ON A CLASS OF GENERALIZED SASAKIAN-SPACE-FORMS

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Abstract. The object of the present paper is to study quasi-conformally flat generalized Sasakian-space-forms. Also we study quasi-conformally semisymmetric generalized Sasakian-space-forms. As a consequence of the results, we obtain some important corollaries.

1. Introduction

The nature of a Riemannian manifold mostly depends on the curvature tensor $R$ of the manifold. It is well known that the sectional curvatures of a manifold determine its curvature tensor completely. A Riemannian manifold with a constant sectional curvature $c$ is known as a real space-form and its curvature tensor is given by

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}.$$  

A Sasakian manifold with constant $\phi$-sectional curvature is a Sasakian space-form and has a specific form of its curvature tensor. Similar notion also holds for Kenmotsu and cosymplectic space-forms. In order to generalize such space-forms in a common frame, Alegre, Blair and Carriazo [1] introduced and studied generalized Sasakian space-forms. These space-forms are defined as follows.

Given an almost contact metric manifold $M(\phi, \xi, \eta, g)$, we say that $M$ is a generalized Sasakian space-form if there exist three functions $f_1, f_2, f_3$ on $M$ such that the curvature tensor $R$ is given by

$$R(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\}$$

$$+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\}$$

$$+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X$$

$$+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \tag{1.1}$$

for any vector fields $X, Y, Z$ on $M$. In such a case we denote the manifold as $M(f_1, f_2, f_3)$. In [1], the authors cited several examples of generalized Sasakian space-forms. If $f_1 = \frac{c+1}{4}$, $f_2 = \frac{c-1}{4}$ and $f_3 = \frac{-c-1}{4}$, then a generalized Sasakian space-form with Sasakian structure becomes a Sasakian space-form. In [21], Kim studied conformally flat generalized Sasakian space-forms and locally symmetric...
generalized Sasakian space-forms. He proved some geometric properties of generalized Sasakian space-form which depends on the nature of the functions $f_1$, $f_2$ and $f_3$. A large number of geometers studied generalized Sasakian space-forms in the papers ([2], [3], [4], [5], [14], [16], [17], [19]). In [13], De and Sarkar studied locally $\phi$-symmetric generalized Sasakian space-forms and generalized Sasakian space-forms with $\eta$-recurrent Ricci tensor. Also De and Sarkar [12] studied projectively flat, projectively semisymmetric generalized Sasakian space-forms. Motivated by these studies, in this paper, we study quasi-conformally flat generalized Sasakian space-forms. Beside these we study quasi-conformally semisymmetric generalized Sasakian space-forms.

The present paper is organized as follows. After preliminaries in Section 3, we consider quasi-conformally flat generalized Sasakian space-forms. Section 4 deals with quasi-conformally semisymmetric generalized Sasakian space-form. As a consequence of the results, we obtain some important corollaries.

2. preliminaries

In an almost contact metric manifold, we have ([8], [9]):

\begin{align}
\phi^2(X) &= -X + \eta(X)\xi, & \phi \xi &= 0. \\
\eta(\xi) &= 1, & g(X, \xi) &= \eta(X), & \eta(\phi X) &= 0. \\
g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y). \\
g(\phi X, Y) &= -g(X, \phi Y), & g(\phi X, X) &= 0. \\
g(\phi X, \xi) &= 0. \\
2n \eta(X) &= (2nf_1 + 3f_2 - f_3)g(X, Y) - (3f_2 + (2n - 1)f_3)\eta(X)\eta(Y). \\
QX &= (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n - 1)f_3)\eta(X)\xi. \\
R(X, Y)\xi &= (f_1 - f_3)[\eta(Y)X - \eta(X)Y]. \\
R(\xi, X)Y &= (f_1 - f_3)[g(X, Y)\xi - \eta(Y)X]. \\
S(X, \xi) &= 2n(f_1 - f_3)\eta(X). \\
S(\xi, \xi) &= 2nf_1 - f_3. \\
Q\xi &= 2n(f_1 - f_3)\xi. \\
r &= 2n(2n + 1)f_1 + 6nf_2 - 4nf_3,
\end{align}

