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ON TWO-α-CONVEX SEQUENCES OF ORDER THREE

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Abstract. The class of convex sequences came across in several branches of mathematics as well as their generalizations. The present paper introduces a new classes of convex sequences, the class of two-α-convex sequences of order three. Moreover, the characterization of sequences belonging to this class is shown.

1. Introduction

Convex sequences as entirety can be considered as one of the most important subclasses of the class of real sequences. This class is raised as a result of some efforts to solve several problems in mathematics. Of course, the sequences that belong to that class have useful applications in some branches of mathematics, in particular, in mathematical analysis. For instance, such sequences are widely used in theory of inequalities (see [**16**], [**9**], [**10**]), in absolute summability of infinite series (see [**1**], [**2**]), and in theory of Fourier series, related to their uniform convergence and the integrability of their sum functions (see as example [**8**], page 587). Here, in this paper, we are going to introduce a new class of "convex" sequences, which indeed generalizes a class of sequences introduced previously by others. To do this, we need first to recall some notations and notions as follows.

Let $(a_n)_{n=0}^{\infty}$ be a real sequence. It is previously defined that

$$\triangle^1 a_n = a_{n+1} - a_n, \quad \triangle^2 a_n = \triangle(\triangle a_n), \quad \triangle^3 a_n = \triangle(\triangle^2 a_n), \qquad n = 0, 1, \dots,$$

and throughout the paper, we shall write $\triangle a_n$ instead of $\triangle^1 a_n$.

The following definition presents the concept of convexity of order three of a sequence.

Definition 1.1. A sequence $(a_n)_{n=0}^{\infty}$ is said to be convex of order three if

$$\triangle^3 a_n \geq 0$$

for all $n \in \{0, 1, 2, \dots\}$.

Next Lemma has been proved by Gh. Toader, which characterizes the convex sequences of order three, and can be simplified as follows.

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Lemma 1.1 ([**12**])**.** *The sequence* $(a_n)_{n=0}^{\infty}$ *is convex of order three if and only if*

$$a_n = \frac{1}{2}\sum_{k=0}^{n}(n-k+2)(n-k+1)b_k$$

with $b_k \geq 0$ *and* $k \geq 3$.

Various generalizations of convexity heve benn studied by many authors, for instance, p-convexity (see [**7**]), (p,q)-convexity (see [**6**]), $(p,q;r)$-convexity, and α-convexity of higher order (see [**3**]–[**5**]).

Two other classes of sequences, the so-called two-starshaped sequences of order three and α-convex sequences were introduced in [**13**] and [**11**].

Indeed, throughout this paper be $\alpha \in [0,1]$ be.

Definition 1.2. A sequence $(a_n)_{n=0}^{\infty}$ is called α-convex if the sequence

$$\Big(\alpha(a_{n+1}-a_n)+(1-\alpha)\frac{a_n-a_0}{n}\Big)_{n=1}^{\infty}$$

is increasing.

Definition 1.3. A sequence $(a_n)_{n=0}^{\infty}$ is called two-starshaped of order three if it satisfies the relation

$$\frac{a_{n+3}-a_0}{n+3} \geq \frac{a_{n+2}-a_1}{n+1}$$

for $n \geq 0$.

Next Lemma characterizes sequences that are two-starshaped of order three.

Lemma 1.2 ([**13**])**.** *The sequence* $(a_n)_{n=0}^{\infty}$ *is a starshaped two-starshaped of order three if and only if*

$$a_n = n(n-1)\sum_{k=2}^{n} d_k + nd_1 - (n-1)d_0,$$

where $d_k \geq 0$ *and* $k \geq 3$.

Now we are able to introduce a new class of sequences by the following definition.

Definition 1.4. A sequence $(a_n)_{n=0}^{\infty}$ is called two-α-convex of order three if the sequence

$$\Big(\alpha \triangle^3 a_n + (1-\alpha)\Big(\frac{a_{n+3}-a_0}{n+3}-\frac{a_{n+2}-a_1}{n+1}\Big)\Big)_{n=0}^{\infty}$$

is nonnegative for all $n \in \{0,1,2,\dots\}$.

Remark 1.1. We note that for $\alpha = 1$,a two-α-convex sequence of order three is the same as its convexity of order three, while for $\alpha = 0$, a two-α-convex sequence of order three is the same as its two-starshapedness of order three.

