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ABOUT A MEAN VALUE THEOREM FOR PRODUCT OF TWO FUNCTIONS AND SOME RELATED RESULTS

D. Ş. MARINESCU and M. MONEA

Abstract. The aim of this paper is to present some mean value theorems involving Riemann integral. The starting point is represented by the recent result due to Mingarelli et al. We prove a stronger result and obtain the Mingarelli's result as consequence. We show the applicability of our results when solving some problems recently appeared in some mathematical journals.

1. Introduction

The mean value theorems represent one of the most useful mathematical analysis tools. Starting from Rolle or Lagrange Theorem, we can find more results, generalizations or extensions. Sahoo and Riedel's book [**10**] present a large collection of old and new mean value theorems. The readers can consult the references [**4**], [**5**], [**9**], [**11**], or [**12**] to find some recent results.

In [**8**], Mingarelli et al. proved the following integral mean value theorem.

Theorem 1.1. *Let $f, g\ [0,1] \to \mathbb{R}$ be two continuous functions and $w\ [0,1] \to [0,\infty)$ be a continuously differentiable and non-constant function such that $w'(x) \geq 0$ for any $x \in [0,1]$. Then there exists $c \in (0,1)$ such that*

$$\int_0^1 f(x)\mathrm{d}x \cdot \int_0^c w(x) \cdot g(x)\mathrm{d}x = \int_0^1 g(x)\mathrm{d}x \cdot \int_0^c w(x) \cdot f(x)\mathrm{d}x.$$

The aim of this paper is to present some results related to Theorem 1.1. Our main result is given in Theorem 2.1 in Section 2. Also, some consequences of this theorem are presented in that section, including Theorem 1.1 under less restrictive conditions on w. In Section 3, we discuss some recent results included in the references list of this paper. We present a new proof or a new version of these results (Corollaries 3.1 and 3.2).

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2. Generalizations of the theorem 1.1

First, we recall that $C[0,1]$ represents the set of all real continuous functions defined on $[0,1]$, and $C^1[0,1]$ represents the set of all real continuously differentiable functions defined on the same interval. Denote W, the set of all non-constant and non-decreasing functions $w\,[0,1] \to [0,\infty)$. The main result of this paper is following now.

Theorem 2.1. *Let $w \in W$ and $f \in C^1[0,1]$ be such that $f(0) = f(1)$. Then there exists $c \in (0,1)$ such that*

$$\int_0^c w(x)f'(x)\mathrm{d}x = 0. \tag{2.1}$$

Proof. We can assume that the function f is non-constant, otherwise the result is trivial. First, we consider the case $f(0) = 0$. We introduce the function $F[0,1] \to \mathbb{R}$ defined by

$$F(t) = \int_0^t w(x)f'(x)\mathrm{d}x$$

for any $t \in [0,1]$. Since f is continuous and w is non-decreasing, then f is Riemann-Stieltjes integrable on $[0,t]$ with respect to w for any $t \in [0,1]$ (see [**1**, Theorem 7.27]). Also, w is Riemann-Stieltjes integrable on $[0,t]$ with respect to f (see [**1**, Theorem 7.6]). Moreover, f is continuously differentiable, so we have

$$\int_0^t w(x)f'(x)\mathrm{d}x = \int_0^t w(x)\mathrm{d}f(x).$$

(see [**1**, Theorem 7.8]). Due to these facts, we can use the integration by parts formula and we obtain

$$F(t) = \int_0^t w(x)\mathrm{d}f(x) = w(t)f(t) - w(0)f(0) - \int_0^t f(x)\mathrm{d}w(x).$$

Since we assume $f(0) = 0$, we get

$$F(t) = w(t)f(t) - \int_0^t f(x)\mathrm{d}w(x), \qquad t \in [0,1], \tag{2.2}$$

and

$$F(1) = -\int_0^1 f(x)\mathrm{d}w(x). \tag{2.3}$$

Now, we can assume that $w(x) > 0$ for any $x \in (0,1]$. If not, we find $a \in (0,1]$ such that $w(a) = 0$ and from monotonicity of w, we get $w(x) = 0$ for any $x \in [0,a]$ and the equality (2.1) holds for any point $c \in (0,a)$.

Further, we assume by contradiction that $F(t) \neq 0$ for any $t \in (0,1)$. The function F is continuous (see [**1**, Theorem 7.32]), so we can assume that F is positive on $(0,1)$. Then, the equality (2.3) goes to

$$F(1) = \int_0^1 f(x)\mathrm{d}w(x) \leq 0.$$

From Weierstrass theorem, there exists $a \in (0,1)$ such that $\min\limits_{x\in[a,b]} f(x) = f(a)$ and $f(a) < 0$. Further, we have

$$F(a) = f(a)w(a) - \int_0^a f(x)\mathrm{d}w(x).$$

Since

$$\int_0^a f(x)\mathrm{d}w(x) \geq f(a)\int_0^a \mathrm{d}w(x) = f(a)w(a) - f(a)w(0),$$

we obtain

$$F(a) \leq f(a)w(0) \leq 0.$$

This contradicts the fact that F is positive on $[0,1]$ and concludes this part of the proof.