The present paper is organized as follows. After preliminaries in Section 3, we consider quasi-conformally flat generalized Sasakian space-forms. As a consequence of the results, we obtain some important corollaries.
where $R$, $S$ and $r$ are the curvature tensor, Ricci tensor and scalar curvature of the space-form, respectively, and $Q$ is the Ricci operator defined by $g(QX,Y) = S(X,Y)$. We know that [1] the $\phi$-sectional curvature of a generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is $f_1 + 3f_2$.

An almost contact metric manifold $M$ is said to be an $\eta$-Einstein manifold if the Ricci tensor satisfies the condition $S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$, where $a$ and $b$ are smooth functions on $M$. In [7], Bejan studied geometric properties of $\eta$-Einstein manifolds.

Let $(M, g)$ be a Riemannian manifold and let $\nabla$ be the Levi-Civita connection of $(M, g)$. A Riemannian manifold is called locally symmetric [10] if $\nabla R = 0$, where $R$ is the Riemannian curvature tensor of $(M, g)$.

The notion of the quasi-conformal curvature tensor was given by Yano and Sawaki [22]. The quasi-conformal curvature tensor $\tilde{C}$ is defined by

\[
\tilde{C}(X,Y)Z = aR(X,Y)Z + bS(Y,Z)X - S(X,Z)Y + g(Y,Z)QX
- g(X,Z)QY \frac{r}{2n+1} + \frac{a}{2n} + 2b \right] g(Y,Z)X - g(X,Z)Y,
\]

where $a$ and $b$ are constants, and $R$, $Q$ and $r$ are the Riemannian curvature tensor of type $(1,3)$, the Ricci operator defined by $g(QX,Y) = S(X,Y)$ and the scalar curvature, respectively. If $a = 1$ and $b = -\frac{1}{2n-1}$, then (2.15) takes the form

\[
\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{1}{2n-1}|S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX
- g(X,Z)QY + \frac{r}{2n(2n-1)} g(Y,Z)X - g(X,Z)Y |
= C(X,Y)Z,
\]

where $C$ is the conformal curvature tensor [18]. Thus the conformal curvature tensor $C$ is the particular case of the tensor $\tilde{C}$. For this reason, $\tilde{C}$ is called a quasi-conformal curvature tensor. A manifold $(M^n, g)$ $(n > 3)$ is called quasi-conformally flat if $\tilde{C} = 0$. It is known [6] that a quasi conformally flat manifold is either conformally flat if $a \neq 0$ or Einstein if $a = 0$ and $b \neq 0$. Since they give no restrictions for manifolds if $a = 0$ and $b = 0$, it is essential for us to consider the case of $a \neq 0$ or $b \neq 0$. Quasi-conformally flat Riemannian manifolds with different structures have been studied by U. C. De at el. ([10], [11], [20]) and many others.

**Definition 2.1.** A Riemannian manifold is said to be quasi-conformally semi-symmetric if $R(X, Y) \cdot \tilde{C} = 0$.

In [21], U. K. Kim proved that for a $(2n+1)$-dimensional generalized Sasakian space-form, the following results hold:

1. If $n > 1$, then $M$ is conformally flat if and only if $f_2 = 0$.
2. If $M$ is conformally flat and $\xi$ is a Killing vector field, then it is flat, or of a constant curvature, or locally the product $N^1 \times N^{2n}$, where $N^1$ is a 1-dimensional manifold and $N^{2n}$ is a $2n$-dimensional Hermitian manifold of
constant curvature. In any case, \( M \) is locally symmetric and has a constant \( \phi \)-sectional curvature.