The main aim of this paper is to characterize the two-α-convex sequences of order three as well as to show some of their basic properties.

2. Main Results

At first, we begin with the following statement. It gives some equivalent conditions under which a sequence is a two-α-convex sequence of order three.

Theorem 2.1. *The sequence $(a_n)_{n=0}^{\infty}$, is two-α-convex sequence of order three if and only if*

$$(a_n - a_0 + \alpha[n(a_n - 2a_{n-1} + a_{n-2}) - (a_n - a_0)])_{n=0}^{\infty}, \qquad (a_{-1} := a_{-2} := 0),$$

is a two starshaped sequence of order three and $a_1 \geq 2a_0 + \triangle^2 a_n$, $n \in \{0, 1, \dots\}$.

Proof. For the sake of brevity, we denote

$$A_n := a_n - a_0 + \alpha[n(a_n - 2a_{n-1} + a_{n-2}) - (a_n - a_0)], \qquad n \in \{0, 1, \dots\},$$

which can be rewritten as

$$A_n = \alpha n(a_n - 2a_{n-1} + a_{n-2}) + (1-\alpha)(a_n - a_0), \qquad n \in \{0, 1, \dots\}.$$

Since $A_0 = 0$, $A_1 = a_1 - (1+\alpha)a_0$, and $a_1 \geq 2a_0 + \triangle^2 a_n$, $n \in \{0, 1, \dots\}$, we have

$$\begin{aligned}
& \frac{A_{n+3} - A_0}{n+3} - \frac{A_{n+2} - A_1}{n+1} \geq 0 \\
\Longleftrightarrow \quad & \frac{\alpha(n+3)(a_{n+3} - 2a_{n+2} + a_{n+1}) + (1-\alpha)(a_{n+3} - a_0)}{n+3} \\
& - \frac{\alpha(n+2)(a_{n+2} - 2a_{n+1} + a_n) + (1-\alpha)(a_{n+2} - a_0) - a_1 + (1+\alpha)a_0}{n+1} \geq 0 \\
\Longleftrightarrow \quad & \alpha(a_{n+3} - 2a_{n+2} + a_{n+1}) + (1-\alpha)\frac{a_{n+3} - a_0}{n+3} \\
& - \alpha(a_{n+2} - 2a_{n+1} + a_n) - (1-\alpha)\frac{a_{n+2} - a_1}{n+1} \\
& - \frac{\alpha(a_{n+2} - 2a_{n+1} + a_n) + (1-\alpha)(a_1 - a_0) - a_1 + (1+\alpha)a_0}{n+1} \geq 0 \\
\Longleftrightarrow \quad & \alpha \triangle^3 a_n + (1-\alpha)\left(\frac{a_{n+3} - a_0}{n+3} - \frac{a_{n+2} - a_1}{n+1}\right) + \alpha\frac{a_1 - 2a_0 - \triangle^2 a_n}{n+1} \geq 0.
\end{aligned}$$

for all $n \in \{0, 1, 2, \dots\}$.

The proof is completed. □

Next theorem characterizes a type of two-α-convex sequence of order three under some natural conditions.

Theorem 2.2. *The sequence $(a_n)_{n=0}^{\infty}$, is two-α-convex sequence of order three if and only if it may be represented by*

$$a_n = n(n-1)\sum_{k=2}^{n} d_k + nd_1 - (n-1)d_0 \qquad \textit{for } n \geq 2, \tag{1}$$

and $a_0 = d_0$, $a_1 = d_0$ with

$$d_{n+3} \geq 2\left[1 - \frac{2\alpha+1}{\alpha(n+2)+1}\right](\triangle d_{n+1}), \tag{2}$$

and $d_n \geq 0$, $n \in \{3, 4, \dots\}$.

Proof. On one hand, taking (1) into account as in [**13**, page 4], we easily get

$$\triangle^3(a_n) = (n+2)(n+3)d_{n+3} - 2n(n+2)d_{n+2} + n(n-1)d_{n+1} \tag{3}$$

and

$$\frac{a_{n+3}-a_0}{n+3} - \frac{a_{n+2}-a_1}{n+1} = (n+2)d_{n+3}. \tag{4}$$

Based on (3) and (4), we obtain

$$\begin{aligned}&\alpha\triangle^3(a_n) + (1-\alpha)\Big(\frac{a_{n+3}-a_0}{n+3} - \frac{a_{n+2}-a_1}{n+1}\Big)\\ &= (n+2)[\alpha(n+2)+1]d_{n+3} - 2\alpha n(n+2)d_{n+2} + \alpha n(n-1)d_{n+1}.\end{aligned}$$

