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STABILITY OF A CLASS OF FRACTIONAL INTEGRO-DIFFERENTIAL EQUATION WITH NONLOCAL INITIAL CONDITIONS

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Abstract. The aim of the present paper is to investigate the Hyers-Ulam stability and generalized Hyers-Ulam stability of a new class of a fractional integro-differential equation with nonlocal initial conditions.

1. Introduction

"Under what conditions does there exist an additive mapping near an approximately additive mapping?", this is the problem proposed by Ulam [**16**] in 1940. In the next year, the first positive answer for additive functions defined on Banach spaces was given by Hyers [**8**]. The generalization of Hyers result was given by Rassias [**15**] in 1978. Since this pioneering result, the stability concept had been rapidly devoloped and become one of the central subjects in mathematical analysis.

Motivated by this result, S. M. Jung [**9**] initiated the application of these concepts in differential equations and integral equations via a fixed point method by using some ideas of Cadariu and Radu [**2**]. Following this, many authors have proved the stability of differential equations, integral equations and integro-differential equations (see [**1**], [**3**], [**6**], [**7**] etc.) using the fixed point approach in Banach spaces.

On the other hand, fractional differential equations have arisen as a major field of research in recent years. The commentable development in this area is finding the existence and uniqueness results of linear, nonlinear and integro-differential equations of fractional order. By contrast, the stability concepts of fractional order differential equations are very slow. There are very few works only available on the stability of FDE.

In 2012, Wang [**17**] proved the Hyers-Ulam stability of the following problem

$$D^{\alpha}y(t) = F(t, y(t)), \qquad 0 < \alpha < 1.$$

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For more results, one can see ([**5**], [**12**], [**14**], [**13**], [**18**] etc.). This paper is concerned with the stability of the following fractional integro-differential equation with the given initial condition

$$ {}^cD^\alpha y(t) = F\left(t, y(t), \int_0^t k(t,s,y(s))\mathrm{d}s, \int_0^1 h(t,s,y(s))\mathrm{d}s\right), \tag{1.1}$$

$$ x(0) = \int_0^1 g(s)x(s)\mathrm{d}s, \tag{1.2}$$

where ${}^cD^\alpha$ is Caputo derivative of order α, $0 < \alpha < 1$, $t \in I = [0,1]$, $g(t) \in (0,1]$, $g \in L^1(I, \mathbb{R}_+)$, $y \in Y = C(I,X)$ is a continuous function on I with values in the Banach space X, $\|y\|_Y = \max_{t\in I} \|y(t)\|_X$, $F\colon I \times X \times X \times X \to X$, $k\colon D \times X \to X$, and $h\colon D_0 \times X \to X$ are continuous X valed functions.Here we note that $D = \{(t,s) \in \mathbb{R}^2 : 0 \le s \le t \le 1\}$ and $D_0 = I \times I$. For our convenience let us denote $Ky(t) = \int_0^t k(t,s,y(s))\mathrm{d}s$, $Hy(t) = \int_0^1 h(t,s,y(s))\mathrm{d}s$.

In this paper, authors prove the Hyers-Ulam stability of a class of the fractional order integro-differential equation (1.1) with the given initial condition (1.2) by applying the fixed point method.

This paper is organized as follows: In Section 2, the Hyers-Ulam stability of the fractional integro-differential equation (1.1) with the nonlocal initial condition (1.2) is proved. In Section 3, the generalized Hyers-Ulam stability of the fractional integro-differential equation (1.1) with the nonlocal initial condition (1.2) is proved.

2. Priliminaries

Assume the following:

(H1) If $f \in (C[0,1] \times X \times X \times X, X)$ and a nonnegative, bounded $p_f \in L^1([0,1], \mathbb{R})$, there exists $M > 0$, $p_f(t) \le M$ for $t \in [0,1]$, such that

$$\|f(t,x,Kx,Hx)\| \le p_f(t)\,\|x\| \qquad \text{for } x \in X.$$

(H2) There exist positive constants L_1, L_2, and L such that

$$\|f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)\| \le L_1\left(\|x_1 - x_2\| + \|y_1 - y_2\| + \|z_1 - z_2\|\right)$$

for all $x_1, y_1, z_1, x_2, y_2, z_2 \in Y$, $L_2 = \max_{t\in I} \|f(t,0,0,0)\|$, and $L = \max\{L_1, L_2\}$.

