ON A CLASS OF GENERALIZED SASAKIAN-SPACE-FORMS

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Abstract. The object of the present paper is to study quasi-conformally flat generalized Sasakian-space-forms. Also we study quasi-conformally semisymmetric generalized Sasakian-space-forms. As a consequence of the results, we obtain some important corollaries.

1. Introduction

The nature of a Riemannian manifold mostly depends on the curvature tensor $R$ of the manifold. It is well known that the sectional curvatures of a manifold determine its curvature tensor completely. A Riemannian manifold with a constant sectional curvature $c$ is known as a real space-form and its curvature tensor is given by

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}.$$  

A Sasakian manifold with constant $\phi$-sectional curvature is a Sasakian space-form and has a specific form of its curvature tensor. Similar notion also holds for Kenmotsu and cosymplectic space-forms. In order to generalize such space-forms in a common frame, Alegre, Blair and Carriazo [1] introduced and studied generalized Sasakian space-forms. These space-forms are defined as follows.

Given an almost contact metric manifold $M(\phi, \xi, \eta, g)$, we say that $M$ is a generalized Sasakian space-form if there exist three functions $f_1, f_2, f_3$ on $M$ such that the curvature tensor $R$ is given by

$$R(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(X)\xi - g(Y,Z)\eta(X)\xi\}$$  

(1.1)

for any vector fields $X, Y, Z$ on $M$. In such a case we denote the manifold as $M(f_1, f_2, f_3)$. In [1], the authors cited several examples of generalized Sasakian space-forms. If $f_1 = \frac{c+3}{4}, f_2 = \frac{c-1}{4}$ and $f_3 = \frac{c-1}{4}$, then a generalized Sasakian space-form with Sasakian structure becomes a Sasakian space-form. In [21], Kim studied conformally flat generalized Sasakian space-forms and locally symmetric...
generalized Sasakian space-forms. He proved some geometric properties of generalized Sasakian space-form which depends on the nature of the functions $f_1$, $f_2$ and $f_3$. A large number of geometers studied generalized Sasakian space-forms in the papers ([2], [3], [4], [5], [14], [16], [17], [19]). In [13], De and Sarkar studied locally $\phi$-symmetric generalized Sasakian space-forms and generalized Sasakian space-forms with $\eta$-recurrent Ricci tensor. Also De and Sarkar [12] studied projectively flat, projectively semisymmetric generalized Sasakian space-forms. Motivated by these studies, in this paper, we study quasi-conformally flat generalized Sasakian space-forms. Beside these we study quasi-conformally semisymmetric generalized Sasakian space-forms.

The present paper is organized as follows. After preliminaries in Section 3, we consider quasi-conformally flat generalized Sasakian space-forms. Section 4 deals with quasi-conformally semisymmetric generalized Sasakian space-forms. As a consequence of the results, we obtain some important corollaries.

2. Preliminaries

In an almost contact metric manifold, we have ([8], [9]):

(2.1) $\phi^2(X) = -X + \eta(X)\xi$, $\phi\xi = 0$.
(2.2) $\eta(\xi) = 1$, $g(X, \xi) = \eta(X)$, $\eta(\phi X) = 0$.
(2.3) $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$.
(2.4) $g(\phi X, Y) = -g(X, \phi Y)$, $g(\phi X, X) = 0$.
(2.5) $g(\phi X, \xi) = 0$.

Again for a $(2n+1)$-dimensional generalized Sasakian space-form, we have [1]:

$$R(X, Y)Z = f_1 \{g(Y, Z)X - g(X, Z)Y\}$$
$$+ f_2 \{g(\phi X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\}$$
$$+ f_3 \{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\}$$
$$+ g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}.$$  

(2.6)

(2.7) $S(X, Y) = (2nf_1 + 3f_2 - f_3)g(X, Y) - (3f_2 + (2n - 1)f_3)\eta(X)\eta(Y)$.
(2.8) $QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n - 1)f_3)\eta(X)\xi$.
(2.9) $R(X, Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y]$.
(2.10) $R(\xi, X)Y = (f_1 - f_3)[g(X, Y)\xi - \eta(Y)X]$.
(2.11) $S(\xi, \xi) = 2n(f_1 - f_3)$.
(2.12) $Q\xi = 2n(f_1 - f_3)\xi$.
(2.13) $r = 2n(2n + 1)f_1 + 6nf_2 - 4nf_3$.  

