Acta Math. Univ. Comenianae
Vol. LXXXIV, 1 (2015), pp. 79–96

EQUILIBRIA AND STABLE PATHS IN INFINITE HORIZON NONLINEAR CONTROL PROBLEMS: THE LINEAR-QUADRATIC APPROXIMATION

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Abstract. Nonlinear discrete time infinite horizon problems with discount are discussed. It is assumed that the problem without discount admits a nondegenerate steady state "extremal" solution. Under this and certain additional hypotheses it is proved that for sufficiently mild discounts the steady state solution exists, for initial conditions sufficiently close to it the problem has a solution of the stable path type and that the solution can be approximated by the linear-quadratic truncation of the problem.

1. Introduction

For several decades, dynamic models based on optimal control theory have been popular in macroeconomics. In the spirit of the early Ramsey model the interest focused on the presence of equilibria of infinite horizon problems as well as existence and uniqueness of stable (also called saddle) paths converging to them. Both discrete time and continuous time problems have been employed, the former e. g. in real business cycle theory [**7**], [**14**], the latter in growth and convergence economic models [**14**], [**2**]. Because equilibria not always and stable paths rarely admit a closed form expression, approximative methods had to be employed. To this end the problem has commonly been locally at the equilibrium truncated to a linear-quadratic one [**14**], [**4**]. The purpose of this paper is to provide a rigorous justification of this method. We have chosen to focus on the discrete time option. The continuous time is to a large extent analogous, the differences being discussed in the concluding Section 5.

The optimal control problem discussed in this paper is as follows: Let $X \subseteq \mathbb{R}^n$, $U \subseteq \mathbb{R}^m$ be open, $x_0 \in X$, $f \in C^3(X \times U, \mathbb{R})$, $F \in C^3(X \times U, X)$ and $\beta \in (0, 1]$. By a *feasible control/response pair* we understand a pair of sequences $\{u_t\}_{t=0}^{\infty}$,

Received April 11, 2014; revised August 22, 2014.
2010 *Mathematics Subject Classification.* Primary 91B62, 49N10.
Key words and phrases. Optimal control; infinite horizon; discrete time; stable path.

$\{x_t\}_{t=0}^{\infty}$ satisfying

$$x_t \in X, \tag{1}$$

$$u_t \in U, \tag{2}$$

$$x_{t+1} = F(x_t, u_t) \qquad \text{for } t = 0, 1, \dots \tag{3}$$

such that the series

$$J(x_0, \{u_t\}) = \sum_{t=0}^{\infty} \beta^t f(x_t, u_t) \tag{4}$$

converges. We denote by $\phi(\beta, x_0)$ the set of all feasible controls and by $\Phi(\beta, x_0)$ the set of all feasible control/response pairs.

The control/response pair $(\{\hat{x}_t\}, \{\hat{u}_t\}) \in \Phi(\beta, x_0)$ is called *optimal* if

$$\sum_{t=0}^{\infty} \beta^t f(\hat{x}_t, \hat{u}_t) \geq \sum_{t=0}^{\infty} \beta^t f(x_t, u_t) \tag{5}$$

holds for all $(\{x_t\}, \{u_t\}) \in \Phi(\beta, x_0)$. The problem of finding an optimal control we will label by $(\mathbf{U}_{\beta,x_0})$, the family of problems $(\mathbf{U}_{\beta,x_0})$ with $x_0 \in X$ by $(\mathbf{U}_\beta)$.

The crucial assumption of this paper is that there exists an $\overline{x}_0$ such that the problem $(\mathbf{U}_{\beta,\overline{x}_0})$ has an *equilibrium* solution, i. e. an optimal control/response pair $\{\hat{x}_t\}, \{\hat{u}_t\}$ such that $\hat{x}_t \equiv \overline{x}_0$ and $\hat{u}_t \equiv \overline{u}_0$ for some $\overline{u}_0$ and all t. As a rule, in macroeconomic applications this is commonly the case and this equilibrium solution can be determined either analytically or numerically.

The interest is to find hypotheses under which the problem $(\mathbf{U}_{\beta,x_0})$ has a unique "stable path", i. e., optimal control/response pair for any $x_0 \in X$ (where typically $X = \mathbb{R}^n$ or X is a neighborhood of $\overline{x}_0$), with the control and response sequences converging to $(\overline{x}_0, \overline{u}_0)$, respectively.

The principal goal of the present paper is to formulate and prove verifiable and interpretable hypotheses of general nature under which this methodology is justified. Although in economic applications $\beta < 1$ as a rule, in order to establish such hypotheses we imbed the problem $(\mathbf{U}_\beta)$ for fixed β into a family of problems including the limit case $\beta = 1$. The hypotheses are formulated in terms of this limit case.

