional derivative, $\alpha, \beta \in (0,1)$ such that $0 < \alpha + \beta \le 1$, $p(t)$, $h(t)$ on $[0,T]$ and $\phi(t)$ on $[-r, r]$, r a small positive real, are real given functions, $f\colon\ [0,T] \times \mathbb{R} \to \mathbb{R}$ is a given function.

In the article [**6**], Du and Wang initialized an initial value problem involving Riemann-Liouville fractional order derivative by using an initial history over a small interval. Following this, we impose to our problem a similar condition (2) which is more appropriate for this type of initial value problem involving Caputo fractional two-orders derivatives. The unknown function $u(t)$ is given on a small interval $[-r, 0]$. $\phi(t)$ is its initial history. This is more helpful for us to obtain the necessary condition on $u(t)$ when converting the equation (1) into an integral equation. We need to know the values of $u(t)$ and ${}^C D^{\alpha}_{0+} u(t)$ at the initial time $t = 0$ that can be obtained from the initial history data of $u(t)$ over the small interval $[-r, 0]$.

The organization of the manuscript is given below. In Section 2, we give some basic definitions of fractional calculus which are used in this paper. In Section 3, we study the existence and uniqueness of the solution of the problem (1)–(2). In Section 4, we develop two types of Ulam stability, namely Ulam-Hyers stability and Ulam-Hyers-Rassias stability of the fractional differential problem under consideration by using some inequalities. Finally, Section 5 is devoted to the example.

2. Preliminaries

In this section, we present some definitions and properties from fractional calculus that used throughout this paper. For more details see [**13**].

Definition 1. The Riemann-Liouville fractional (arbitrary) integral of order $\alpha > 0$ of the function $f \in L^1[0,T]$ is formally defined by

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \mathrm{d}s,$$

where Γ is the classical Gamma function.

Definition 2. The Caputo fractional derivative of order $\alpha > 0$ for a given function $f(t)$ on $[0,T]$ is defined by

$$ {}^{C}D_{0^+}^{\alpha} f(t) = D_{0^+}^{\alpha} \Big[f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k \Big], \tag{3}$$

where $n = [\alpha]+1$, $[\alpha]$ means the integer part of α and $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative operator of order α defined by

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t (t-s)^{n-1-\alpha} f(s) \mathrm{d}s = D^n I_{0^+}^{n-\alpha} f(t) \qquad \text{for } t > 0.$$

The Caputo fractional derivative ${}^{C}D_{0^+}^{\alpha} f(t)$ exists for $f(t)$ belonging to $AC^n[0,T]$ the space of functions which have continuous derivatives up to order $(n-1)$ on $[0,T]$ such that $f^{(n-1)} \in AC^1([0,T],\mathbb{R})$. $AC^1([0,T],\mathbb{R})$ also denoted $AC[0,T]$ is the space of absolutely continuous functions. In this case, Caputo's fractional derivative is defined by

$${}^{C}D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-1-\alpha} f^{(n)}(s) \mathrm{d}s = I_{0^+}^{n-\alpha} D^n f(t) \qquad \text{for } t > 0.$$

Remark that when $\alpha = n$, we have ${}^{C}D_{0^+}^{n} f(t) = D^n f(t)$.

Lemma 3. *The fractional integration operator is bounded on $C([0,T],\mathbb{R})$, in the sense that for each $f \in C([0,T],\mathbb{R})$ there exists a positive constant A such that*

$$\| I_{0^+}^{\alpha} f \|_{\infty} \leq A \| f \|_{\infty}.$$

Furthermore,

$$I_{0^+}^{\alpha} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{\mu+\alpha}, \qquad \mu > -1,\ \alpha > 0. \tag{4}$$

Lemma 4. *Let $f \in AC^n[0,T]$, then the Caputo fractional derivative of order $\alpha > 0$ such that $n = [\alpha]+1$ is continuous on $[0,T]$ and*

$${}^{C}D_{0^+}^{\alpha} I_{0^+}^{\alpha} f(t) = f(t), \qquad I_{0^+}^{\alpha} {}^{C}D_{0^+}^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k. \tag{5}$$

In particular, when $0 < \alpha \leq 1$ we have $I_{0^+}^{\alpha} {}^{C}D_{0^+}^{\alpha} f(t) = f(t) - f(0)$.