Subsequently, it follows that

$$\alpha\triangle^3(a_n) + (1-\alpha)\Big(\frac{a_{n+3}-a_0}{n+3} - \frac{a_{n+2}-a_1}{n+1}\Big) \geq 0$$

if and only if

$$d_{n+3} \geq 2\Big[1 - \frac{2\alpha+1}{\alpha(n+2)+1}\Big]d_{n+2} - \frac{\alpha n(n-1)}{(n+2)[\alpha(n+2)+1]}d_{n+1}.$$

Last condition can be estimated as

$$d_{n+3} \geq 2\Big[1 - \frac{2\alpha+1}{\alpha(n+2)+1}\Big]\left(d_{n+2} - d_{n+1}\right),$$

since $d_k \geq 0$ and

$$2\Big[1 - \frac{2\alpha+1}{\alpha(n+2)+1}\Big] > \frac{\alpha n(n-1)}{(n+2)[\alpha(n+2)+1]},$$

which can be simplified in the form $n+3>0$, which is always true.

The proof is completed. □

Using the above statement, we are able to prove an inclusion theorem, pertaining to a two-α-convex sequence of order three, and of course, based on different values of α.

Theorem 2.3. *Let the sequence $(d_n)_{n=3}^{\infty}$ in the representation* (1) *be increasing. If the sequence $(a_n)_{n=0}^{\infty}$ is a two-α-convex sequence of order three, then it is two-β-convex sequence of order three for $0 \leq \beta \leq \alpha$.*

Proof. The proof follows from Theorem 2.2. Indeed, let the sequence $(a_n)_{n=0}^{\infty}$ be a two-α-convex sequence of order three. Then, it may be represented by (1) with

$$d_{n+3} \geq 2\Big[1 - \frac{2\alpha+1}{\alpha(n+2)+1}\Big]\left(d_{n+2} - d_{n+1}\right)$$

and $d_n \geq 0$, $n \geq 3$.

Since $0 \leq \beta \leq \alpha$ and $(d_n)_{n=3}^{\infty}$ is increasing, we also have

$$d_{n+3} \geq 2\Big[1 - \frac{2\beta+1}{\beta(n+2)+1}\Big]\left(d_{n+2} - d_{n+1}\right),$$

with the same conditions as above, which shows that the sequence $(a_n)_{n=0}^{\infty}$ is a two-β-convex sequence of order three as well.

The proof is completed. □

The authors of [**14**] introduced the following mean

$$A_n(p) = \frac{a_n + pa_{n+1}}{p+1} \tag{5}$$

for an arbitrary real sequence $a := (a_n)_{n=0}^{\infty}$, where p is a given nonnegative real number.

Assume that P is a property which a sequence may or may not posse. It is written $P(a)$ to indicate that the sequence $(a_n)_{n=0}^{\infty}$ has the property P. There exist s properties P such that the following implication

$$P(a) \Longrightarrow P(A(p)) \tag{6}$$

holds true, where $A(p)$ is defined by (5).

Just to give some examples, we have these two simple cases:

1) If a is a positive sequence, then $A(p)$ is also a positive sequence;
2) If a is a convergent sequence and $\lim_{n\to\infty} a_n = s$, then

$$(A_n)_{n=0}^{\infty} := \Big(\frac{1}{n+1}\sum_{m=0}^{n} a_m\Big)_{n=0}^{\infty}$$

is also a convergent sequence and $\lim_{n\to\infty} A_n = s$.

Now, we are going to extend the class of properties P satisfying (6).

Theorem 2.4. *Let the sequence $(a_n)_{n=0}^{\infty}$ be a two-α-convex sequence of order three and*

(7)
$$\begin{aligned}&(n+2)(n+3)(n+4)a_2 - (n+1)(n+4)(2n+5)a_1 + (n+1)(n+2)(n+3)a_0\\ &\geq (n+3)(n+4)a_{n+3} - (n+1)(n+2)a_{n+4},\end{aligned}$$

where $n \geq 0$. Then for any real number $p \geq 0$, the sequence $(A_n(p))_{n=0}^{\infty}$ is also a two-α-convex of order three.