The paper is organized as follows: In Section 2 we introduce the concept of an extremal control/response pair and the basic hypotheses. In Section 3 we discuss the linear-quadratic truncation of the problem. In Section 4 we extend the results of Section 3 to their local version for the full problem. Finally, in the concluding Section 5 we discuss several issues including the relation of our results to economic models and to the continuous time version of the problem.

2. The necessary condition of optimality and the extremal steady state

In this section we introduce the concept of normal extremal steady state for the extension $(\mathbf{U}_1)$ of the problem $(\mathbf{U}_\beta)$ to $\beta = 1$. The problem $(\mathbf{U}_1)$ will serve as an anchor for the family of problems $(\mathbf{U}_\beta)$ for β sufficiently close to 1. Since the convergence of the series in (4) for $\beta = 1$ requires more stringent assumptions on f than those for the case $\beta < 1$ we phrase the anchor problem in terms of necessary conditions of optimality rather than optimality itself. We derive this condition under the assumption

Q1: $D_x F(x, u)$ is nonsingular for all $x \in X$ and $u \in U$.

This assumption as well as additional hypotheses labeled by **Q** introduced later will be kept throughout the paper unless said otherwise.

By [**5**], under **Q1**, a necessary condition of optimality of a control/response pair $(\{\hat{x}_t\}, \{\hat{u}_t\})$ for the problem $(\mathbf{U}_{\beta,x_0})$ is that there exists a constant $\psi^0 \geq 0$ and a sequence $\{\psi_t\}$, $\psi_t \in \mathbb{R}^n$ for $t = 0, 1, 2, \dots$ such that $(\psi^0, \psi_t) \neq 0$ and the system of equations

$$x_{t+1} = F(x_t, u_t), \tag{6}$$

$$0 = \psi^0 D_u^T f(x_t, u_t) + \beta D_u^T F(x_t, u_t)\psi_{t+1}, \tag{7}$$

$$\psi_t = \psi^0 D_x^T f(x_t, u_t) + \beta D_x^T F(x_t, u_t)\psi_{t+1} \tag{8}$$

is satisfied for $u_t = \hat{u}_t$, $x_t = \hat{x}_t$ and all $t = 0, 1, \dots$. Given $\beta \in (0, 1]$, a triple of sequences $\{x_t\}_{t=0}^{\infty}$, $\{u_t\}_{t=0}^{\infty}$ and $\{\psi_t\}_{t=0}^{\infty}$ satisfying the system (6), (7), (8) (the superscript T meaning transposition) will be called *extremal*, equations (7), (8) being called *maximum condition, adjoint equation* respectively. The extremal triple will be called *normal* if $\psi^0 \neq 0$ (in which case one can without loss of generality take $\psi^0 = 1$). By a *steady state extremal triple* we will understand a triple $(\overline{x}, \overline{u}, \overline{\psi}) \in X \times U \times \mathbb{R}^n$ of constants such that $x_t \equiv \overline{x}, u_t \equiv \overline{u}, \psi_t \equiv \overline{\psi}$ is a normal extremal triple.

That is, a steady state extremal of $(\mathbf{U}_\beta)$ is a solution $(\overline{x}, \overline{u}, \overline{\psi}) \in X \times U \times \mathbb{R}^n$ of the system of equations

$$\overline{x} = F(\overline{x}, \overline{u}), \tag{9}$$

$$0 = D_u^T f(\overline{x}, \overline{u}) + \beta D_u^T F(\overline{x}, \overline{u})\overline{\psi}, \tag{10}$$

$$\overline{\psi} = D_x^T f(\overline{x}, \overline{u}) + \beta D_x^T F(\overline{x}, \overline{u})\overline{\psi}. \tag{11}$$

Denote

$$H(x, u, \psi) = f(x, u) + \beta\psi^T F(x, u) \tag{12}$$

the *Hamiltonian* of the problem $(\mathbf{U}_\beta)$. Then, we can write (6), (7), (8) for $\psi^0 = 1$ as

$$\beta x_{t+1} = D_\psi H(x_t, u_t, \psi_{t+1}), \tag{13}$$

$$0 = D_u^T H(x_t, u_t, \psi_{t+1}), \tag{14}$$

$$\psi_t = D_x^T H(x_t, u_t, \psi_{t+1}) \tag{15}$$

and (9), (10), (11) as

$$D_\psi H(\overline{x}, \overline{u}, \overline{\psi}) = \beta\overline{x}, \tag{16}$$

$$D_u^T H(\overline{x}, \overline{u}, \overline{\psi}) = 0, \tag{17}$$

(18) $$D_x^T H(\overline{x},\overline{u},\overline{\psi}) = \overline{\psi}.$$

We assume

Q2: There exists a steady state extremal triple $(\overline{x}_1,\overline{u}_1,\overline{\psi}_1)$ for $(\mathbf{U}_1)$.