3. Quasi-conformally flat generalized Sasakian-space-form

Assume that \( M(f_1, f_2, f_3) \) is a quasi-conformally flat generalized Sasakian-space-form. Then

\[
\tilde{C}(X, Y)Z = 0
\]

for all vector field \( X, Y \) and \( Z \). Using (2.5) in (3.1), we have

\[
\frac{1}{2n + 1} \left[ (-3a + 6b)f_2 + (2a + 2(2n - 1)b)f_3 \right] \{ g(Y, Z)X - g(X, Z)Y \}
\]

\[
+ af_2 \{ g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z \}
\]

\[
+ \{ (a + (2n - 1)b)f_3 + 3bf_2 \} \{ \eta(\phi Y)\eta(Z)Y - \eta(Y)\eta(Z)\phi Y
\]

\[+ g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi \} = 0.
\]

If we put \( X = \phi Y \) in (3.2), we get

\[
\frac{1}{2n + 1} \left[ (-3a + 6b)f_2 + (2a + 2(2n - 1)b)f_3 \right] \{ g(Y, Z)\phi Y - g(\phi Y, Z)Y \}
\]

\[
+ af_2 \{ g(\phi Y, \phi Z)\phi Y - g(Y, \phi Z)\phi^2 Y + 2g(\phi Y, \phi Y)\phi Z \}
\]

\[
+ \{ (a + (2n - 1)b)f_3 + 3bf_2 \} \{ \eta(\phi Y)\eta(Z)Y - \eta(Y)\eta(Z)\phi Y
\]

\[+ g(\phi Y, Z)\eta(Y)\xi - g(Y, Z)\eta(\phi Y)\xi \} = 0.
\]

If we choose a unit vector \( U \) such that \( g(U, \xi) = 0 \) and substitute \( Y = U \) in (3.3), we have

\[
\frac{1}{2n + 1} \left[ \{ (2n - 1)a + 6b \} f_2 + 2(a + (2n - 1)b)f_3 \right]
\]

\[
\{ g(U, Z)\phi U - g(\phi U, Z)U \} + 2af_2\phi Z = 0.
\]

Putting \( Z = U \) in (3.4), we have

\[
\{ (2n - 1)a + 6b + 2(2n + 1)a \} f_2 + 2(a + (2n - 1)b)f_3 \phi U = 0.
\]

Thus we have

\[
(2n - 1)a + 6b + 2(2n + 1)a \} f_2 + 2(a + (2n - 1)b)f_3 = 0.
\]

It follows that

\[
f_2 = -\frac{(a + (2n - 1)b)}{3(an + b)}f_3.
\]

Conversely, if \( f_2 = -\frac{(a + (2n - 1)b)}{3(an + b)}f_3 \), then from (3.1), we have \( \tilde{C}(X, Y)Z = 0 \) and \( M(f_1, f_2, f_3) \) is a quasi-conformally flat. Thus when \( n > 1 \), \( M(f_1, f_2, f_3) \) is quasi-conformally flat if and only if \( f_2 = -\frac{(a + (2n - 1)b)}{3(an + b)}f_3 \). Therefore, we can state the following theorem.
Theorem 3.1. Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional generalized Sasakian-space-form. If $n > 1$, then $M(f_1, f_2, f_3)$ is quasi-conformally flat if and only if $f_2 = \frac{(a+2(n-1)b)}{a(2n+1)+2b} f_3$, provided $(an+b) \neq 0$.

Also if $a = 1$ and $b = \frac{1}{2n-1}$, then a quasi-conformal curvature tensor reduces to a conformal curvature tensor. Therefore, in view of the result of Kim [21] and (3.7), we get $f_2 = 0$ and conversely. Therefore we conclude the following corollary.

Corollary 3.1. A generalized Sasakian-space-form $M(f_1, f_2, f_3)$, $n > 1$, is conformally flat if and only if $f_2 = 0$.

The above corollary have been proved by Kim [21].