Corollary 5. *For $\alpha > 0$, $\beta > 0$ the Gamma function satisfies*

$$\alpha\Gamma(\alpha) = \Gamma(\alpha+1), \qquad \int_0^1 (1-\theta)^{\alpha-1}\,\theta^{\beta-1}\mathrm{d}\theta = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

To define Ulam's stability, we consider the following fractional differential equation

$$ {}^{C}D_{0+}^{\alpha}u(t) = f(t,u(t)), \qquad 0 < \alpha \leq 1, \;\; t \in [0,T] \tag{6}$$

Definition 6. The equation (6) is said to be Ulam-Hyers stable if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each $y \in AC([0,T],\mathbb{R})$ solution of the inequality

$$\left|{}^{C}D_{0+}^{\alpha}y(t) - f\left(t,y(t)\right)\right| \leq \varepsilon, \qquad t \in [0,T], \tag{7}$$

there exists a solution $u \in AC\left([0,T],\mathbb{R}\right)$ of the equation (6) with

$$|y(t)-u(t)| \leq c_f\varepsilon, \qquad t \in [0,T].$$

Definition 7. The equation (6) is Ulam-Hyers-Rassias stable with respect to $\psi \in C\left([0,T],\mathbb{R}_+\right)$ if there exists a real number $c_f > 0$ such that for each $\varepsilon > 0$ and for each $y \in AC\left([0,T],\mathbb{R}\right)$ solution of the inequality

$$\left|{}^{C}D_{0+}^{\alpha}y(t) - f\left(t,y(t)\right)\right| \leq \varepsilon\psi(t), \qquad t \in [0,T]$$

there exists a solution $u \in AC\left([0,T],\mathbb{R}\right)$ of the equation (6) with

$$|y(t)-u(t)| \leq c_f\psi(t)\varepsilon, \qquad t \in [0,T].$$

Definition 8. A function $y \in AC\left([0,T],\mathbb{R}\right)$ is a solution of the inequality (7) if and only if there exists a function $g \in C\left([0,T],\mathbb{R}\right)$ such that for every $t \in [0,T]$, $|g(t)| \leq \varepsilon$ and ${}^{C}D_{0+}^{\alpha}y(t) = f\left(t,y(t)\right) + g(t)$.

3. Existence and Uniqueness

In this section, we are concerned with the existence of a unique solution for the problem (1)–(2). Let us start by recalling what we mean by a solution.

Definition 9. A function $u \in AC\left([0,T],\mathbb{R}\right) \cap C\left([-r,T],\mathbb{R}\right)$ is said to be a solution of the initial value problem (1)–(2) if it satisfies the equation (1) on $[0,T]$ and the initial condition (2) on the small interval $[-r,0]$.

In the sequel, we introduce the following assumptions:

(H1) $f(t,u)$ is continuous, bounded for any $(t,u) \in [0,T] \times \mathbb{R}$ and there exists constant $L > 0$ such that for any $u, v \in \mathbb{R}$ and $t \in [0,T]$, we have

$$|f(t,u) - f(t,v)| \leq L\,|u - v|\,.$$

(H2) $p \in AC([0,T],\mathbb{R})$ such that $p(t) \neq 0$, $t \in [0,T]$, $h \in C([0,T],\mathbb{R})$, $\phi \in C^1([-r,r],\mathbb{R})$ with $p(0) = p_0$,

$$\phi(0) = \phi_0 \quad \text{and} \quad {}^{C}D^{\alpha}_{0+}\phi(t)\big|_{t=0} = \phi_\alpha, \tag{8}$$

where ϕ_0, p_0 and ϕ_α are real constants.