We denote $A_1 = D_xF(\overline{x}_1,\overline{u}_1)$, $B_1 = D_uF(\overline{x}_1,\overline{u}_1)$, $P_1 = D^2_{xx}H(\overline{x}_1,\overline{u}_1,\overline{\psi}_1)$, $Q_1 = D^2_{ux}H(\overline{x}_1,\overline{u}_1,\overline{\psi}_1)$ and $R_1 = D^2_{uu}H(\overline{x}_1,\overline{u}_1,\overline{\psi}_1)$.

Further, we assume

Q3: $D^2_{(x,u)}H(\overline{x}_1,\overline{u}_1,\overline{\psi}_1) = \begin{pmatrix} P_1 & Q_1^T \\ Q_1 & R_1 \end{pmatrix}$ is negative definite.[1]

Q4: The pair of matrices (A_1,B_1) is controllable.

Recall that the pair of matrices (A,B) is called controllable if $\operatorname{rank}(B\ AB\ \dots\ A^{n-1}B) = n$. Further, if (A,B) is controllable then it is *stabilizable* i.e. there exists an $m\times n$ matrix Z such that all the eigenvalues of $A+BZ$ are inside the unit circle (see [**9**]). Note that if $n = m$ and rank $B = n$, then (A,B) is trivially controllable.

Further, note that if is a controllable pair of matrices then this remains true for all pairs of matrices in a sufficiently small neighborhood of (A,B). Therefore, if O has been chosen sufficiently small, controllability (and, thus, stabilizability) of (A_1,B_1) assumed in **Q4** extends to (A_β,B_β) for all $\beta\in O$. Finally,we have

Proposition 2.1. *Let A be $n\times n$ and let (A,B) be controllable. Then,* $\operatorname{rank}(A-I\ B) = n$.

Proof. Let $c(A-I\ B) = 0$. Then, $cB = 0$, $cA = c$. Substituting for c from the second equality we recurrently obtain $cAB = 0,\dots,cA^{n-1}B = 0$, so $c = 0$ because of controllability. □

Corollary 2.1. *From* **Q3** *and* **Q4** *it follows that the Jacobian matrix*

$$K = \begin{pmatrix} P_1 & Q_1^T & ({A_1}^T - I) \\ Q_1 & R_1 & {B_1}^T \\ (A_1 - I) & B_1 & 0 \end{pmatrix}$$

of the system of equations (16), (17), (18) *for* (x,u,ψ) *at* $(\overline{x}_1,\overline{u}_1,\overline{\psi}_1)$ *is nonsingular.*

Because K is nonsingular, from the implicit function theorem it follows that the steady state extremal $(\overline{x}_1,\overline{u}_1,\overline{\psi}_1)$ is locally unique and extends to a locally unique C^2 family of steady state extremals $\beta\mapsto(\overline{x}_\beta,\overline{u}_\beta,\overline{\psi}_\beta)$ of $(\mathbf{U}_\beta)$ for β from a neighborhood O of 1.

For fixed $0<\beta\le 1$ denote $y = x-\overline{x}_\beta$, $v = u-\overline{u}_\beta$, $\mu = \psi-\overline{\psi}_\beta$, $A_\beta = D_xF(\overline{x}_\beta,\overline{u}_\beta)$, $B_\beta = D_uF(\overline{x}_\beta,\overline{u}_\beta)$, $k_\beta = D_xf(\overline{x}_\beta,\overline{u}_\beta)$, $l_\beta = D_uf_\beta(\overline{x}_\beta,\overline{u}_\beta)$, $P_\beta = D^2_{xx}H(\overline{x}_\beta,\overline{u}_\beta,\overline{\psi}_\beta)$, $Q_\beta = D^2_{ux}H(\overline{x}_\beta,\overline{u}_\beta,\overline{\psi}_\beta)$ and $R_\beta = D^2_{uu}H(\overline{x}_\beta,\overline{u}_\beta,\overline{\psi}_\beta)$; A_β, B_β, k_β, l_β, P_β, Q_β, R_β depend C^2 continuously on β.