In [1], P. Alegre, D. Blair and A. Carriazo proved that if a generalized Sasakian space-form $M(f_1, f_2, f_3)$ is a Sasakian manifold, then the functions $f_1, f_2, f_3$ are constant and $f_1 - 1 = f_2 = f_3$.

Now, in our case, $f_2 = \frac{(a+(2n-1)b)}{a(2n+1)+2b} f_3$ implies $f_2 = 0$ provided $3(n+1)a + (2n+1)b \neq 0$. Thus $f_2 = 0$ implies $f_3 = 0$ and $f_1 = 1$. Thus from (1.1), we obtain

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y ,$$

that is, the manifold is of constant curvature 1. Hence we can state the following corollary.

Corollary 3.2. A $(2n+1)$-dimensional $(n > 1)$ quasi-conformally flat Sasakian manifold is a manifold of a constant curvature 1, that is, locally isometric to the unit sphere $S^{(2n+1)}(1)$.

Remark 1. If $an + b = 0$, then from (3.7), we infer $\{a+(2n-1)b\} f_3 = 0$, which implies that either $f_3 = 0$ or $a + (2n-1)b = 0$. Suppose $a + (2n-1)b = 0$, then from the definition of quasi-conformal curvature tensor, it follows that $\tilde{C} = bC$, hence quasi conformally flatness and conformally flatness are equivalent, since by hypothesis $b \neq 0$.

Moreover, the quasi-conformally flat generalized Sasakian space-form implies

$$R(X, Y)Z = -\frac{a}{b} |S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY|$$

(3.8) 

$$+ \frac{r}{a(2n+1)} \left[ \frac{a}{2n} + 2b \right] g(Y, Z)X - g(X, Z)Y.$$ 

Puting $Y = Z = \xi$ in (3.8), we get

$$R(X, \xi)\xi = -\frac{a}{b} |S(\xi, \xi)X - S(X, \xi)\xi + g(\xi, \xi)QX - g(X, \xi)Q\xi|$$

(3.9) 

$$+ \frac{r}{a(2n+1)} \left[ \frac{a}{2n} + 2b \right] g(\xi, \xi)X - g(X, \xi)\xi.$$ 

Substituting (2.9), (2.10) and (2.11) in (3.9), we have

$$QX = -\frac{b}{a} (f_1 - f_3) + 2a(f_1 - f_3) \frac{a}{b} - \frac{r}{a(2n+1)} \left[ \frac{a}{2n} + 2 \right] X$$

(3.10) 

$$+ \frac{b}{a} (f_1 - f_3) + 4a(f_1 - f_3) \frac{a}{b} - \frac{r}{a(2n+1)} \left[ \frac{a}{2n} + 2b \right] \eta(X)\xi.$$
Therefore, (3.10) is of the form
\[ S(X,W) = g(QX,W) = Ag(X,W) + B\eta(X)\eta(W), \]
where
\[ A = -\frac{b}{a}\left[ (f_1 - f_3) + 2n(f_1 - f_3)\frac{a}{b} - \frac{r}{a(2n + 1)}\left( \frac{a}{2n} + 2b \right) \right] \]
and
\[ B = \frac{b}{a}\left[ (f_1 - f_3) + 4n(f_1 - f_3)\frac{a}{b} - \frac{r}{a(2n + 1)}\left( \frac{a}{2n} + 2b \right) \right]. \]
Therefore, the manifold is an \( \eta \)-Einstein manifold. Now comparing (2.7) and (3.11) yields
\[ 2nf_1 + 3f_2 - f_3 = -\frac{b}{a}\left[ (f_1 - f_3) + 2n(f_1 - f_3)\frac{a}{b} \right] \]
\[ -\frac{r}{a(2n + 1)}\left( \frac{a}{2n} + 2b \right) \]
\[ -(3f_2 + (2n - 1)f_3) = \frac{b}{a}\left[ (f_1 - f_3) + 4n(f_1 - f_3)\frac{a}{b} \right] \]
\[ -\frac{r}{a(2n + 1)}\left( \frac{a}{2n} + 2b \right). \]
The above two equations can be written as
\[ \left( 4n + \frac{b}{a} \right)f_1 + 3f_2 + \left( -1 - 2n - \frac{b}{a} \right)f_3 = -\frac{rb}{a^2(2n + 1)}\left( \frac{a}{2n} + 2b \right) \]
\[ -\left( 4n + \frac{b}{a} \right)f_1 - 3f_2 + \left( 1 + 2n + \frac{b}{a} \right)f_3 = -\frac{rb}{a^2(2n + 1)}\left( \frac{a}{2n} + 2b \right). \]
Since by hypothesis \( a \neq 0, b \neq 0 \), then from the above equations (3.14) and (3.15), we obtain \( r = 0 \), provided \( a + 4nb \neq 0 \). In view of the above, we are in a position to state the following.