(2.14)
where $R$, $S$ and $r$ are the curvature tensor, Ricci tensor and scalar curvature of the space-form, respectively, and $Q$ is the Ricci operator defined by $g(QX, Y) = S(X, Y)$. We know that [1] the $\phi$-sectional curvature of a generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is $f_1 + 3f_2$.

An almost contact metric manifold $M$ is said to be an $\eta$-Einstein manifold if the Ricci tensor satisfies the condition $S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$, where $a$ and $b$ are smooth functions on $M$. In [7], Bejan studied geometric properties of $\eta$-Einstein manifolds.

Let $(M, g)$ be a Riemannian manifold and let $\nabla$ be the Levi-Civita connection of $(M, g)$. A Riemannian manifold is called locally symmetric [10] if $\nabla R = 0$, where $R$ is the Riemannian curvature tensor of $(M, g)$.

The notion of the quasi-conformal curvature tensor was given by Yano and Sawaki [22]. The quasi-conformal curvature tensor $\tilde{C}$ is defined by

$$\tilde{C}(X, Y)Z = aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] - \frac{r}{2n+1} \left[ \frac{a}{2n} + 2b \right] g(Y, Z)X - g(X, Z)Y,$$

where $a$ and $b$ are constants, and $R$, $Q$ and $r$ are the Riemannian curvature tensor of type $(1, 3)$, the Ricci operator defined by $g(QX, Y) = S(X, Y)$ and the scalar curvature, respectively. If $a = 1$ and $b = -\frac{1}{2n-1}$, then (2.15) takes the form

$$\tilde{C}(X, Y)Z = R(X, Y)Z - \frac{1}{2n-1} [S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] - \frac{r}{2n(2n-1)} [g(Y, Z)X - g(X, Z)Y] = C(X, Y)Z,$$

where $C$ is the conformal curvature tensor [18]. Thus the conformal curvature tensor $C$ is the particular case of the tensor $\tilde{C}$. For this reason, $\tilde{C}$ is called a quasi-conformal curvature tensor. A manifold $(M^n, g)$ $(n > 3)$ is called quasi-conformally flat if $C = 0$. It is known [6] that a quasi conformally flat manifold is either conformally flat if $a \neq 0$ or Einstein if $a = 0$ and $b \neq 0$. Since they give no restrictions for manifolds if $a = 0$ and $b = 0$, it is essential for us to consider the case of $a \neq 0$ or $b \neq 0$. Quasi-conformally flat Riemannian manifolds with different structures have been studied by U. C. De et al. ([10], [11], [20]) and many others.

**Definition 2.1.** A Riemannian manifold is said to be quasi-conformally semi-symmetric if $R(X, Y) \cdot \tilde{C} = 0$.

In [21], U. K. Kim proved that for a $(2n+1)$-dimensional generalized Sasakian space-form, the following results hold:

1. If $n > 1$, then $M$ is conformally flat if and only if $f_2 = 0$.
2. If $M$ is conformally flat and $\xi$ is a Killing vector field, then it is flat, or of a constant curvature, or locally the product $N^1 \times N^{2n}$, where $N^1$ is a 1-dimensional manifold and $N^{2n}$ is a $2n$-dimensional Hermitian manifold of

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constant curvature. In any case, $M$ is locally symmetric and has a constant \( \phi \)-sectional curvature.