Proof. Using the assumption of Theorem 2.4, we get

$$\alpha \bigtriangleup^3 (a_n) + (1-\alpha)\left(\frac{a_{n+3}-a_0}{n+3} - \frac{a_{n+2}-a_1}{n+1}\right) \geq 0$$

and

$$\alpha \bigtriangleup^3 (a_{n+1}) + (1-\alpha)\left(\frac{a_{n+4}-a_0}{n+4} - \frac{a_{n+3}-a_1}{n+2}\right) \geq 0.$$

Subsequently, taking into account the last two inequalities, we obtain

$$\begin{aligned}
&\alpha \triangle^3 (A_n(p)) + (1-\alpha)\left(\frac{A_{n+3}(p)-A_0(p)}{n+3} - \frac{A_{n+2}(p)-A_1(p)}{n+1}\right)\\
&= \alpha\left(-\frac{a_n+pa_{n+1}}{p+1} + 3\frac{a_{n+1}+pa_{n+2}}{p+1} - 3\frac{a_{n+2}+pa_{n+3}}{p+1} + \frac{a_{n+3}+pa_{n+4}}{p+1}\right)\\
&\quad + (1-\alpha)\left(\frac{\frac{a_{n+3}+pa_{n+4}}{p+1} - \frac{a_0+pa_1}{p+1}}{n+3} - \frac{\frac{a_{n+2}+pa_{n+3}}{p+1} - \frac{a_1+pa_2}{p+1}}{n+1}\right)\\
&= \frac{1}{p+1}\left[\alpha(-a_n+3a_{n+1}-3a_{n+2}+a_{n+3})\right]\\
&\quad + \frac{p}{p+1}\left[\alpha(-a_{n+1}+3a_{n+2}-3a_{n+3}+a_{n+4})\right]\\
&\quad + \frac{1}{p+1}\left[(1-\alpha)\left(\frac{a_{n+3}-a_0}{n+3} - \frac{a_{n+2}-a_1}{n+1}\right)\right]\\
&\quad + \frac{p}{p+1}\left[(1-\alpha)\left(\frac{a_{n+4}-a_1}{n+3} - \frac{a_{n+3}-a_2}{n+1}\right)\right]\\
&= \frac{1}{p+1}\left[\alpha \triangle^3 (a_n) + (1-\alpha)\left(\frac{a_{n+3}-a_0}{n+3} - \frac{a_{n+2}-a_1}{n+1}\right)\right]\\
&\quad + \frac{p}{p+1}\left[\alpha \triangle^3 (a_{n+1}) + (1-\alpha)\left(\frac{a_{n+4}-a_1}{n+3} - \frac{a_{n+3}-a_2}{n+1}\right)\right]\\
&= \frac{1}{p+1}\left[\alpha \triangle^3 (a_n) + (1-\alpha)\left(\frac{a_{n+3}-a_0}{n+3} - \frac{a_{n+2}-a_1}{n+1}\right)\right]\\
&\quad + \frac{p}{p+1}\left[\alpha \triangle^3 (a_{n+1}) + (1-\alpha)\left(\frac{a_{n+4}-a_0}{n+4} - \frac{a_{n+3}-a_1}{n+2}\right)\right]\\
&\quad + \frac{p}{p+1}\frac{(1-\alpha)}{(n+1)(n+2)(n+3)(n+4)}\\
&\quad \times \Big((n+1)(n+2)a_{n+4} - (n+3)(n+4)a_{n+3} + (n+2)(n+3)(n+4)a_2\\
&\qquad - (n+1)(n+4)(2n+5)a_1 + (n+1)(n+2)(n+3)a_0\Big) \geq 0,
\end{aligned}$$

because of (7), $\frac{1}{p+1} > 0$ and $\frac{p}{p+1} \geq 0$ $(p \geq 0,\ \alpha \in [0,1])$.
The proof is completed. □

The above theorem holds true also for α-convex sequences. Namely,

Theorem 2.5. *Let the sequence $(a_n)_{n=0}^{\infty}$ be an α-convex sequence such that $na_{n+2} - (n+2)a_{n+1} \geq na_0 - (n+2)a_1$, $n \geq 1$. Then for any real number $p \geq 0$, the $(A_n(p))_{n=0}^{\infty}$ is also an α-convex sequence.*

Proof. Since $(a_n)_{n=0}^{\infty}$ is an α-convex sequence, then

$$\alpha \triangle^2 (a_n) + (1-\alpha)\left(\frac{a_{n+1}-a_0}{n+1} - \frac{a_n-a_0}{n}\right) \geq 0$$

and

$$\alpha \triangle^2 (a_{n+1}) + (1-\alpha)\left(\frac{a_{n+2}-a_0}{n+2} - \frac{a_{n+1}-a_0}{n+1}\right) \geq 0$$

hold true.