(H3) For $\sup\limits_{t\in[0,T]} |h(t)| = \eta$ and $\inf\limits_{t\in[0,T]} |p(t)| = q$, we have

$$k := T^{\alpha+\beta}\frac{[L+\eta]}{\Gamma(\alpha+\beta+1)\,q} < 1. \tag{9}$$

Now we convert the initial value problem to an integral equation which is also used in the existence and the stability studies.

Indeed, we are interested in the solution of the problem (1) with (8) on the interval $[0,T]$ in view of the supplementary data of $u(t)$ on the interval $[-r,0]$.

Lemma 10. *A function $u \in C([-r,T],\mathbb{R})$ is a solution of the following fractional integral equation for $t \in [0,T]$*

$$\begin{aligned} u(t) &= \phi_0 + p_0\phi_\alpha \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)}\mathrm{d}s \\ &\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)} \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}\left[f(\tau,u(\tau)) - h(\tau)\,u(\tau)\right]\mathrm{d}\tau\mathrm{d}s, \end{aligned} \tag{10}$$

with $u(t) = \phi(t)$ for $t \in [-r,0]$, if and only if u is a solution of the fractional two-orders initial value problem (1)–(2)*.*

Proof. First, we apply ${}^{C}D^{\alpha}_{0+}$ to (10) and obtain with ${}^{C}D^{\alpha}_{0+}\phi_0 = 0$

$${}^{C}D^{\alpha}_{0+}u(t) = \frac{1}{p(t)}I^{\beta}_{0+}\left[f(t,u(t)) - h(t)u(t)\right] + \frac{p_0}{p(t)}\phi_\alpha.$$

Then, we apply ${}^{C}D^{\beta}_{0+}$ to $p(t){}^{C}D^{\alpha}_{0+}u(t)$ to get (1). For $t = 0$, (10) coincides with the initial function $\phi_0 = u(0)$. Furthermore, under (H1)–(H2) we conclude that u is in $AC([0,T],\mathbb{R})$.

Conversely, we apply the fractional integral I^{β}_{0+} to (1) to obtain, in view of Lemma 4,

$$p(t){}^{C}D^{\alpha}_{0+}u(t) - p(0){}^{C}D^{\alpha}_{0+}u(0) = I^{\beta}_{0+}\left[f(t,u(t)) - h(t)u(t)\right]. \tag{11}$$

Using the fact that $u \in C([-r,T],\mathbb{R})$ and $\phi \in C^1([-r,r],\mathbb{R})$, we obtain

$${}^{C}D^{\alpha}_{0+}u(0) = {}^{C}D^{\alpha}_{0+}u(t)\big|_{t=0} = {}^{C}D^{\alpha}_{0+}\phi(t)\big|_{t=0} = \phi_\alpha.$$

Then, after dividing (11) by $p(t)$, we apply $I_{0^+}^{\alpha}$ to get

$$u(t) = \phi_0 + I_{0^+}^{\alpha}\left[\frac{1}{p(t)} I_{0^+}^{\beta}\left[f\left(t, u(t)\right) - h(t)u(t)\right] + \frac{p_0}{p(t)}\phi_\alpha\right],$$

where $u(0) = \phi(0) = \phi_0$, which is the solution and this completes the proof. □

Now, we give the existence result based on the Banach contraction fixed point theorem.

Theorem 11. *Assume that* (H1)–(H3) *are satisfied. Then the problem* (1)–(2) *has a unique solution.*

Proof. First, we denote by $X = C\left([0,T], \mathbb{R}\right)$ the Banach space of all continuous functions from $[0,T]$ into $\mathbb{R}$ with the sup norm $\|u\|_\infty = \sup_{t\in[0,T]} |u(t)|$

Define the operator $A\colon X \to X$ by

$$Au(t) = \phi_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)} \times \left[\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}\left[f\left(\tau, u\left(\tau\right)\right) - h\left(\tau\right)u\left(\tau\right)\right]\mathrm{d}\tau + p_0\phi_\alpha\right]\mathrm{d}s.$$

It is clear that the fixed points of the operator A are solutions of the problem (1), (8) with the additional condition (2).