**Theorem 3.2.** In a \((2n + 1)\)-dimensional \((n > 1)\) quasi-conformally flat generalized Sasakian-space-form, the scalar curvature vanishes, provided \( a + 4nb \neq 0 \).

### 4. QUASI-CONFORMALLY SEMISYMMETRIC GENERALIZED SASAKIAN-SPACE-FORMS

Let us consider a quasi-conformally semisymmetric generalized Sasakian-space-form \( M(f_1, f_2, f_3) \). Therefore, we have
\[ R(X,Y)\cdot C = 0 \]
holds on \( M \) for every vector fields \( X, Y \). Hence we have
\[ (R(X,Y)\cdot C)(U,V)W = R(X,Y)C(U,V)W - C(R(X,Y)U,V)W - \tilde{C}(U,R(X,Y)V)W = 0. \]
Putting \( X = \xi \) in (4.2), we get
\[ R(\xi,Y)C(U,V)W - C(R(\xi,Y)U,V)W - \tilde{C}(U,R(\xi,Y)V)W \]
\[ -\tilde{C}(U,V)R(\xi,Y)W = 0. \]
Using (2.9) in (4.3), we have

\[(f_1 - f_3)\{g(Y, \tilde{C}(U, V)W)\xi - \eta(\tilde{C}(U, V)W)Y - g(Y, U)\tilde{C}(\xi, V)W + \eta(U)\tilde{C}(Y, \xi)W + g(Y, V)\tilde{C}(U, \xi)W - g(\xi, Y)\tilde{C}(U, V) + \eta(W)\tilde{C}(U, V)Y\} = 0.\]  

(4.4)

Taking inner product of (4.4) by \(\xi\), we have

\[(f_1 - f_3)\{g(Y, \tilde{C}(U, V)W) - \eta(\tilde{C}(U, V)W)\eta(Y) - g(Y, U)\eta(\tilde{C}(\xi, V)W) + \eta(U)\eta(\tilde{C}(Y, \xi)W) + g(Y, V)\eta(\tilde{C}(U, \xi)W) + \eta(V)\eta(\tilde{C}(U, Y)W) + \eta(W)\eta(\tilde{C}(U, V)Y)\} = 0.\]  

(4.5) - (4.6)

Putting \(Y = U\) in (4.5), we get

\[(f_1 - f_3)\{g(U, \tilde{C}(U, V)W) - \eta(\tilde{C}(U, U)W)\eta(U) + \eta(U)\eta(\tilde{C}(\xi, V)W) - g(U, V)\eta(\tilde{C}(U, \xi)W) + \eta(V)\eta(\tilde{C}(U, U)W) + \eta(W)\eta(\tilde{C}(U, V)Y)\} = 0.\]  

(4.7)

It follows that

\[\eta(\tilde{C}(X, Y)Z) = \frac{a + (2n + b)}{(2n + 1)} [-3f_2 + (1 - 2n)f_3]\{g(Y, Z)\eta(Y) - g(X, Z)\eta(Y)\} = 0.\]  

(4.8)

Putting \(Z = \xi\) in equation (4.7) yields

\[\eta(\tilde{C}(X, Y)\xi) = 0.\]  