[1]The brackets in the subscript of the symbol $D^2_{(x,u)}$ mean the second derivative with respect to the variables x,u with the variable ψ fixed rather that the mixed second derivatives in the variables x,u.

In this notation the steady state extremal equations (17), (18) read

$$l_\beta + \beta B_\beta^T \overline{\psi}_\beta = 0, \tag{19}$$

$$k_\beta + (\beta A_\beta^T - I)\overline{\psi}_\beta = 0. \tag{20}$$

Furthermore the conditions of normal extremality (13), (14), (15) can be written as

$$x_{t+1} = \overline{x}_\beta + A_\beta y_t + B_\beta v_t + \Xi(y_t, v_t), \tag{21}$$

$$0 = l_\beta + \beta B_\beta^T \overline{\psi}_\beta + Q_\beta y_t + R_\beta v_t + \beta B_\beta^T \mu_{t+1} + \Theta(y_t, v_t, \mu_{t+1}), \tag{22}$$

$$\psi_t = k_\beta + \beta A_\beta^T \overline{\psi}_\beta + P_\beta y_t + Q_\beta^T v_t + \beta A_\beta^T \mu_{t+1} + \Psi(y_t, v_t, \mu_{t+1}) \tag{23}$$

with

$$\Xi = o(\|y_t\| + \|v_t\|), \quad \Theta, \quad \Psi = o(\|y_t\| + \|v_t\| + \|\mu_{t+1}\|). \tag{24}$$

Because of (19) and (20), equations (21), (22), (23) turn to

$$y_{t+1} = A_\beta y_t + B_\beta v_t + \Xi(y_t, v_t), \tag{25}$$

$$0 = Q_\beta y_t + R_\beta v_t + \beta B_\beta^T \mu_{t+1} + \Theta(y_t, v_t, \mu_{t+1}), \tag{26}$$

$$\mu_t = P_\beta y_t + Q_\beta^T v_t + \beta A_\beta^T \mu_{t+1} + \Psi(y_t, v_t, \mu_{t+1}). \tag{27}$$

By **Q3** and continuity, the matrix

$$\begin{pmatrix} P_\beta & Q_\beta^T \\ Q_\beta & R_\beta \end{pmatrix} \tag{28}$$

is negative definite provided O has been chosen sufficiently small. Therefore R_β is nonsingular for $\beta \in O$ and by the implicit function theorem, from (26) we can express v_t as

$$v_t(y_t, \mu_{t+1}) = -R_\beta^{-1} Q_\beta y_t - \beta R_\beta^{-1} B_\beta^T \mu_{t+1} + o(\|y_t\| + \|\mu_{t+1}\|). \tag{29}$$

Substituting (29) into (25), (27) we eliminate v_t to obtain the reduced system

$$y_{t+1} = \mathbf{A}_\beta y_t + \beta \mathbf{B}_\beta \mu_{t+1} + o(\|y_t\| + \|\mu_{t+1}\|), \tag{30}$$

$$\mu_t = \mathbf{P}_\beta y_t + \beta \mathbf{A}_\beta^T \mu_{t+1} + o(\|y_t\| + \|\mu_{t+1}\|), \tag{31}$$

where $\mathbf{P}_\beta = (P_\beta - Q_\beta^T R_\beta^{-1} Q_\beta)$, $\mathbf{B}_\beta = -B_\beta R_\beta^{-1} B_\beta^T$, $\mathbf{A}_\beta = (A_\beta - B_\beta R_\beta^{-1} Q_\beta)$.

Now, we assume

Q5: The matrix $\mathbf{A}_1$ is nonsingular.

Then, $\mathbf{A}_\beta$ is nonsingular for $\beta \in O$ as well, provided O has been chosen small enough. Applying the implicit function theorem once more we can express μ_{t+1} from (31) to obtain

$$\mu_{t+1} = \frac{1}{\beta}(\mathbf{A}_\beta^T)^{-1}(\mu_t - \mathbf{P}_\beta y_t) + o(\|y_t\| + \|\mu_t\|)$$