Let us denote $\delta := \frac{\eta T^{\alpha+\beta}}{q\Gamma\left(\alpha+\beta+1\right)}$ which satisfies $0 < \delta < k$ defined by (9) and $M := \sup_{(t,u)\in[0,T]\times\mathbb{R}} |f\left(t,u\right)|$.

Then, we define the nonempty convex closed set of X as follows

$$B_R = \{u \in X : \|u - \phi_0\|_\infty \le R\}$$

such that

$$R \ge \left[\frac{\left[M + \eta\left|\phi_0\right|\right]T^{\alpha+\beta}}{q\Gamma\left(\alpha+\beta+1\right)} + \frac{\left|p_0\phi_\alpha\right|T^\alpha}{\Gamma(\alpha+1)q}\right]\frac{1}{(1-\delta)}. \tag{12}$$

To show that $AB_R \subset B_R$ for each $u \in B_R$

$$\begin{aligned}
&\|Au - \phi_0\|_\infty \\
&\le \sup_{t\in[0,T]}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)\left|p(s)\right|}\left[\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma\left(\beta\right)}\left|f\left(\tau,u\left(\tau\right)\right)\right| + \left|h\left(\tau\right)\right|\left|u\left(\tau\right)\right|\mathrm{d}\tau + \left|p_0\phi_\alpha\right|\right]\mathrm{d}s \\
&\le \frac{1}{\Gamma(\alpha)\Gamma\left(\beta+1\right)q}\left[M + \eta\left\|u-\phi_0\right\|_\infty + \eta\left|\phi_0\right|\right]\sup_{t\in[0,T]}\int_0^t (t-s)^{\alpha-1}s^\beta\mathrm{d}s \\
&\quad + \left|p_0\phi_\alpha\right|\frac{\sup_{t\in[0,T]} t^\alpha}{\Gamma(\alpha+1)q},
\end{aligned}$$

which yields

$$\begin{aligned}
&\|Au-\phi_0\|_\infty \\
&\le \frac{1}{\Gamma(\alpha)\Gamma(\beta+1)\,q}\left[M+\eta R+\eta\left|\phi_0\right|\right]\sup_{t\in[0,T]}\int_0^1 (t-\theta t)^{\alpha-1}(\theta t)^{\beta}\,t\mathrm{d}\theta \\
&\quad+\frac{\left|p_0\phi_\alpha\right|T^\alpha}{\Gamma(\alpha+1)q} \\
&\le \frac{\eta T^{\alpha+\beta}}{q\Gamma(\alpha+\beta+1)}R+\frac{\left[M+\eta\left|\phi_0\right|\right]T^{\alpha+\beta}}{q\Gamma(\alpha+\beta+1)}+\frac{\left|p_0\phi_\alpha\right|T^\alpha}{\Gamma(\alpha+1)q}.
\end{aligned}$$

We conclude from (12) that

$$\|Au-\phi_0\|_\infty\le R.$$

B_R is stable by A. We proceed to prove that A is a contraction mapping. For each $u,v\in B_R$ and for all $t\in[0,T]$ we have

$$\begin{aligned}
&\left|(Au)(t)-(Av)(t)\right| \\
&\le \sup_{t\in[0,T]}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)\left|p(s)\right|} \\
&\qquad\times\left[\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}\left(\left|f(\tau,u(\tau))-f(\tau,v(\tau))\right|+\left|h(\tau)\right|\left|u(\tau)-v(\tau)\right|\right)\mathrm{d}\tau\right]\mathrm{d}s \\
&\le \sup_{t\in[0,T]}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)\left|p(s)\right|}\left[\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}\left[L+\eta\right]\left|u(\tau)-v(\tau)\right|\mathrm{d}\tau\right]\mathrm{d}s \\
&\le \sup_{t\in[0,T]} t^{\alpha+\beta}\frac{\left[L+\eta\right]}{\Gamma(\alpha+\beta+1)\,q}\left\|u-v\right\|_\infty.
\end{aligned}$$