Now, we remove the condition $f(0) = 0$. We define the function $g\,[0,1] \to \mathbb{R}$ by

$$g(x) = f(x) - f(0)$$

for any $x \in [0,1]$. By using the previous reasoning, we find $c \in (0,1)$ such that

$$\int_0^c w(x)g'(x)\mathrm{d}x = 0.$$

Since $g'(x) = f'(x)$ for any $x \in [0,1]$, we get the desired result. □

We remark that this result fails for constant functions w. For example, if we choose $f(x) = x^2 - x$ and $w(x) = 1$ for any $x \in [0,1]$, then for any $c \in (0,1)$, we obtain

$$\int_0^c w(x)f'(x)\mathrm{d}x = f(c) - f(0) = c^2 - c < 0.$$

As a consequence of Theorem 2.1, we obtain the following results.

Corollary 2.2. *Let $w \in W$ and $h \in C[0,1]$ such that $\int_0^1 h(t)\mathrm{d}t = 0$. Then there exists $c \in (0,1)$ such that*

$$\int_0^c w(x)h(x)\mathrm{d}x = 0.$$

Proof. We apply Theorem 2.1. to the functions w and $f\,[0,1] \to \mathbb{R}$, defined by

$$f(x) = \int_0^x h(t)\mathrm{d}t$$

for any $x \in [0,1]$. □

Corollary 2.3. *Let $w \in W$ and $f, g \in C^1[0,1]$ be such that $f(0) = g(0) = 0$. Then there exists $c \in (0,1)$ such that*

$$f(1)\cdot\int_0^c w(x)g'(x)\mathrm{d}x = g(1)\cdot\int_0^c w(x)f'(x)\mathrm{d}x.$$

Proof. If $f(1) = g(1) = 0$, the conclusion is trivial. Further, we assume that $g(1) \neq 0$. Let the function $h\,[0,1] \to \mathbb{R}$, be defined by the formula

$$h(x) = \frac{f(1)}{g(1)} \cdot g(x) - f(x).$$

We have $h \in C^1[0,1]$ and $h(0) = h(1) = 0$. From Theorem 2.1, there exists $c \in (0,1)$ such that

$$\int_0^c w(x)h'(x)\mathrm{d}x = 0,$$

which is equivalent to

$$\frac{f(1)}{g(1)} \int_0^c w(x)g'(x)\mathrm{d}x = \int_0^c w(x)f'(x)\mathrm{d}x,$$

and the conclusion follows now. □

Observe that the function w from Theorem 1.1 is continuously differentiable with a non-negative derivative. This means that this function is non-decreasing. Due to this fact, we can weaken the hypothesis of this theorem to obtain the following form.

Proposition 2.4. *Let $w \in W$ and $f, g \in C[0,1]$. Then there exists $c \in (0,1)$ such that*

$$\int_0^1 f(x)\mathrm{d}x \cdot \int_0^c w(x)g(x)\mathrm{d}x = \int_0^1 g(x)\mathrm{d}x \cdot \int_0^c w(x)f(x)\mathrm{d}x. \tag{2.4}$$

Proof. We consider the functions $F, G\,[0,1] \to \mathbb{R}$ defined by $F(t) = \int_0^t f(x)\mathrm{d}x$ and $G(t) = \int_0^t g(x)\mathrm{d}x$. These functions satisfy the hypothesis of Corollary 2.3 and we find $c \in (0,1)$ such that

$$F(1) \cdot \int_0^c w(x)G'(x)\mathrm{d}x = G(1) \cdot \int_0^c w(x)F'(x)\mathrm{d}x.$$

Since $F(1) = \int_0^1 f(x)\mathrm{d}x$, $G(1) = \int_0^1 g(x)\mathrm{d}x$, and $F'(t) = f(t)$, $G'(t) = g(t)$, we obtain the conclusion. □

If we choose $w(x) = x$ for any $x \in [0,1]$, we find $c \in (0,1)$ such that

$$\int_0^1 f(x)\mathrm{d}x \cdot \int_0^c xg(x)\mathrm{d}x = \int_0^1 g(x)\mathrm{d}x \cdot \int_0^c xf(x)\mathrm{d}x.$$

This equality, proposed in [**6**], represents the starting point of Mingarelli's paper.

3. Solutions of some recently posed problems

In this section we present two problems, recently published in the American Mathematical Monthly. In fact, we obtain the solutions for these problems as consequences of our work.

The next corollary represents a generalization of the mean value theorem from [**2**].

Corollary 3.1. *Let* $w \in W \cap C^1[0,1]$ *with* $w(0) = 0$ *and* $f \in C^1[0,1]$ *with* $f(0) = f(1)$*. Then, there exists* $c \in (0,1)$ *such that*

$$\int_0^c w'(x)f(x)\mathrm{d}x = w(c)f(c).$$

Proof. Due to the integration by parts formula, we obtain

$$\int_0^t w'(x)f(x)\mathrm{d}x = w(t)f(t) - \int_0^t w(x)f'(x)\mathrm{d}x$$

for any $t \in [0,1]$. By using Theorem 2.1, we find $c \in (0,1)$ such that

$$\int_0^c w(x)f'(x)\mathrm{d}x = 0,$$

and the conclusion follows. □

If we choose $w(x) = x^2$ for any $x \in [0,1]$, we find a point $c \in (0,1)$ such that

$$\int_0^c 2xf(x)\mathrm{d}x = c^2 f(c),$$

which is the equality from [**2**].