Whence, we have

$$\begin{aligned}
&\alpha \triangle^2 (A_n(p)) + (1-\alpha)\left(\frac{A_{n+1}(p)-A_0(p)}{n+1} - \frac{A_n(p)-A_0(p)}{n}\right)\\
&= \alpha\left(\frac{a_n+pa_{n+1}}{p+1} - 2\frac{a_{n+1}+pa_{n+2}}{p+1} + \frac{a_{n+2}+pa_{n+3}}{p+1}\right)\\
&\quad + (1-\alpha)\left(\frac{\frac{a_{n+1}+pa_{n+2}}{p+1} - \frac{a_0+pa_1}{p+1}}{n+1} - \frac{\frac{a_n+pa_{n+1}}{p+1} - \frac{a_0+pa_1}{p+1}}{n}\right)\\
&= \frac{\alpha}{1+p}\triangle^2 (a_n) + \frac{\alpha p}{1+p}\triangle^2 (a_{n+1})\\
&\quad + \frac{1-\alpha}{1+p}\left(\frac{a_{n+1}-a_0}{n+1} - \frac{a_n-a_0}{n}\right) + \frac{(1-\alpha)p}{1+p}\left(\frac{a_{n+2}-a_1}{n+1} - \frac{a_{n+1}-a_1}{n}\right)\\
&= \frac{1}{1+p}\left[\alpha \triangle^2 (a_n) + (1-\alpha)\left(\frac{a_{n+1}-a_0}{n+1} - \frac{a_n-a_0}{n}\right)\right]\\
&\quad + \frac{p}{1+p}\left[\alpha \triangle^2 (a_{n+1}) + (1-\alpha)\left(\frac{a_{n+2}-a_1}{n+1} - \frac{a_{n+1}-a_1}{n}\right)\right]\\
&= \frac{1}{1+p}\left[\alpha \triangle^2 (a_n) + (1-\alpha)\left(\frac{a_{n+1}-a_0}{n+1} - \frac{a_n-a_0}{n}\right)\right]\\
&\quad + \frac{p}{1+p}\left[\alpha \triangle^2 (a_{n+1}) + (1-\alpha)\left(\frac{a_{n+2}-a_0}{n+2} - \frac{a_{n+1}-a_0}{n+1}\right)\right]\\
&\quad + \frac{p(1-\alpha)}{1+p}\,\frac{na_{n+2}-(n+2)a_{n+1}+(n+2)a_1-na_0}{n(n+1)(n+2)} \geq 0,
\end{aligned}$$

based on the assumptions of the theorem, $\frac{1}{p+1} > 0$ and $\frac{p}{p+1} \geq 0$ ($p \geq 0$, $\alpha \in [0,1]$).

The proof is completed. □

Putting $\alpha = 1$ in Theorem 2.5, we immediately obtain

Corollary 2.1 ([**14**])**.** *If the sequence $(a_n)_{n=0}^{\infty}$ is a convex sequence, then for any real number $p \geq 0$, the $(A_n(p))_{n=0}^{\infty}$ is also a convex sequence.*

Let us finish our study with a result connected to "convexity" of some weighted arithmetic means. Indeed, let the sequence $(X_n)_{n=0}^{\infty}$ be given by

$$X_n = \frac{p_0x_0 + \cdots + p_nx_n}{p_0 + \cdots + p_n} \tag{8}$$

for $n \geq 0$ and a given sequence $(x_n)_{n=0}^{\infty}$, where the sequence $(p_n)_{n=0}^{\infty}$ is a positive one.