Thus,

$$\|Au-Av\|_\infty\le k\left\|u-v\right\|_\infty.$$

A is a contraction by (9). The conclusion of the theorem follows by the Banach fixed point theorem. This completes the proof. □

4. Ulam stability

In this section, we study two types of Ulam stability of the two-orders fractional differential equation (1) which are Ulam-Hyers and Ulam-Hyers-Rassias stabilities.

Lemma 12. *If $y\in AC([0,T],\mathbb{R})$ is a solution of the fractional differential inequality for each $\varepsilon>0$*

$$\left|{}^{C}D^{\beta}_{0+}\left(p(t)\,{}^{C}D^{\alpha}_{0+}y(t)\right)+h(t)y(t)-f(t,y(t))\right|<\varepsilon \tag{13}$$

and the initial condition (2) *then y is a solution of the following integral inequality*

$$\Bigg| y(t) - \phi_0 - p_0\phi_\alpha \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)} \mathrm{d}s - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)} \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} \left[f(\tau, y(\tau)) - h(\tau)\, y(\tau) \right] \mathrm{d}\tau \mathrm{d}s \Bigg| < \frac{T^{\alpha+\beta}}{q\Gamma(\alpha+\beta+1)}\varepsilon.$$

Proof. Let $y \in AC([0,T],\mathbb{R})$ be a solution of the inequality (13) for each $\varepsilon > 0$. Then, from Definition 8 and Lemma 10 for some continuous function $g(t)$ such that $|g(t)| < \varepsilon$, $t \in [0,T]$, we have

$$y(t) = \phi_0 + p_0\phi_\alpha \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)} \mathrm{d}s + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)} \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} \left[f(\tau, y(\tau)) - h(\tau)\, y(\tau) + g(\tau) \right] \mathrm{d}\tau \mathrm{d}s.$$

Then, we use properties of I_{0+}^{α} to get

$$\left| I_{0+}^{\alpha}\left(\frac{1}{p(t)} I_{0+}^{\beta} g(t) \right) \right| \leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)} \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} |g(\tau)| \mathrm{d}\tau \mathrm{d}s < \varepsilon \frac{T^{\alpha+\beta}}{q\Gamma(\alpha+\beta+1)},$$

which is (12). The proof is complete. □

Theorem 13. *Assume that the assumptions* (H1)–(H3) *hold. Then the problem* (1)–(2) *is Ulam-Hyers stable.*

Proof. Under (H1)–(H3), the problem (1)–(2) has a unique solution in $AC([0,T],\mathbb{R}) \cap C([-r,T],\mathbb{R})$. Let $y \in AC([0,T],\mathbb{R})$ be a solution of the inequality (13), then for each $t \in [0,T]$

$$|y(t) - u(t)| \leq \Bigg| y(t) - \phi_0 - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)} \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} \left[f(\tau, y(\tau)) - h(\tau)\, y(\tau) \right] \mathrm{d}\tau \mathrm{d}s - p_0\phi_\alpha \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)} \mathrm{d}s \Bigg|$$

$$
\begin{aligned}
&+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)} \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} \left|f(\tau, y(\tau)) - f(\tau, u(\tau))\right| \mathrm{d}\tau \mathrm{d}s \\
&+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)} \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} \left|h(\tau)\right| \left|y(\tau) - u(\tau)\right| \mathrm{d}\tau \mathrm{d}s \\
\leq{}& \varepsilon \frac{T^{\alpha+\beta}}{q\Gamma(\alpha+\beta+1)} + T^{\alpha+\beta} \frac{[L+\eta]}{\Gamma(\alpha+\beta+1)q} \|y-u\|_\infty .
\end{aligned}
$$