### 3. Quasi-conformally flat generalized Sasakian-space-form

Assume that \( M(f_1, f_2, f_3) \) is a quasi-conformally flat generalized Sasakian-space-form. Then

\[
\tilde{C}(X, Y)Z = 0
\]

for all vector field \( X, Y \) and \( Z \). Using (2.5) in (3.1), we have

\[
\frac{1}{2n + 1} \left[ (-3a + 6b) f_2 + (2a + 2(2n - 1)b) f_3 \right] \{g(Y, Z)X - g(X, Z)Y\}
\]

\[
+ af_2 \{g(X, \phi Z) \phi Y - g(Y, \phi Z) \phi X + 2g(X, \phi Y) \phi Z\}
\]

\[
+ \left[ (a + (2n - 1)b) f_3 + 3b f_2 \right] \{\eta(\phi Y) \eta(Y)X - \eta(Y) \eta(Z)X
\]

\[
+ g(X, \phi Z) \eta(\phi Y) \xi - g(Y, Z) \eta(X) \xi \} = 0.
\]

If we put \( X = \phi Y \) in (3.2), we get

\[
\frac{1}{2n + 1} \left[ (-3a + 6b) f_2 + (2a + 2(2n - 1)b) f_3 \right] \{g(Y, Z) \phi Y - g(\phi Y, Z)Y\}
\]

\[
+ af_2 \{g(\phi Y, \phi Z) \phi Y - g(Y, \phi Z) \phi Y + 2g(\phi Y, \phi Y) \phi Z\}
\]

\[
+ \left[ (a + (2n - 1)b) f_3 + 3b f_2 \right] \{\eta(\phi Y) \eta(Y)X - \eta(Y) \eta(Z) \phi Y
\]

\[
+ g(\phi Y, Z) \eta(Y) \xi - g(Y, Z) \eta(\phi Y) \xi \} = 0.
\]

If we choose a unit vector \( U \) such that \( g(U, \xi) = 0 \) and substitute \( Y = U \) in (3.3), we have

\[
\frac{1}{2n + 1} \left[ (2(n - 1)a + 6b) f_2 + 2(a + (2n - 1)b) f_3 \right]
\]

\[
\{g(U, Z) \phi U - g(\phi U, Z)U\} + 2af_2 \phi Z = 0.
\]

Putting \( Z = U \) in (3.4), we have

\[
(2(n - 1)a + 6b + 2(2n + 1)a) f_2 + 2(a + (2n - 1)b) f_3 \phi U = 0.
\]

Thus we have

\[
(2(n - 1)a + 6b + 2(2n + 1)a) f_2 + 2(a + (2n - 1)b) f_3 = 0.
\]

It follows that

\[
f_2 = -\frac{(a + (2n - 1)b)}{3(an + b)} f_3.
\]

Conversely, if \( f_2 = -\frac{(a + (2n - 1)b)}{3(an + b)} f_3 \), then from (3.1), we have \( \tilde{C}(X, Y)Z = 0 \) and \( M(f_1, f_2, f_3) \) is a quasi-conformally flat. Thus when \( n > 1 \), \( M(f_1, f_2, f_3) \) is quasi-conformally flat if and only if \( f_2 = -\frac{(a + (2n - 1)b)}{3(an + b)} f_3 \). Therefore, we can state the following theorem.
Theorem 3.1. Let \( M(f_1, f_2, f_3) \) be a \((2n + 1)\)-dimensional generalized Sasakian-space-form. If \( n > 1 \), then \( M(f_1, f_2, f_3) \) is quasi-conformally flat if and only if \( f_2 = -\frac{(a + 2n - 1)b}{a(2n + 1)} f_3 \), provided \((an + b) \neq 0\).

Also if \( a = 1 \) and \( b = -\frac{1}{2n - 1} \), then a quasi-conformal curvature tensor reduces to a conformal curvature tensor. Therefore, in view of the result of Kim \[21\] and (3.7), we get \( f_2 = 0 \) and conversely. Therefore we conclude the following corollary.

Corollary 3.1. A generalized Sasakian-space-form \( M(f_1, f_2, f_3) \), \( n > 1 \), is conformally flat if and only if \( f_2 = 0 \).

The above corollary have been proved by Kim \[21\].

In [1], P. Alegre, D. Blair and A. Carriazo proved that if a generalized Sasakian space-form \( M(f_1, f_2, f_3) \) is a Sasakian manifold, then the functions \( f_1, f_2, f_3 \) are constant and \( f_1 = 1 = f_2 = f_3 \).