Substituting this expression for μ_{t+1} into (30) we end up with

Proposition 2.2. *The triple* $(\{\hat{x}_t\}, \{\hat{u}_t\}, \{\hat{\psi}_t\})$ *is a normal extremal triple if and only if* $\hat{y}_t = \hat{x}_t - \overline{x}_\beta$, $\hat{\mu}_t = \hat{\psi}_t - \overline{\psi}_\beta$ *satisfy the system of equations*

$$y_{t+1} = [\boldsymbol{A}_\beta - \boldsymbol{B}_\beta (\boldsymbol{A}_\beta^T)^{-1} \boldsymbol{P}_\beta] y_t + \boldsymbol{B}_\beta (\boldsymbol{A}_\beta^T)^{-1} \mu_t + o(\|y_t\| + \|\mu_t\|), \tag{32}$$

$$\mu_{t+1} = -\frac{1}{\beta} (\boldsymbol{A}_\beta^T)^{-1} \boldsymbol{P}_\beta y_t + \frac{1}{\beta} (\boldsymbol{A}_\beta^T)^{-1} \mu_t + o(\|y_t\| + \|\mu_t\|) \tag{33}$$

and $\hat{v}_t = \hat{u}_t - \overline{u}_\beta$ *is given by the formula* (29) *with* $\boldsymbol{P}_\beta = (P_\beta - Q_\beta^T R_\beta^{-1} Q_\beta)$, $\boldsymbol{B}_\beta = -B_\beta R_\beta^{-1} B_\beta^T$, $\boldsymbol{A}_\beta = (A_\beta - B_\beta R_\beta^{-1} Q_\beta)$ *and* $y_t = \hat{y}_t$, $\mu_t = \hat{\mu}_t$. *The right hand sides of* (32), (33) *are* C^2.

3. The linear-quadratic approximation

In this section we deal in detail with the problem $(\mathbf{U}_{\beta,x_0})$ with linear dynamics and the objective function vanishing at 0 and terms of order higher than two missing. Under the standing hypotheses **Q1**–**Q5** we establish existence and uniqueness of the optimal control as well as the presence of the linear space consisting of stable paths for particular initial values x_0. The results will be of global nature, i. e. we take $X = \mathbb{R}^n$, $U = \mathbb{R}^m$.

The truncation

$$y_{t+1} = A_\beta y_t + B_\beta v_t, \tag{34}$$

$$0 = Q_\beta y_t + R_\beta v_t + \beta B_\beta^T \mu_{t+1}, \tag{35}$$

$$\mu_t = P_\beta y_t + Q_\beta^T v_t + \beta A_\beta^T \mu_{t+1} \tag{36}$$

of the system (25)–(27) obtained by dropping the higher order terms represents the system of conditions of normal extremality for the "linear-quadratic" optimal control problem (1)–(4) with $X = \mathbb{R}^n$, $U = \mathbb{R}^m$, $x = y$, $u = v$

$$F(y, v) = F_\beta^{lq}(y, v) = A_\beta y + B_\beta v, \tag{37}$$

$$f(y, v) = f_\beta^{lq}(y, v) = \frac{1}{2} y^T P_\beta y + v^T Q_\beta y + \frac{1}{2} v^T R_\beta v \tag{38}$$

and denote this problem by $(\mathbf{U}_\beta^{lq})$.

For $(\mathbf{U}_\beta^{lq})$, the systems of equations (30)–(31) and (32)–(33) read

$$y_{t+1} = \mathbf{A}_\beta y_t + \beta \mathbf{B}_\beta \mu_{t+1}, \tag{39}$$

$$\mu_t = \mathbf{P}_\beta y_t + \beta \mathbf{A}_\beta^T \mu_{t+1}, \tag{40}$$

and

$$y_{t+1} = [\mathbf{A}_\beta - \mathbf{B}_\beta (\mathbf{A}_\beta^T)^{-1} \mathbf{P}_\beta] y_t + \mathbf{B}_\beta (\mathbf{A}_\beta^T)^{-1} \mu_t, \tag{41}$$

$$\mu_{t+1} = -\frac{1}{\beta} (\mathbf{A}_\beta^T)^{-1} \mathbf{P}_\beta y_t + \frac{1}{\beta} (\mathbf{A}_\beta^T)^{-1} \mu_t, \tag{42}$$

respectively.

Furthermore, (29) truncates to

$$v_t(y_t, \mu_{t+1}) = -R_\beta^{-1} Q_\beta y_t - \beta R_\beta^{-1} B_\beta^T \mu_{t+1}. \tag{43}$$

Before exploiting extremality as a necessary condition of optimality we settle the problem of existence of optimal control.

Proposition 3.1. *For every feasible control/response pair* $(\{y_t\}, \{v_t\})$ *of the problem* $(\boldsymbol{U}_\beta^{lq})$ *with* $1 \geq \beta \in O$ *one has* $\beta^t(\|y_t\| + \|v_t\|) \to 0$ *for* $t \to \infty$.

Proof. Since β will be fixed in this proof we will drop it as a subscript.