(H3) There exist positive constants N_1, N_2, and N such that

$$\|k(t,s,x_1) - k(t,s,x_2)\| \le N_1\,\|x_1 - x_2\|$$

for all $x_1, x_2 \in Y$, $N_2 = \max_{(t,s)\in D} \|k(t,s,0)\|$, and $N = \max\{N_1, N_2\}$.

(H4) There exist positive constants C_1, C_2 and C such that

$$\|h(t,s,x_1) - h(t,s,x_2)\| \le C_1\,\|x_1 - x_2\|$$

for all $x_1, x_2 \in Y, C_2 = \max_{(t,s)\in D_0} \|h(t,s,0)\|$, and $C = \max\{C_1, C_2\}$.

(H5) $p = \frac{L}{\Gamma(\alpha+1)}\left(1 + C + \frac{N}{(\alpha+1)}\right)$ is such that $0 \le p \le 1$.

Definition 2.1 ([**17**])**.** For a nonempty set X, a function $d\colon X \times X \to [0, \infty]$ is called generalized metric on X if and only if d satisfies:

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

This concept differs from the usual concept of a complete metric space by the fact that not every two points in X have necessarily a finite distance. One might call such space a generalized complete metric space.

Theorem 2.1 ([**4**])**.** *Let (X, d) be a generalized complete metric space. Assume that $\Lambda\colon X \to X$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $d(\Lambda^{k+1}x, \Lambda^k x) < \infty$ for some $x \in X$, then the following statements are true:*

(a) *The sequence $\{\Lambda^n x\}$ converges to a fixed end point x^* of Λ.*

(b) *x^* is the unique fixed point of Λ in $X^* = \big\{y \in X | d(\Lambda^k x, y) < \infty\big\}$.*

(c) *If $y \in X^*$, then $d(y, x^*) \leq \frac{1}{1-L} d(\Lambda y, y)$.*

Lemma 1 ([**11**])**.** *If $Q(\tau, \alpha) = \Gamma(\alpha) I_{1-}^{\alpha} g(\tau) = \int_\tau^1 g(s)(s-\tau)^{\alpha-1} \mathrm{d}s$ for $\tau \in [0, 1]$, and if $g \in [I, R]$ satisfies $0 \leq g(s) \leq 1$ for $0 \leq s \leq 1$ and $\alpha > 0$, then*

$$\frac{Q(\tau, \alpha)}{\Gamma(\alpha)} < \mathrm{e} \qquad \text{and} \qquad \frac{\int_0^t (t-s)^{\alpha-1} \mathrm{d}s}{\Gamma(\alpha)} < \mathrm{e}\,.$$

Theorem 2.2 ([**11**])**.** *If* (H1)–(H5) *are satisfied, then the fractional integro-differential equation* (1.1)*, with the initial condition* (1.2) *has a unique solution in I defined by*

$$\begin{aligned} y(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, f(s), Kf(s), Hf(s)) \mathrm{d}s \\ & - \frac{1}{(1-\mu)\Gamma(\alpha)} \int_0^1 Q(\tau, \alpha) F\left(\tau, f(\tau), Kf(\tau), Hf(\tau)\right) \mathrm{d}\tau. \end{aligned}$$

3. Hyers-Ulam Stability

In this section, authors investigate the Hyers-Ulam stability of the fractional integro-differential equation (1.1) with the integral initial condition (1.2).