Now, in our case, \( f_2 = -\frac{(a + 2n - 1)b}{a(2n + 1)} f_3 \) implies \( f_2 = 0 \) provided \( 3(n + 1)a + (2n + 1)b \neq 0 \). Thus \( f_2 = 0 \) implies \( f_3 = 0 \) and \( f_1 = 1 \). Thus from (1.1), we obtain \( R(X, Y)Z = g(Y, Z)X - g(X, Z)Y \), that is, the manifold is of constant curvature 1. Hence we can state the following corollary.

Corollary 3.2. A \((2n + 1)\)-dimensional \((n > 1)\) quasi-conformally flat Sasakian manifold is a manifold of a constant curvature 1, that is, locally isometric to the unit sphere \( S^{(2n+1)}(1) \).

Remark 1. If \( an + b = 0 \), then from (3.7), we infer \( \{a + (2n - 1)b\} f_3 = 0 \), which implies that either \( f_3 = 0 \) or \( a + (2n - 1)b = 0 \). Suppose \( a + (2n - 1)b = 0 \), then from the definition of quasi-conformal curvature tensor, it follows that \( \tilde{C} = bC \), hence quasi conformally flatness and conformally flatness are equivalent, since by hypothesis \( b \neq 0 \).

Moreover, the quasi-conformally flat generalized Sasakian space-form implies

\[
R(X, Y)Z = -\frac{a}{b} [S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] \\
+ \frac{r}{a(2n + 1)} \left[ \frac{a}{2n} + 2b \right] g(Y, Z)X - g(X, Z)Y.
\]

(3.8)

Putting \( Y = Z = \xi \) in (3.8), we get

\[
R(X, \xi)\xi = -\frac{a}{b} [S(\xi, \xi)X - S(X, \xi)\xi + g(\xi, \xi)QX - g(X, \xi)Q\xi] \\
+ \frac{r}{a(2n + 1)} \left[ \frac{a}{2n} + 2b \right] g(\xi, \xi)X - g(X, \xi)\xi.
\]

(3.9)

Substituting (2.9), (2.10) and (2.11) in (3.9), we have

\[
QX = -\frac{b}{a} (f_1 - f_3) + 2n(f_1 - f_3)\frac{a}{b} - \frac{r}{a(2n + 1)} \left[ \frac{a}{2n} + 2b \right] X \\
+ \frac{b}{a} (f_1 - f_3) + 4n(f_1 - f_3)\frac{a}{b} - \frac{r}{a(2n + 1)} \left[ \frac{a}{2n} + 2b \right] \eta(X)\xi.
\]

(3.10)
Therefore, (3.10) is of the form

\[ S(X, W) = g(QX, W) = Ag(X, W) + B\eta(X)\eta(W), \]

where \( A = -\frac{b}{a}[(f_1 - f_3) + 2n(f_1 - f_3)\frac{a}{b} - \frac{r}{a(2n+1)} \left( \frac{a}{2n} + 2b \right)] \) and \( B = \frac{b}{a}[(f_1 - f_3) + 4n(f_1 - f_3)\frac{a}{b} - \frac{r}{a(2n+1)} \left( \frac{a}{2n} + 2b \right)] \). Therefore, the manifold is an \( \eta \)-Einstein manifold.

Now comparing (2.7) and (3.11) yields

\[ 2nf_1 + 3f_2 - f_3 = -\frac{b}{a} \left( f_1 - f_3 \right) + 2n(f_1 - f_3)\frac{a}{b} \]

(3.12)

\[ -(3f_2 + (2n-1)f_3) = \frac{b}{a} \left( f_1 - f_3 \right) + 4n(f_1 - f_3)\frac{a}{b} \]

(3.13)

The above two equations can be written as

\[ (4n + \frac{b}{a})f_1 + 3f_2 + \left( -1 - 2n - \frac{b}{a} \right)f_3 = -\frac{rb}{a^2(2n+1)} \left( \frac{a}{2n} + 2b \right). \]

(3.14)

\[-\left( 4n + \frac{b}{a} \right)f_1 - 3f_2 + \left( 1 + 2n + \frac{b}{a} \right)f_3 = -\frac{rb}{a^2(2n+1)} \left( \frac{a}{2n} + 2b \right). \]

(3.15)

Since by hypothesis \( a \neq 0, b \neq 0 \), then from the above equations (3.14) and (3.15), we obtain \( r = 0 \), provided \( a + 4nb \neq 0 \). In view of the above, we are in a position to state the following.