By **Q3**, f^{lq} is negative definite. Hence, there is a $q > 0$ such that

$$f^{lq}(y, v) \leq -q(\|y\| + \|v\|)^2. \tag{44}$$

Suppose that $\{\beta^t(y_t, v_t)\}_{t=0}^\infty$ does not converge to $(0, 0)$.
Consider first $\beta < 1$. Then there is a sequence $t_j \to \infty$ such that

$$\beta^{t_j}(\|y_{t_j}\| + \|v_{t_j}\|) > \eta > 0, \tag{45}$$

therefore there exists t_{j0} such that

$$\|y_{t_j}\| + \|v_{t_j}\| > \frac{\eta}{\beta^{t_j}} > \frac{\eta}{\beta^{t_{j0}}} \geq 1 \tag{46}$$

for every $t_j > t_{j0}$.

From (46) it follows that

$$(\|y_{t_j}\| + \|v_{t_j}\|)^2 \geq \|y_{t_j}\| + \|v_{t_j}\|,$$

hence

$$\beta^{t_j} f^{lq}(y_{t_j}, v_{t_j}) \leq -q\eta < 0.$$

For $N = 0, 1, 2, \dots$ denote

$$J_N^{lq}(y_0, \{v_t\}) = \sum_{t=0}^{N} \beta^t f^{lq}(y_t, v_t). \tag{47}$$

Because $f^{lq}(y_t, v_t) \leq 0$ for all t, we have

$$\begin{aligned} J_N^{lq}(y_0, \{v_t\}) &\leq \sum_{t_j \leq N} \beta^{t_j} f^{lq}(y_{t_j}, v_{t_j}) \\ &\leq -(\#\{t_j : t_j \leq N\}) q\eta \to -\infty, \end{aligned}$$

for $N \to \infty$, a contradiction.

Let now $\beta = 1$. Then, using (44) and (45), one obtains

$$\begin{aligned} J_N^{lq}(y_0, \{v_t\}) &\leq \sum_{t_j \leq N} f^{lq}(y_{t_j}, v_{t_j}) \\ &\leq -(\#\{t_j : t_j \leq N\}) q\eta^2 \to -\infty, \end{aligned}$$

for $N \to \infty$, a contradiction. □

Proposition 3.2. *For every* $1 \geq \beta \in O$ *and every* y_0 *there exists an optimal control sequence for the problem* $(\mathbf{U}_{\beta, y_0}^{lq})$.

Proof. Since we will deal with fixed y_0, β in our proof, we will drop them as subscripts. Denote

$$J^{lq}(y_0, \{v_t\}) = \sum_{t=0}^{\infty} \beta^t f^{lq}(y_t, v_t), \tag{48}$$

then $J^{lq}(y_0, \{v_t\}) = \lim_{N\to\infty} J^{lq}_N(y_0, \{v_t\})$ with $J^{lq}_N(y_0, \{v_t\})$ defined in (47).

Further denote

$$\sigma = \sup_{\{v_t\}\in\phi(\beta,y_0)} J^{lq}(y_0, \{v_t\}). \tag{49}$$

Because of **Q3**, $\sigma \le 0$.

Because (A, B) is stabilizable, there exists a matrix Z such that the spectrum of $A + BZ$ is inside the unit circle. Consequently, there exist $0 < \overline{\lambda} < 1$ and $C > 0$ such that

$$\|(A + BZ)^t\| \le C\overline{\lambda}^t$$

for every t. The response $\{y_t\}$ of the control $\{v_t\}$ generated recurrently by the feedback rule $v_t = Zy_t$ satisfies $y_t = (A + BZ)^t y_0$. Therefore,

$$\|y_t\| \le C\overline{\lambda}^t \|y_0\|$$

for every t.

Hence we have

$$\begin{aligned}
|J^{lq}(y_0, \{Zy_t\})| &\le \sum_{t=0}^{\infty} \frac{1}{2}\beta^t |y_t^T P y_t + 2y_t^T Q^T Z y_t + y_t^T Z^T R Z y_t| \\
&= \sum_{t=0}^{\infty} \frac{1}{2}\beta^t |{y_0}^T (A^T + Z^T B^T)^t (P + 2Q^T Z + Z^T R Z)(A + BZ)^t y_0| \\
&\le \sum_{t=0}^{\infty} \frac{1}{2} C^2 \beta^t \overline{\lambda}^{2t} \|P + 2Q^T Z + Z^T R Z\| \|y_0\|^2 \\
&= D \sum_{t=0}^{\infty} \beta^t \overline{\lambda}^{2t}
\end{aligned}$$

for some constant $D > 0$. Thus, $\Phi(\beta, y_0)$ is not empty and σ is finite.