Theorem 3.1. *Set $l := (L(1 + N + M)) < 1$. Let $L, M,$ and N be positive constants with $0 < \frac{t^\alpha(1-\mu)}{\Gamma(\alpha+1)(1-\mu-le)-(1-\mu)lt^\alpha} < 1$ and $I\colon [0, 1]$ denote a closed and bounded interval. Suppose that $F\colon I \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying a Lipschitz condition*

$$|F(t, x_1, y_1, z_1) - F(t, x_2, y_2, z_2)| \leq L\left(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|\right) \tag{3.1}$$

$k\colon I \times I \times \mathbb{R} \to \mathbb{R}$ is a continous function satisfying a Lipschitz condition

$$|k(t, s, f) - k(t, s, g)| \leq N\left[|f - g|\right] \qquad \text{for all } \ t, s \in I \text{ and } f, g \in \mathbb{R}. \tag{3.2}$$

and $h\colon I\times I\times\mathbb{R}\to\mathbb{R}$ *is a continous function satisfying a Lipschitz condition*

$$|h(t,s,f)-h(t,s,g)|\le M\,[|f-g|]\qquad \textit{for all}\ \ t,s\in I\ \textit{and}\ f,g\in\mathbb{R}. \tag{3.3}$$

If for $\varepsilon\ge 0$*, a continuously differential function* $y\colon I\to\mathbb{R}$ *satisfies*

$$|{}^cD^\alpha y(t)-F(t,y(t),Ky(t),Hy(t))|\le\varepsilon \tag{3.4}$$

for all $t\in I$*, then there exists a unique continuous function* $y_0\colon I\to\mathbb{R}$ *such that*

$$\begin{aligned} y_0(t)=&\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}F(s,f(s),Kf(s),Hf(s))\mathrm{d}s\\ &-\frac{1}{(1-\mu)\Gamma(\alpha)}\int_0^1 Q(\tau,\alpha)F\left(\tau,f(\tau),Kf(\tau)Hf(\tau)\right)\mathrm{d}\tau. \end{aligned} \tag{3.5}$$

and

$$|y(t)-y_0(t)|\le\frac{t^\alpha(1-\mu)}{\Gamma(\alpha+1)(1-\mu-le)-(1-\mu)lt^\alpha}\varepsilon. \tag{3.6}$$

Proof. Let X denote the set of all real valued continuous functions on I. We define a generalized complete metric (see [**9**]) on X as follows:

$$d(f,g)=\inf\{C\in[0,\infty]||f(t)-g(t)|\le C\ \ \text{for all } t\in I\}. \tag{3.7}$$

Now, define an operator $\Lambda\colon X\to X$ by

$$\begin{aligned} (\Lambda f)(t)=&\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}F(s,f(s),K(s),Hf(s))\mathrm{d}s\\ &-\frac{1}{(1-\mu)\Gamma(\alpha)}\int_0^1 Q(\tau,\alpha)F\left(\tau,f(\tau),Kf(\tau),Hf(\tau)\right)\mathrm{d}\tau \end{aligned} \tag{3.8}$$

for all $f\in X$.

Next we check that Λ is strictly contractive on X.

Let $f,g\in X$ and let $C_{fg}\in[0,\infty]$ be an arbitrary constant such that $d(f,g)\le C_{fg}$. Then, by (3.7), we get

$$|f(t)-g(t)|\le C_{fg} \tag{3.9}$$

for any $t\in I$.

Using (3.1), (3.2), (3.3), (3.8), and (3.9), we have

$$\begin{aligned} &|(\Lambda f)t-(\Lambda g)t|\\ &\le\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left|F(s,f(s),Kf(s),Hf(s))-F(s,g(s),Kg(S),Hg(s))\right|\mathrm{d}s\\ &\quad+\frac{\mathrm{e}}{(1-\mu)}\int_0^1\left|F(\tau,f(\tau),Kf(\tau),Hf(\tau))-F(\tau,g(\tau),Kg(\tau),Hg(\tau))\right|\mathrm{d}\tau \end{aligned}$$