Thus, in view of (H3)

$$
\|y-u\|_\infty \leq \varepsilon \frac{T^{\alpha+\beta}}{q\Gamma(\alpha+\beta+1)(1-k)}.
$$

Then, there exists a real number $K_f = \dfrac{T^{\alpha+\beta}}{q\Gamma(\alpha+\beta+1)(1-k)} > 0$ such that

$$
|y(t) - u(t)| \leq K_f \varepsilon.
$$

Thus (1)–(2) has the Ulam-Hyers stability, which completes the proof. □

In the next, we introduce the following function

(H4) $\psi \in C([0,T], \mathbb{R})$ an increasing function which satisfies the property $I_{0^+}^{\gamma} \psi(t) \leq \lambda_{\psi,\gamma} \psi(t)$, $0 < \gamma < 1$ for some constant $\lambda_{\psi,\gamma} > 0$.

Lemma 14. *Assume that ψ satisfies* (H4)*. If $y \in AC([0,T], \mathbb{R})$ is a solution of the inequality*

$$
\left|{}^{C}D_{0^+}^{\beta}\left(p(t)^{C}D_{0^+}^{\alpha} y(t)\right) + h(t)y(t) - f(t, y(t))\right| < \varepsilon\psi(t) \quad \text{for each } \varepsilon > 0 \tag{14}
$$

and the initial condition (2) *then y is a solution of the following integral inequality*

$$
\begin{aligned}
&\Bigg| y(t) - \phi_0 - p_0\phi_\alpha \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)} \mathrm{d}s \\
&\quad - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)} \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} \left[f(\tau, y(\tau)) - h(\tau)y(\tau)\right] \mathrm{d}\tau \mathrm{d}s \Bigg| \\
&< \varepsilon \frac{\lambda_\varphi^2}{q} \psi(t).
\end{aligned} \tag{15}
$$

Proof. Let $y \in AC([0,T], \mathbb{R})$ be a solution of the inequality (14) for each $\varepsilon > 0$. From Definition 8 and Lemma 10, for some continuous function $g(t)$ such that

$|g(t)| < \varepsilon\psi(t)$ for each $\varepsilon > 0$, $t \in [0, T]$,

$$\begin{aligned}
&\left| y(t) - \phi_0 - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)} \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} \left[f(\tau, y(\tau)) - h(\tau) y(\tau) \right] \mathrm{d}\tau \mathrm{d}s \right. \\
&\qquad \left. - p_0\phi_\alpha \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)} \mathrm{d}s \right| \\
&\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)} \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} |g(\tau)| \mathrm{d}\tau \mathrm{d}s < \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)} \varepsilon I_{0^+}^{\beta} \psi(s) \mathrm{d}s \\
&\le \varepsilon \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)} \lambda_{\psi,\beta} \psi(s) \mathrm{d}s \le \varepsilon \frac{\lambda_{\psi,\beta}\lambda_{\psi\alpha}}{q} \psi(t) < \varepsilon \frac{\lambda_\psi^2}{q} \psi(t),
\end{aligned}$$

where $\lambda_\psi = \max(\lambda_{\psi,\alpha}, \lambda_{\psi,\beta})$. This completes the proof. □

Theorem 15. *Assume that the assumptions* (H1)–(H4) *hold, then the problem* (1)–(2) *is Ulam-Hyers-Rassias stable with respect to* ψ.