It follows that there exists a sequence $\{\{(y_t^k, v_t^k)\}_{t=0}^{\infty}\}_{k=0}^{\infty}$ of feasible control/response pairs such that:

$$J^{lq}(y_0, \{v_t^k\}) = \sum_{t=0}^{\infty} \beta^t f^{lq}(y_t^k, v_t^k) \nearrow \sigma \tag{50}$$

for $k \to \infty$.

Fix t. Because of (44), the sequence $\{(y_t^k, v_t^k)\}_{k=0}^{\infty}$ is bounded. Therefore there exists a convergent subsequence of $\{(y_t^k, v_t^k)\}_{k=0}^{\infty}$. We denote the limit of this subsequence as $(y_t^{\infty}, v_t^{\infty})$.

Employing the well known diagonal sequence construction we obtain a subsequence $\{\{(y_t^{k_j}, v_t^{k_j})\}_{t=0}^{\infty}\}_{j=0}^{\infty}$ of the sequence $\{\{(y_t^k, v_t^k)\}_{t=0}^{\infty}\}_{k=0}^{\infty})$, which converges to the sequence $\{(y_t^\infty, v_t^\infty)\}_{t=0}^\infty$ pointwise. Obviously, it satisfies $y_0^\infty = y_0$ and, by continuity of F, we have $y_{t+1}^\infty = F(y_t^\infty, v_t^\infty)$ for all t.

Without loss of generality assume that the convergent subsequence coincides with the original one. Then we have

$$J_N^{lq}(y_0, \{v_t^\infty\}) = \lim_{k\to\infty} J_N^{lq}(y_0, \{v_t^k\}) = \sum_{t=0}^{N} \beta^t f^{lq}(y_t^\infty, v_t^\infty). \tag{51}$$

Because $f^{lq}(y,v)$ is negative definite, the sequences $\{J_N^{lq}(y_0, \{v_t^k\})\}_{N=0}^\infty$ are nonincreasing and satisfy $J_N^{lq}(y_0, \{v_t^k\}) \le 0$ for all N, k, including $k = \infty$. Therefore, from (50) it follows that

$$J_N^{lq}(y_0, \{v_t^\infty\}) \ge \sigma$$

for all N. Since $\{J_N^{lq}(y_0, \{v_t^\infty\})\}_{N=0}^\infty$ is nonincreasing and bounded from below, it converges, so

$$J^{lq}(y_0, \{v_t^\infty\}) = \lim_{N\to\infty} J_N^{lq}(y_0, \{v_t^\infty\}) \ge \sigma.$$

The reverse inequality is trivial, hence

$$J^{lq}(y_0, \{v_t^\infty\}) = \sigma = \max_{\{v_t\}\in\phi(\beta, y_0)} J^{lq}(y_0, \{v_t\}). \tag{52}$$

□

Proposition 3.3. *If $(\hat{y}_t, \hat{v}_t)$ is an optimal control/response pair then there is a solution $\hat{\mu}_t$ of the adjoint equation* (36) *such that $\{(\hat{y}_t, \hat{v}_t, \hat{\mu}_t)\}$ is a normal extremal triple.*

Proof. Recall that, because of **Q1**, existence of a solution $\{(\hat{\mu}^0, \hat{\mu}_t)\}$ completing $\{(\hat{y}_t, \hat{v}_t)\}$ to an extremal triple, i. e. satisfying $(\mu^0, \mu_t) \ne 0$ and

$$y_{t+1} = A_\beta y_t + B_\beta v_t, \tag{53}$$

$$0 = \mu^0(Q_\beta y_t + R_\beta v_t) + \beta B_\beta^T \mu_{t+1}, \tag{54}$$

$$\mu_t = \mu^0(P_\beta y_t + Q_\beta^T v_t) + \beta A_\beta^T \mu_{t+1} \tag{55}$$

follows from [**5**]. It remains to prove $\hat{\mu}^0 \ne 0$.

Suppose the contrary. Then, because of (54), (55) we have

$$B_\beta^T \hat{\mu}_t = 0, \tag{56}$$

$$\hat{\mu}_t = \beta A_\beta^T \hat{\mu}_{t+1}, \tag{57}$$

respectively for all t. For any chosen t it follows $0 = B_\beta^T \hat{\mu}_{t-j} = \beta^j (A_\beta^j B_\beta)^T \hat{\mu}_t$ for $j = 0, \dots, n-1$, hence $\hat{\mu}_t = 0$ because of **Q**4. Then $\{(\hat{\mu}^0, \hat{\mu}_t)\} = 0$ which contradicts the extremality definition. □

Since optimal responses of $\{v_t\}$ for $\beta = 1$ have to converge to zero by Proposition 3.1 and satisfy system of equations (41), (42) with a suitable sequence $\{\mu_t\}$, the following corollary of Proposition 3.2 holds

Corollary 3.1. *For each y_0 there exists a solution $\{(y_t, \mu_t)\}$ of the system* (41), (42) *with $\beta = 1$ such that $y_t \to 0$ for $t \to \infty$.*

Next we show that such a solution is unique.