Corollary 3.2 ([**7**, C. Lupu])**.** *Let* $\varphi \in C^1[0,1]$ *with* $\varphi(0) = 0$ *and* $\varphi(1) = 1$*, and suppose that* $\varphi'(x) \neq 0$ *for any* $x \in [0,1]$*. Let* $f \in C[0,1]$ *be such that*

$$\int_0^1 f(x)\mathrm{d}x = \int_0^1 \varphi(x)f(x)\mathrm{d}x.$$

Then there exists $c \in (0,1)$ *such that*

$$\int_0^c \varphi(x)f(x)\mathrm{d}x = 0.$$

Proof. We observe that under the hypothesis, the function φ is increasing and positive. Now, let $F\,[0,1] \to \mathbb{R}$ be defined by

$$F(x)\int_0^x f(t)\mathrm{d}t.$$

By using the integration by parts formula, we obtain

$$\begin{aligned}\int_0^1 \varphi(x)f(x)\mathrm{d}x &= \varphi(1)F(1) - \varphi(0)F(0) - \int_0^1 \varphi'(x)F(x)\mathrm{d}x \\ &= \int_0^1 f(x)\mathrm{d}x - \int_0^1 \varphi'(x)F(x)\mathrm{d}x.\end{aligned}$$

The assumption yields

$$\int_0^1 \varphi'(x)F(x)\mathrm{d}x = 0.$$

Since $\varphi' \cdot F \in C[0,1]$, by the mean value theorem for Riemann integral, there exists $\alpha \in (0,1)$ such that

$$0 = \int_0^1 \varphi'(x)F(x)\mathrm{d}x = \varphi'(\alpha)F(\alpha).$$

Since $\varphi'(\alpha) \geq 0$, we conclude that $F(\alpha) = 0$, which means that

$$\int_0^\alpha f(x)\mathrm{d}x = 0.$$

Further, we consider the functions $g[0,1] \to \mathbb{R}$, $g(x) = f(\alpha x)$, and $u[0,1] \to [0,\infty)$, $u(x) = \varphi(\alpha x)$. It is clear that $g \in C[0,1]$ and $u \in W$. Moreover,

$$\int_0^1 g(x)\mathrm{d}x = \int_0^1 f(\alpha x)\mathrm{d}x = \frac{1}{\alpha}\int_0^\alpha f(y)\,\mathrm{d}y = 0.$$

Corollary 2.2 gives us a point $\beta \in (0,1)$ such that

$$0 = \int_0^\beta u(x)g(x)\,\mathrm{d}x = \int_0^\beta \varphi(\alpha x)f(\alpha x)\,\mathrm{d}x = \frac{1}{\alpha}\int_0^{\alpha\beta} \varphi(y)f(y)\mathrm{d}y.$$

Now, the proof is complete if we choose $c = \alpha\beta$. □

We conclude our paper with the next corollary.

Corollary 3.3 ([**3**, P. C. Le Van])**.** *Let n be a positive integer and $f \in C[0,1]$ be such that $\int_0^1 f(x)\mathrm{d}x = 0$. Then there exists $c \in (0,1)$ such that*

$$n\int_0^c x^n f(x)\mathrm{d}x = c^{n+1}f(c).$$

Proof. Applying Corollary 2.2 to the functions $w(x) = x^n$ and f, we find $d \in (0,1)$ such that

$$\int_0^d x^n f(x)\mathrm{d}x = 0.$$

We consider the function $F\,[0,1] \to \mathbb{R}$,

$$F(t) = \begin{cases} \dfrac{1}{t^n}\displaystyle\int_0^t x^n f(x)\mathrm{d}x, & t \in (0,1], \\ 0, & t = 0. \end{cases}$$

We have $\lim\limits_{x\to 0^+} F(x) = 0$, so the function F is continuous on $[0,d]$. Moreover, $F(d) = 0$ and F is differentiable on $(0,1)$. Then there exists $c \in (0,d)$ such that $F'(c) = 0$.

Since

$$F'(t) = \frac{t^n f(t)\cdot t^n - nt^{n-1}\cdot\int_0^t x^n f(x)\mathrm{d}x}{t^{2n}}$$
$$= \frac{t^{n+1}f(t) - n\int_0^t x^n f(x)\mathrm{d}x}{t^{n+1}}$$

for any $t \in [0,1]$, the equality $F'(c) = 0$ yields the conclusion. □

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D. Ş. Marinescu, National College "IANCU DE HUNEDOARA", Hunedoara, Romania,
e-mail: marinescuds@gmail.com

M. Monea, National College "DECEBAL", Deva, Romania,
e-mail: mihaimonea@yahoo.com