$$
\begin{aligned}
&\le \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[|f(s)-g(s)| + |Kf(s)-Kg(s)| + |Hf(s)-Hg(s)|\right] \mathrm{d}s \\
&\quad + \frac{\mathrm{e}\, L}{(1-\mu)} \int_0^1 \left[|f(\tau)-g(\tau)| + |Kf(\tau)-Kg(\tau)| + |Hf(\tau)-Hg(\tau)|\right] \mathrm{d}s \\
&\le \frac{L(1+N+M)}{\Gamma(\alpha)} C_{fg} \int_0^t (t-s)^{\alpha-1} \mathrm{d}s + \frac{\mathrm{e}\, L(1+N+M)}{(1-\mu)} C_{fg} \int_0^1 \mathrm{d}s \\
&\le \frac{L(1+N+M)t^{\alpha}}{\Gamma(\alpha+1)} C_{fg} + \frac{\mathrm{e}\, L_1(1+N+M)}{(1-\mu)} C_{fg} \\
&\le L(1+N+M) \left[\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{\mathrm{e}}{1-\mu}\right] C_{fg}
\end{aligned}
$$

for all $t \in I$. That is

$$
d\,(\Lambda f, \Lambda g) \le \left[\frac{l t^{\alpha}}{\Gamma(\alpha+1)} + \frac{l\,\mathrm{e}}{1-\mu}\right] C_{fg}.
$$

Hence we can conclude that

$$
d\,(\Lambda f, \Lambda g) \le \left[\frac{l t^{\alpha}}{\Gamma(\alpha+1)} + \frac{l\,\mathrm{e}}{1-\mu}\right] C_{fg} \le \left[\frac{l t^{\alpha}}{\Gamma(\alpha+1)} + \frac{l\,\mathrm{e}}{1-\mu}\right] d(f,g)
$$

for all $f, g \in X$. Let g_0 be any arbitrary element in X. Then there exists a constant $0 < C < \infty$ with

$$
\begin{aligned}
|(\Lambda g_0)(t) - g_0(t)| = \Big| &\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, f(s), Kf(s), Hf(s)) \mathrm{d}s - \frac{1}{(1-\mu)\Gamma(\alpha)} \\
&\times \int_0^1 Q(\tau,\alpha) F\left(\tau, f(\tau), Kf(\tau), Hf(\tau)\right) \mathrm{d}\tau - g_0(t) \Big| \le C
\end{aligned}
$$

for all $t \in I$, since $F(t, (g_0)(t), Kg_0(t), Hg_0(t))$ and $(g_0)(t)$ are bounded on I. Thus, (3.7) implies that

$$
d(\Lambda g_0, g_0) < \infty. \tag{3.10}
$$

Therefore according to Theorem 2.1, there exists a continuous function $y_0 \colon I \to \mathbb{R}$ such that the sequence $\{\Lambda^n g_0\}$ converges to y_0 and $\Lambda y_0 = y_0$, that is, y_0 is a solution of (1.1), (1.2). We will now verify that

$$
\{g \in X \mid d(g_0, g) < \infty\} = X.
$$

Since g and g_0 are bounded on I, for any $g \in X$, there exists a constant $0 < C_g < \infty$ such that

$$
|g_0(t) - g(t)| \le C_g
$$

Hence, we have $d(g_0, g) < \infty$ for all $g \in X$. That is $\{g \in X \mid d(g_0, g) < \infty\} = X$.

Therefore, in view of Theorem 2.1, we conclude that y_0 given by (3.5) is the unique continuous function. From (3.4) we have

$$
-\varepsilon \le^{c} D^{\alpha}_{a+} y(t) - F(t, y(t), Ky(t), Hy(t)) \le \varepsilon
$$

for all $t \in I$.

If we integrate each term in the above inequality from 0 to t and substitute the initial condition, we obtain

$$\Big|y(t) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} F(s, f(s), Kf(s), Hf(s))\mathrm{d}s - \frac{1}{(1-\mu)\Gamma(\alpha)}\int_0^1 Q(\tau,\alpha) F\left(\tau, f(\tau), Kf(\tau), Hf(\tau)\right)\mathrm{d}\tau\Big| \leq \frac{t^\alpha}{\Gamma(\alpha+1)}\varepsilon$$

for any $t \in I$.