(4.9)

Thus using (4.8) in (4.6), we get

\[(f_1 - f_3)\{g(U, \tilde{C}(U, V)W) - g(U, U)\eta(\tilde{C}(\xi, V)W) - g(U, V)\eta(\tilde{C}(U, \xi)W) + \eta(W)\eta(\tilde{C}(U, V)U)\} = 0.\]  

(4.10)

Let \(\{e_i\}, 1 \leq i \leq 2n + 1, \{e_{2n+1}\} = \xi\) be an orthonormal basis of the tangent space at each point of the manifold. Then summing for \(U = e_i, 1 \leq i \leq 2n + 1\), the relation (4.9) give us

\[(f_1 - f_3)\{g(e_i, \tilde{C}(e_i, V)W) - g(e_i, e_i)\eta(\tilde{C}(\xi, V)W) - g(e_i, V)\eta(\tilde{C}(e_i, \xi)W) + \eta(W)\eta(\tilde{C}(e_i, V)e_i)\} = 0.\]  

(4.11)

On the other hand, from (4.7), we have

\[\eta(\tilde{C}(\xi, V)W) = \frac{a + (2n + b)}{(2n + 1)} [-3f_2 + (1 - 2n)f_3] \times \{g(W, V) - \eta(W)\eta(V)\} = 0.\]
Using (4.11) in (4.10), we have
\[
(f_1 - f_3) \left\{ g(e_i, \tilde{C}(e_i, V) W) + 2n a + \frac{2n - 1}{2n + 1} [3f_2 + (1 - 2n) f_3] g(W, V) \right\} = 0.
\]

(4.12)

Now
\[
g(\tilde{C}(e_i, V) W, e_i) = \frac{a + (2n - 1)b}{2n + 1} (3f_2 + (2n - 1)f_3) \times [g(W, V) - (2n + 1)\eta(W)\eta(V)].
\]

(4.13)

Using (4.13) in (4.12), we have
\[
(f_1 - f_3) (a + (2n - 1)b) (3f_2 + (2n - 1)f_3) [g(W, V) - \eta(W)\eta(V)] = 0.
\]

(4.14)

Therefore, we can state the following theorem.

**Theorem 4.1.** Let \(M(f_1, f_2, f_3)\) \((n > 1)\) be a quasi-conformally semisymmetric generalized Sasakian-space-form. Then one of the following statements holds:

1. \(f_1 = f_3\)
2. \(b = \frac{a}{2n - 1}\)
3. \(3f_2 = -(2n - 1)f_3\).

Now we consider the case (3). If \(b = \frac{a}{2n - 1}\), then substituting it in (2.15), we observe that \(\tilde{C}(X, Y) Z = aC(X, Y) Z\). Therefore, in this case, quasi-conformally semisymmetric and conformally semisymmetric generalized Sasakian-space-forms are equivalent. In a recent paper De and Majhi [15] proved that for a \((2n + 1)\)-dimensional \((n > 1)\) conformally semisymmetric generalized Sasakian-space-form \(M(f_1, f_2, f_3)\), either \(f_1 = f_3\) or \(f_2 = 0\). Thus, we can state the following corollary.

**Corollary 4.1.** In a \((2n + 1)\)-dimensional \((n > 1)\) quasi-conformally semisymmetric generalized Sasakian-space-form \(M(f_1, f_2, f_3)\), either \(f_1 = f_3\) or \(f_2 = 0\).

Again, in view of the second part of the Kim’s [21] theorem, we have the following corollary.

**Corollary 4.2.** For a \((2n+1)\)-dimensional \((n > 1)\) quasi-conformally semisymmetric generalized Sasakian-space-form \(M(f_1, f_2, f_3)\) with \(\xi\) as a Killing vector field, either \(f_1 = f_3\) or the space-form is locally symmetric and has a constant \(\varphi\)-sectional curvature.

**References**


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