For some fixed sequences $a, b, c, d, e\colon \mathbb{N} \to \mathbb{R}$, let

$$Tx_n = a(n)x_{n+3} + b(n)x_{n+2} + c(n)x_{n+1} + d(n)x_n + e(n)x_0$$

and

$$S^2=\{(x_n)_{n=0}^{\infty}: Tx_n\geq 0 \text{ for all } n\geq 0\}.$$

Theorem 2.6. *If $(kn)_{n=0}^{\infty}\in S^2$ for any $k\in\mathbb{R}^+\cup\{0\}$, and $(X_n)_{n=0}^{\infty}\in S^2$ for any $(x_n)_{n=0}^{\infty}\in S^2$, then there exists $u>0$ such that the weights p_n are of the form*

$$p_n=p_0\binom{u+n-1}{n}, \tag{9}$$

where $p_0,p_1>0$ and $n\geq 0$.

Proof. For the proof we will use mathematical induction. It is obvious that if the condition

$$a(n)(n+3)+b(n)(n+2)+c(n)(n+1)+d(n)n=0 \tag{10}$$

is satisfied, then $(kn)_{n=0}^{\infty}\in S^2$ for any $k\in\mathbb{R}^+\cup\{0\}$.

For the same sequence, the relation (8) takes the form

$$X_n=k\cdot\frac{1\cdot p_1+\cdots+n\cdot p_n}{p_0+\cdots+p_n}\geq 0,$$

which by assumption, belongs to S^2. We have

$$p_1=p_0\cdot\frac{p_1}{p_0}=p_0u=p_0\binom{u+1-1}{1},$$

where we put $u=p_1/p_0>0$.

Putting $n=0$ in (10) we find

$$c(0)=-3a(0)-2b(0), \tag{11}$$

and

$$X_0=0,\quad X_1=k\frac{u}{1+u},\quad X_2=k\frac{up_0+2p_2}{p_0(1+u)+p_2},\quad X_3=k\frac{up_0+2p_2+3p_3}{p_0(1+u)+p_2+p_3}.$$

Using (11) and above equalities, we obtain

$$\begin{aligned}TX_0&=a(0)X_3+b(0)X_2+c(0)X_1\\&=k\left[a(0)\frac{up_0+2p_2+3p_3}{p_0(1+u)+p_2+p_3}+b(0)\frac{up_0+2p_2}{p_0(1+u)+p_2}+c(0)\frac{u}{1+u}\right]\\&=ka(0)\left[\frac{up_0+2p_2+3p_3}{p_0(1+u)+p_2+p_3}-\frac{3u}{1+u}\right]+kb(0)\left[\frac{up_0+2p_2}{p_0(1+u)+p_2}-\frac{2u}{1+u}\right].\end{aligned}$$

Therefore, if

$$\begin{cases}\dfrac{up_0+2p_2}{p_0(1+u)+p_2}=\dfrac{2u}{1+u}\\[2mm]\dfrac{up_0+2p_2+3p_3}{p_0(1+u)+p_2+p_3}=\dfrac{3u}{1+u},\end{cases}$$

then $TX_0 \geq 0$ for any $k \in \mathbb{R}^+ \cup \{0\}$. Solving the above system of equations, for p_2 and p_3, we easily find

$$p_2 = p_0 \frac{u(1+u)}{2} = p_0 \binom{u+2-1}{2} \qquad \text{and}$$
$$p_3 = p_0 \frac{u(1+u)(u+2)}{6} = p_0 \binom{u+3-1}{3}.$$

Assume that (9) holds true for all $n \leq m+1$. Following [**15**, page 4], we use the following relations

$$X_m = k\frac{um}{u+1}, \qquad X_{m+1} = k\frac{u(m+1)}{u+1}$$

and

$$X_{m+2} = k\frac{up_0\binom{u+m+1}{m} + (m+2)p_{m+2}}{p_0\binom{u+m+1}{m} + p_{m+2}}.$$

Now we note that

$$\begin{aligned}
X_{m+3} &= k\frac{\sum_{i=0}^{m} ip_i + (m+2)p_{m+2} + (m+3)p_{m+3}}{\sum_{i=0}^{m} p_i + p_{m+2} + p_{m+3}} \\
&= k\frac{\sum_{i=0}^{m} ip_0\binom{u+i-1}{i} + (m+2)p_{m+2} + (m+3)p_{m+3}}{\sum_{i=0}^{m} p_0\binom{u+i-1}{i} + p_{m+2} + p_{m+3}} \\
&= k\frac{up_0\sum_{i=0}^{m}\binom{u+i-1}{i-1} + (m+2)p_{m+2} + (m+3)p_{m+3}}{p_0\sum_{i=0}^{m}\binom{u+i-1}{i} + p_{m+2} + p_{m+3}} \\
&= k\frac{up_0\binom{u+m+1}{m} + (m+2)p_{m+2} + (m+3)p_{m+3}}{p_0\binom{u+m+1}{m+1} + p_{m+2} + p_{m+3}},
\end{aligned}$$