That is, it holds that

$$|y(t) - (\Lambda y)(t)| \leq \frac{t^\alpha}{\Gamma(\alpha+1)}\varepsilon,$$

i.e,

$$d(y, \Lambda y) \leq \frac{t^\alpha}{\Gamma(\alpha+1)}\varepsilon \tag{3.11}$$

for each $t \in I$.

Finally, Theorem 2.1 together with (3.11) implies that

$$d(y, y_0) \leq \frac{1}{1 - \left[\frac{lt^\alpha}{\Gamma(\alpha+1)} + \frac{l\,\mathrm{e}}{1-\mu}\right]} d(y, \Lambda y) \leq \frac{t^\alpha(1-\mu)}{\Gamma(\alpha+1)(1-\mu-l\,\mathrm{e}) - (1-\mu)lt^\alpha}\varepsilon,$$

that is, the inequality (3.6) is true for all $t \in I$. □

4. Generalized Hyers-Ulam stability

In this section, authors established generalized Hyers-Ulam stability of the fractional integro-differential equation (1.1) with initial condition (1.2).

Theorem 4.1. *Set* $l := (L(1+N+M)) < 1$. *Let* $I = [0,1]$ *be a closed and bounded interval, and* L, M, N, P_1, *and* P_2 *be positive constants with* $0 < \left[lP_1 + \frac{leP_2}{1-\mu}\right] < 1$. *Assume that* $F\colon I \times \mathbb{R} \to \mathbb{R}$ *is a continuous function satisfying the Lipschitz condition* (3.1) *and* $K\colon I \times I \times \mathbb{R} \to \mathbb{R}$ *is a continuous function satisfying a Lipschitz condition* (3.2), $H\colon I \times I \times \mathbb{R} \to \mathbb{R}$ *is a continuous function satisfying a Lipschitz condition* (3.3). *If a continuously differential function* $y\colon I \to \mathbb{R}$ *satisfies*

$$|{}^cD^\alpha y(t) - F(t, y(t), Ky(t), Hy(t))| \leq \varphi(t) \tag{4.1}$$

for all $t \in I$, *where* $\varphi\colon I \to (0,\infty)$ *is a continuous function with*

$$\left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi(s)\mathrm{d}s\right| \leq P_1\varphi(t) \quad \text{and} \quad \int_0^1 \varphi(s)\mathrm{d}s \leq P_2\varphi(t) \tag{4.2}$$

for all $t \in I$, *then there exists a unique continuous function* $y_0\colon I \to \mathbb{R}$ *such that*

$$\begin{aligned} y_0(t) = {} & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} F(s, f(s), K(s), Hf(s))\mathrm{d}s \\ & - \frac{1}{(1-\mu)\Gamma(\alpha)}\int_0^1 Q(\tau,\alpha) F\left(\tau, f(\tau), Kf(\tau), Hf(\tau)\right)\mathrm{d}\tau \end{aligned} \tag{4.3}$$

and

$$|y(t)-y_0(t)| \le \frac{(1-\mu)P_1}{(1-\mu)-[l(1-\mu)P_1+leP_2]}\varphi(t) \quad \textit{for all } t \in I. \tag{4.4}$$

Proof. Let X denote the set of all real valued continuous functions on I. We set a generalised complete metric (see [**9**]) on X as follows

$$d(f,g)=\inf\{C\in[0,\infty] \mid |f(t)-g(t)|\le C\varphi(t) \text{ for all} t\in I\} \tag{4.5}$$

Define an operator $\Lambda\colon X\to X$ by

$$\begin{aligned}(\Lambda f)(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s,f(s),K(s),Hf(s))\mathrm{d}s\\ &\quad -\frac{1}{(1-\mu)\Gamma(\alpha)}\int_0^1 Q(\tau,\alpha)F(\tau,f(\tau),Kf(\tau),Hf(\tau))\,\mathrm{d}\tau\end{aligned} \tag{4.6}$$

for all $t\in I$ and $f\in X$.