**Theorem 3.2.** In a \((2n + 1)\)-dimensional \((n > 1)\) quasiconformally flat generalized Sasakian-space-form, the scalar curvature vanishes, provided \( a + 4nb \neq 0 \).

### 4. Quasi-conformally semisymmetric generalized Sasakian-space-forms

Let us consider a quasi-conformally semisymmetric generalized Sasakian-space-form \( M(f_1, f_2, f_3) \). Therefore, we have

\[ R(X, Y) \cdot \tilde{C} = 0 \]

(4.1)

holds on \( M \) for every vector fields \( X, Y \). Hence we have

\[ (R(X, Y) \cdot \tilde{C})(U, V)W = R(X, Y)\tilde{C}(U, V)W - \tilde{C}(R(X, Y)U, V)W - \tilde{C}(U, R(X, Y)V)W + \tilde{C}(U, V)R(X, Y)W = 0. \]

(4.2)

Putting \( X = \xi \) in (4.2), we get

\[ R(\xi, Y)\tilde{C}(U, V)W - \tilde{C}(R(\xi, Y)U, V)W - \tilde{C}(U, R(\xi, Y)V)W \]

\[ -\tilde{C}(U, V)R(\xi, Y)W = 0. \]

(4.3)
Using (2.9) in (4.3), we have

\[(f_1 - f_3)\{g(Y, \tilde{C}(U, V)W)\xi - \eta(\tilde{C}(U, V)W)Y - g(Y, U)\tilde{C}(\xi, V)W
\]

\[\quad + \eta(U)\tilde{C}(Y, V)W - g(Y, V)\tilde{C}(U, \xi)W + \eta(V)\tilde{C}(U, Y)W
\]

\[-g(Y, W)\tilde{C}(U, V)\xi + \eta(W)\tilde{C}(U, V)Y\} = 0.\]  \hspace{1cm} (4.4)

Taking inner product of (4.4) by \(\xi\), we have

\[(f_1 - f_3)\{g(Y, \tilde{C}(U, V)W) - \eta(\tilde{C}(U, V)W)\eta(Y)
\]

\[\quad + (Y)\eta(\tilde{C}(\xi, V)W) + \eta(U)\eta(\tilde{C}(U, V)W) - g(Y, V)\eta(\tilde{C}(U, \xi)W)
\]

\[\quad + \eta(V)\eta(\tilde{C}(U, Y)W) + \eta(W)\eta(\tilde{C}(U, V)Y)\} = 0.\]  \hspace{1cm} (4.5)

Putting \(Y = U\) in (4.5), we get

\[(f_1 - f_3)\{g(U, \tilde{C}(U, V)W) - \eta(\tilde{C}(U, V)W)\eta(U)
\]

\[\quad + (U)\eta(\tilde{C}(\xi, V)W) + \eta(U)\eta(\tilde{C}(U, V)W) - g(U, V)\eta(\tilde{C}(U, \xi)W)
\]

\[\quad + \eta(V)\eta(\tilde{C}(U, U)W) + \eta(W)\eta(\tilde{C}(U, V)U)\} = 0.\]  \hspace{1cm} (4.6)

It follows that

\[\eta(\tilde{C}(X, Y)Z) = \frac{a + (2n_1)b}{(2n + 1)} [-3f_2 + (1 - 2n)f_3] \{g(Y, Z)\eta(X)
\]

\[\quad - g(X, Z)\eta(Y)\} = 0.\]  \hspace{1cm} (4.7)