which along with the above equalities and (10), implies

$$\begin{aligned}
TX_m &= a(m)X_{m+3} + b(m)X_{m+2} + c(m)X_{m+1} + d(m)X_m \\
&= a(m)k\frac{up_0\binom{u+m+1}{m} + (m+2)p_{m+2} + (m+3)p_{m+3}}{p_0\binom{u+m+1}{m+1} + p_{m+2} + p_{m+3}} \\
&\quad + b(m)k\frac{up_0\binom{u+m+1}{m} + (m+2)p_{m+2}}{p_0\binom{u+m+1}{m} + p_{m+2}} \\
&\quad + k\frac{u}{u+1}\left[c(m)(m+1) + d(m)m\right] \\
&= a(m)k\left[\frac{up_0\binom{u+m+1}{m} + (m+2)p_{m+2} + (m+3)p_{m+3}}{p_0\binom{u+m+1}{m+1} + p_{m+2} + p_{m+3}} - \frac{u(m+3)}{u+1}\right] \\
&\quad + b(m)k\left[\frac{up_0\binom{u+m+1}{m} + (m+2)p_{m+2}}{p_0\binom{u+m+1}{m} + p_{m+2}} - \frac{u(m+2)}{u+1}\right].
\end{aligned}$$

Consequently, $TX_m \geq 0$ for any $k \in \mathbb{R}^+ \cup \{0\}$, if

$$\begin{cases} \dfrac{up_0\binom{u+m+1}{m} + (m+2)p_{m+2}}{p_0\binom{u+m+1}{m} + p_{m+2}} = \dfrac{u(m+2)}{u+1} \\ \dfrac{up_0\binom{u+m+1}{m} + (m+2)p_{m+2} + (m+3)p_{m+3}}{p_0\binom{u+m+1}{m+1} + p_{m+2} + p_{m+3}} = \dfrac{u(m+3)}{u+1} \end{cases}.$$

Solving the first equation for p_{m+2}, we can find (after some appropriate calculations)

$$p_{m+2} = p_0\binom{u+m+2-1}{m+2},$$

while putting this into second equality of the last system of equations, we also find

$$\begin{aligned} p_{m+3} &= \frac{up_0}{m+3}\left[(m+3)\binom{u+m+1}{m+1} - (u+1)\binom{u+m+1}{m}\right] \\ &\quad + \frac{u-m-2}{m+3}p_{m+2} \\ &= \frac{up_0}{m+3}\left[(m+3)\binom{u+m+1}{m+1} - (m+1)\binom{u+m+1}{m+1}\right] \\ &\quad + \frac{u-m-2}{m+3}p_0\binom{u+m+1}{m+2} \\ &= \frac{up_0}{m+3}\left[2\binom{u+m+1}{m+1} + \binom{u+m+1}{m+2}\right] - \frac{m+2}{m+3}p_0\binom{u+m+1}{m+2} \\ &= \frac{up_0}{m+3}\left[\binom{u+m+1}{m+1} + \binom{u+m+2}{m+2}\right] - \frac{up_0}{m+3}\binom{u+m+1}{m+1} \\ &= \frac{up_0}{m+3}\binom{u+m+2}{m+2} = p_0\binom{u+m+3-1}{m+3}, \end{aligned}$$

which by induction, proves the assertion of our theorem.

The proof is completed. □

Corollary 2.2. *If for any two-α-convex sequence $(x_n)_{n=0}^{\infty}$ of order three the sequence, $(X_n)_{n=0}^{\infty}$ is a two-α-convex of order three, then there exists $u > 0$ such that the weights p_n are of the form* (9).

Proof. Since for the sequence $(a_n)_{n=0}^{\infty} = (kn)_{n=0}^{\infty}$, we have

$$\alpha \triangle^3 a_n + (1-\alpha)\left(\frac{a_{n+3}-a_0}{n+3} - \frac{a_{n+2}-a_1}{n+1}\right) = 0,$$

then it means that $(a_n)_{n=0}^{\infty}$ is a two-α-convex sequence of order three. Therefore, the assertion of this corollary is an immediate result of the Theorem 2.6. □

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