Now we check that Λ is strictly contractive on X.

For any $f,g\in X$, let $C_{fg}\in[0,\infty]$ be an arbitrary constant with $d(f,g)\le C_{fg}$, that is, by (4.5), we have

$$|f(t)-g(t)|\le C_{fg}\varphi(t) \tag{4.7}$$

for any $t\in I$.

Then from (3.1), (3.2), (3.3), (4.2), (4.6) and (4.7) it follows that

$$\begin{aligned}
&|(\Lambda f)t-(\Lambda g)t|\\
&\quad\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,|F(s,f(s),Kf(s),Hf(s))-F(s,g(s),Kg(S),Hg(s))|\,\mathrm{d}s\\
&\qquad+\frac{\mathrm{e}}{(1-\mu)}\int_0^1 |F(s,f(s),Kf(s),Hf(s))-F(s,g(s),Kg(S),Hg(s))|\,\mathrm{d}s\\
&\quad\le \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,[|f(s)-g(s)|+|Kf(s)-Kg(s)|+|Hf(s)-Hg(s)|]\,\mathrm{d}s\\
&\qquad+\frac{\mathrm{e}\,L}{(1-\mu)}\int_0^1 [|f(\tau)-g(\tau)|+|Kf(\tau)-Kg(\tau)|+|Hf(\tau)-Hg(\tau)|]\,\mathrm{d}s\\
&\quad\le \frac{L(1+N+M)}{\Gamma(\alpha)}C_{fg}\int_0^t (t-s)^{\alpha-1}\varphi(s)\mathrm{d}s+\frac{\mathrm{e}\,L(1+N+M)}{(1-\mu)}C_{fg}\int_0^1 \varphi(s)\mathrm{d}s\\
&\quad\le lP_1C_{fg}\varphi(t)+\frac{l\,\mathrm{e}}{(1-\mu)}P_2C_{fg}\varphi(t)\\
&\quad\le \Big[lP_1+\frac{l\,\mathrm{e}\,P_2}{1-\mu}\Big]C_{fg}\varphi(t)
\end{aligned}$$

for all $t\in I$. That is,

$$d(\Lambda f,\Lambda g)\le \Big[lP_1+\frac{l\,\mathrm{e}\,P_2}{1-\mu}\Big]C_{fg}\varphi(t).$$

Hence we can conclude that

$$d\,(\Lambda f, \Lambda g) \leq \Big[lP_1 + \frac{l\,\mathrm{e}\,P_2}{1-\mu}\Big] d(f,g)$$

for any $f, g \in X$, where we note that $0 < \Big[lP_1 + \frac{l\,\mathrm{e}\,P_2}{1-\mu}\Big] < 1$.

From (4.6), it follows that for an arbitrary $g_0 \in X$, there exists a constant $0 < C < \infty$ with

$$\begin{aligned}|(\Lambda g_0)(t) - g_0(t)| &= \Big|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} F(s, f(s), K(s), Hf(s))\mathrm{d}s \\ &\quad - \frac{1}{(1-\mu)\Gamma(\alpha)}\int_0^1 Q(\tau,\alpha) F\,(\tau, f(\tau), Kf(\tau), Hf(\tau))\,\mathrm{d}\tau - g_0(t)\Big| \\ &\leq C\varphi(t)\end{aligned}$$

for all $t \in I$, since $F(t, g_0(t), Kg_0(t), Hg_0(t))$ and $g_0(t)$ are bounded on I and $\min_{t\in I}\varphi(t) > 0$.

Thus (4.5) implies that

$$d(\Lambda g_0, g_0) < \infty.$$

Therefore, according to theorem 2.1, there exists a continuous function $y_0 \colon I \to \mathbb{R}$ such that the sequence $\{\Lambda^n g_0\}$ converges to y_0 in (X, d) and $\Lambda y_0 = y_0$, that is, y_0 is a solution of (1.1)–(3.2) for every $t \in I$.