Putting \(Z = \xi\) in equation (4.7) yields

\[\eta(\tilde{C}(X, Y)\xi) = 0.\]  \hspace{1cm} (4.8)

Thus using (4.8) in (4.6), we get

\[(f_1 - f_3)\{g(U, \tilde{C}(U, V)W) - g(U, U)\eta(\tilde{C}(\xi, V)W)
\]

\[\quad - g(U, V)\eta(\tilde{C}(U, \xi)W) + \eta(W)\eta(\tilde{C}(U, V)U)\} = 0.\]  \hspace{1cm} (4.9)

Let \(\{e_i\}, 1 \leq i \leq 2n + 1, (e_{2n+1} = \xi)\) be an orthonormal basis of the tangent space at each point of the manifold. Then summing for \(U = e_i, 1 \leq i \leq 2n + 1\), the relation (4.9) give us

\[(f_1 - f_3)\{g(e_i, \tilde{C}(e_i, V)W) - g(e_i, e_i)\eta(\tilde{C}(\xi, V)W)
\]

\[\quad - g(e_i, V)\eta(\tilde{C}(e_i, \xi)W) + \eta(W)\eta(\tilde{C}(e_i, V)e_i)\} = 0.\]  \hspace{1cm} (4.10)

On the other hand, from (4.7), we have

\[\eta(\tilde{C}(\xi, V)W) = \frac{a + (2n_1)b}{(2n + 1)} [-3f_2 + (1 - 2n)f_3]
\]

\[\times \{g(W, V) - \eta(W)\eta(V)\} = 0.\]  \hspace{1cm} (4.11)

Using (4.11) in (4.10), we have

\[(f_1(4.12))\{g(e_i, \tilde{C}(e_i, V)W) + 2n_1\frac{a + (2n - 1)b}{2n + 1} [3f_2 + (1 - 2n)f_3]g(W, V)\} = 0.\]
Now
\begin{equation}
\tilde{C}(e_i, V)W, e_i = \frac{a + (2n - 1)b}{2n + 1} (3f_2 + (2n - 1)f_3) \\
\times [g(W, V) - (2n + 1)\eta(W)\eta(V)].
\tag{4.13}
\end{equation}

Using (4.13) in (4.12), we have
\begin{equation}
(f_1 - f_3)(a + (2n - 1)b)(3f_2 + (2n - 1)f_3)[g(W, V) - \eta(W)\eta(V)] = 0.
\tag{4.14}
\end{equation}

Therefore, we can state the following theorem.

**Theorem 4.1.** Let $M(f_1, f_2, f_3)$ ($n > 1$) be a quasi-conformally semisymmetric generalized Sasakian-space-form. Then one of the following statements holds:

1. $f_1 = f_3$
2. $b = -\frac{a}{2n - 1}$
3. $3f_2 = -(2n - 1)f_3.$

Now we consider the case (3). If $b = -\frac{a}{2n - 1},$ then substituting it in (2.15), we observe that $\tilde{C}(X, Y)Z = aC(X, Y)Z.$ Therefore, in this case, quasi-conformally semisymmetric and conformally semisymmetric generalized Sasakian-space-forms are equivalent. In a recent paper De and Majhi [15] proved that for a $(2n + 1)$-dimensional $(n > 1)$ conformally semisymmetric generalized Sasakian-space-form $M(f_1, f_2, f_3),$ either $f_1 = f_3$ or $f_2 = 0.$ Thus, we can state the following corollary.

**Corollary 4.1.** In a $(2n + 1)$-dimensional $(n > 1)$ quasi-conformally semisymmetric generalized Sasakian-space-form $M(f_1, f_2, f_3),$ either $f_1 = f_3$ or $f_2 = 0.$

Again, in view of the second part of the Kim’s [21] theorem, we have the following corollary.

**Corollary 4.2.** For a $(2n + 1)$-dimensional $(n > 1)$ quasi-conformally semisymmetric generalized Sasakian-space-form $M(f_1, f_2, f_3)$ with $\xi$ as a Killing vector field, either $f_1 = f_3$ or the space-form is locally symmetric and has a constant $\phi$-sectional curvature.

**References**


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