Proof. Under (H1)–(H3), the problem (1)–(2) has a unique solution in $AC([0, T], \mathbb{R}) \cap C([-r, T], \mathbb{R})$. Let $y \in AC([0, T], \mathbb{R})$ be a solution of the inequality (14) then for each $t \in [0, T]$

$$\begin{aligned}
&|y(t) - u(t)| \\
&= \left| y(t) - \phi_0 - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)} \right. \\
&\qquad \left. \times \left[\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} \left[f(\tau, u(\tau)) - h(\tau) u(\tau) \right] \mathrm{d}\tau - p_0\phi_\alpha \right] \mathrm{d}s \right| \\
&\le \varepsilon \frac{\lambda_\psi^2}{q} \psi(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)} \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} \\
&\qquad \times \left(|f(\tau, u(\tau)) - f(\tau, y(\tau))| + |h(\tau)| \, |u(\tau) - y(\tau)| \right) \mathrm{d}\tau \mathrm{d}s \\
&\le \varepsilon \frac{\lambda_\psi^2}{q} \psi(t) + T^{\alpha+\beta} \frac{[L + \eta]}{\Gamma(\alpha+\beta+1)\, q} \|y - u\|_\infty .
\end{aligned}$$

Hence, it follows that there exists a real number $H_f = \dfrac{\lambda_\psi^2}{q(1-k)} > 0$ such that

$$|y(t) - u(t)| \le \varepsilon \frac{\lambda_\psi^2}{q(1-k)} \psi(t) = H_f \varepsilon \psi(t), \qquad t \in [0, T].$$

This gives the wanted result and completes the proof. □

5. Example

Consider the following nonlinear problem

$$\begin{cases} {}^{C}D_{0^+}^{\frac{1}{2}}\left(\frac{1}{1+t}{}^{C}D_{0^+}^{\frac{1}{3}}u(t)\right)+\frac{\sin t}{5}\cdot u(t) & \\ =\frac{t}{100+t}\cos u(t), & t\in[0,1], \\ u(t)=\mathrm{e}^{t}, & t\in[-0,005,0], \end{cases} \tag{16}$$

where

$$\alpha=\frac{1}{3},\qquad \beta=\frac{1}{2},\qquad p(t)=\frac{1}{(1+t)},\qquad h(t)=\frac{\sin t}{5},$$
$$f(t,u(t))=\frac{t}{100+t}\cos u(t),\qquad \phi(t)=\mathrm{e}^{t},$$

which satisfy clearly (H1)–(H2) for any $u,v\in\mathbb{R}$

$$|f(t,u)-f(t,v)|\leq\frac{t}{100+t}|\cos u-\cos v|\leq\frac{1}{100}|\cos u-\cos v|.$$

The unique solution exists for

$$L=\frac{1}{100},\qquad \eta=\sup_{t\in[0,1]}|h(t)|=\frac{1}{5},\qquad q=\inf_{t\in[0,1]}|p(t)|=\frac{1}{2},$$

satisfying the condition (9)

$$k=\frac{[L+\eta]}{\Gamma(\alpha+\beta+1)\,q}=\frac{\frac{1}{100}+\frac{1}{5}}{\Gamma\left(\frac{1}{3}+\frac{1}{2}+1\right)\frac{1}{2}}=0.446\,50<1. \tag{17}$$

It follows from Theorem 13 that the problem (16) is Ulam-Hyers stable on $[0,1]$.

Now, we choose $\psi(t)=t^3$ which satisfies (H4) and in view of (4) we have

$$I_{0^+}^{\gamma}\psi(t)=\frac{\Gamma(4)}{\Gamma(4+\gamma)}t^{3+\gamma}\leq\frac{3\Gamma(3)}{(3+\gamma)\Gamma(3+\gamma)}t^3\leq\frac{1}{\Gamma(\gamma+1)}t^3,\quad 0<\gamma<1.$$

For $\alpha=\frac{1}{3}$ and $\beta=\frac{1}{2}$ we have

$$\lambda_{\psi,\alpha}=\frac{1}{\Gamma\left(\frac{1}{3}+1\right)}=1.119\,8,\qquad \lambda_{\psi,\beta}=\frac{1}{\Gamma\left(\frac{1}{2}+1\right)}=1.128\,4$$

then we take $\lambda_{\psi}=\max(\lambda_{\psi,\alpha},\lambda_{\psi,\beta})=1.128\,4$ to get (15) satisfied. Hence Ulam–Hyers-Rassias stability with respect to ψ is obtained.

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