We will now verify that

$$\{g \in X \mid d(g_0, g) < \infty\} = X,$$

Since g and g_0 are bounded on I for any $g \in X$ and $\min_{t\in I}\varphi(t) > 0$, there exists a constant $0 < C_g < \infty$ such that

$$|g_0(t) - g(t)| \leq C_g$$

Hence, we have $d(g_0, g) < \infty$ for all $g \in X$. That is, $\{g \in X \mid d(g_0, g) < \infty\} = X$.

Therefore, from Theorem 2.1, we conclude that y_0 is the unique continuous function with the property (3.5).

From (4.1), we have

$$-\varphi(t) \leq^c D^\alpha_{a+} y(t) - F(t, y(t), Ky(t), Hy(t)) \leq \varphi(t) \tag{4.8}$$

for all $t \in I$.

If we integrate each term in the above inequality and substitute the boundary conditions, we obtain

$$\begin{aligned}&\Big|y(t) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} F(s, f(s), K(s), Hf(s))\mathrm{d}s \\ &\quad - \frac{1}{(1-\mu)\Gamma(\alpha)}\int_0^1 Q(\tau,\alpha) F\,(\tau, f(\tau), Kf(\tau), Hf(\tau))\,\mathrm{d}\tau\Big| \\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi(s)\mathrm{d}s\end{aligned}$$

for any $t \in I$.

Thus, by (4.2) and (4.6), we get

$$|y(t) - (\Lambda y)(t)| \le P_1\varphi(t)$$

for each $t \in I$, which implies that

$$d(y, \Lambda y) \le P_1\varphi(t). \tag{4.9}$$

Finally, using Theorem 2.1 together with (4.9), we conclude that

$$\begin{aligned} d(y, y_0) &\le \frac{1}{1 - \left[lP_1 + \frac{l\,\mathrm{e}\,P_2}{1-\mu}\right]} d(y, \Lambda y) \\ &\le \frac{(1-\mu)P_1}{(1-\mu) - [l(1-\mu)P_1 + l\,\mathrm{e}\,P_2]}\varphi(t). \end{aligned} \tag{4.10}$$

Consequently, this yields the inequality (4.4) for all $t \in I$. □

In Theorem 4.1, we have examined the generalized Hyers-Ulam stability of the fractional integro-differential equation (1.1) defined on a bounded and closed interval. Now we will show that theorem (4.1) is also valid for the case of unbounded intervals.

Theorem 4.2. *For a given nonnegative real number T, let I denote either $(-\infty, 1]$ or $\mathbb{R}$ or $[1, \infty)$. Let L, M, N, P_1, and P_2 be positive constants with $0 < \left[lP_1 + \frac{leP_2}{1-\mu}\right] < 1$. Suppose that $F\colon I \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying a Lipschitz condition (3.1) for all $t \in I$ and $x, y \in \mathbb{R}$. If a continuously differential function $y\colon I \to \mathbb{R}$ satisfies the differential inequality (4.1) for all $t \in I$, where $\varphi\colon I \to (0, \infty)$ is a continuous function satisfying (4.2) for each $t \in I$, then there exists a unique continuous function $y_0\colon I \to \mathbb{R}$ satisfying (3.5) and (4.4) for all $t \in I$.*

Proof. Let $I = \mathbb{R}$. We first show that y is a unique continuous function. For any $n \in \mathbb{N}$, we define $I_n = [-n, n]$. In accordence with Theorem (4.1), there exists a unique continuous function $y_n\colon I_n \to \mathbb{R}$ such that

$$\begin{aligned} y_n(t) = &\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} F(s, f(s), Kf(s), Hf(s))\mathrm{d}s \\ &- \frac{1}{(1-\mu)\Gamma(\alpha)}\int_0^1 Q(\tau, \alpha)F\left(\tau, f(\tau), Kf(\tau), Hf(\tau)\right)\mathrm{d}\tau \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} |y(t) - y_n(t)| &\le \frac{1}{1 - \left[lP_1 + \frac{l\,\mathrm{e}\,P_2}{1-\mu}\right]} d(y, \Lambda y) \\ &\le \frac{(1-\mu)P_1}{(1-\mu) - [l(1-\mu)P_1 + l\,\mathrm{e}\,P_2]}\varphi(t) \end{aligned}$$

for all $t \in I$.

The uniqueness of y_n implies that if $t \in I_n$, then

$$y_n(t) = y_{n+1}(t) = y_{n+2}(t) = \dots \tag{4.12}$$

For any $t \in \mathbb{R}$, we define $n(t) \in \mathbb{N}$ as

$$n(t) = \min\{n \in \mathbb{N} \mid t \in I_n\}. \tag{4.13}$$

Moreover, let us define a function $y_0 \colon \mathbb{R} \to \mathbb{R}$ by

$$y_0(t) = y_{n(t)}(t). \tag{4.14}$$

We claim that y_0 is continuous. We take the integer $n_1 = n(t_1)$ for an arbitrary $t_1 \in \mathbb{R}$. Then, t_1 belongs to the interior of I_{n_1+1} and there exists $\varepsilon > 0$ such that $y_0(t) = y_{n_1+1}(t)$ for all t with $t_1 - \varepsilon < t < t_1 + \varepsilon$. Since y_{n_1+1} is continuous at t_1, y_0 is continuous at t_1 for any $t_1 \in \mathbb{R}$. Now, we will prove that u_0 satisfies (3.5) and (4.5) for all $t \in \mathbb{R}$. Let $n(t)$ be an integer for an arbitrary $t \in \mathbb{R}$. Then, from (4.11) and (4.14), we have $t \in I_{n(t)}$ and

$$\begin{aligned} y_0(t) = y_n(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, f(s), Kf(s), Hf(s)) \mathrm{d}s \\ &\quad - \frac{1}{(1-\mu)\Gamma(\alpha)} \int_0^1 Q(\tau, \alpha) F\left(\tau, f(\tau), Kf(\tau), Hf(\tau)\right) \mathrm{d}\tau. \end{aligned} \tag{4.15}$$

Since $n(s) \le n(t)$ for any $s \in I_{n(t)}$, the last equality is correct and we have

$$y_{n(t)}(s) = y_{n(s)}(s) = y_0(s)$$

by (4.12) and (4.14).

Since $t \in I_{n(t)}$ for all $t \in \mathbb{R}$, by (4.12) and (4.14), we have

$$\begin{aligned} |y(t) - y_0(t)| &\le \left| y(t) - y_{n(t)}(t) \right| \\ &\le \frac{1}{1 - \left[lP_1 + \frac{leP_2}{1-\mu} \right]} d(y, \Lambda y) \\ &\le \frac{(1-\mu)P_1}{(1-\mu) - [l(1-\mu)P_1 + leP_2]} \varphi(t) \end{aligned}$$

for all $t \in \mathbb{R}$. Finally, we prove that y_0 is unique. Assume that $x_0 \colon \mathbb{R} \to \mathbb{R}$ is another continuous function satisfying (3.5) and (4.5) with x_0 in place of y_0 for all $t \in \mathbb{R}$. Let $t \in \mathbb{R}$ be a discretionary number. Since the restrictions $x_0|I_{n(t)}$ and $y_0|I_{n(t)}$ satisfy (3.5) and (4.5) for all $t \in I_{n(t)}$, the uniqueness of $y_{n(t)} = y_0|_{I_{n(t)}}$ suggests that

$$y_0(t) = y_0|_{I_{n(t)}}(t) = x_0|_{I_{n(t)}}(t) = x_0(t). \tag{4.16}$$

Similarly, the proof can be done for the classes $I = (-\infty, T]$ and $I = [0, \infty)$. □

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