Let us denote by

$$M_\beta = \begin{pmatrix} \mathbf{A}_\beta - \mathbf{B}_\beta (\mathbf{A}_\beta^{\mathrm{T}})^{-1} \mathbf{P}_\beta & \mathbf{B}_\beta (\mathbf{A}_\beta^{\mathrm{T}})^{-1} \\ -\frac{1}{\beta} (\mathbf{A}_\beta^{\mathrm{T}})^{-1} \mathbf{P}_\beta & \frac{1}{\beta} (\mathbf{A}_\beta^{\mathrm{T}})^{-1} \end{pmatrix} \tag{58}$$

the matrix of the system of equations (41), (42).

A $(2n \times 2n)$ matrix M is called symplectic, if it satisfies

$$M^{\mathrm{T}} \Omega M = \Omega$$

with

$$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

and I the $(n \times n)$ identity matrix.

The eigenvalues of a symplectic matrix (including multiplicity) form reciprocal pairs, i. e., if λ is an eigenvalue of matrix M, then so is $\frac{1}{\lambda}$, with the same multiplicity (see [**1**]).

A block matrix:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

A, B, C and D being $(n \times n)$, is symplectic if and only if the identities

$$A^{\mathrm{T}} D - C^{\mathrm{T}} B = I, \tag{59}$$

$$A^{\mathrm{T}} C = C^{\mathrm{T}} A, \tag{60}$$

$$D^{\mathrm{T}} B = B^{\mathrm{T}} D \tag{61}$$

are satisfied.

In a straightforward way we can verify that M_1 is symplectic.

Uniqueness of the optimal control will be obtained by combining the existence theorem of optimal control for the problem $(\mathbf{U}_1^{lq})$ and the symplectic structure of M_1.

Denote

$$E_\beta^S = \{(y_0, \mu_0) \mid (y_t(y_0, \mu_0), \mu_t(y_0, \mu_0)) \to (0,0) \quad \text{for} \quad t \to \infty\}, \tag{62}$$

where $(y_t(y_0, \mu_0), \mu_t(y_0, \mu_0))$ is a solution of the system of equations (41), (42) with initial point (y_0, μ_0). Because of linearity of the system (41), (42), E_β^S is a linear subspace of $\mathbb{R}^{n+n}$. It is the invariant subspace of M_β corresponding to the part of the spectrum inside the unit sphere. Correspondingly, we denote by E_β^U the complementary invariant subspace of the complement of the spectrum.

Theorem 3.1. *The spectrum of M_1 does not contain eigenvalues with absolute value* 1*; n of its eigenvalues have moduli smaller than* 1*, n ones have moduli exceeding* 1*. There exists an $(n \times n)$ matrix L_1 such that*

$$E_1^S = \{(y, \mu) \in \mathbb{R}^n \times \mathbb{R}^n \mid \mu = L_1 y\}. \tag{63}$$

Proof. From Corollary 3.1 we know that for every $y_0 \in \mathbb{R}^n$ there exists a $\mu_0 \in \mathbb{R}^n$ such that $(y_t(y_0, \mu_0), \mu_t(y_0, \mu_0)) \to (0, 0)$ for $t \to \infty$. In other words, the natural projection $(y, \mu) \to y$ projects E_1^S surjectively. Consequently, dim $E_1^S \geq n$.

On the other hand, since M_1 is symplectic, dim $E_1^S \leq n$, so dim $E_1^S = n$ and $(y, \mu) \to y$ is an injection. The representation (63) follows. □

Corollary 3.2. *For given y_0, the solution $\{(y_t, \mu_t)\}$ of the system* (41), (42) *with $\beta = 1$ satisfying $y_t \to 0$ for $t \to \infty$ is unique, μ_0 being given as $\mu_0 = L_1 y_0$*

Because of continuous dependence of the spectra of matrices on their entries, the matrices M_β for β from a sufficiently small neighborhood of 1 inherit the splitting of the spectrum of M_1. That is, n of their eigenvalues lie inside the unit circle whereas n of them lie outside. Furthermore the invariant subspaces E_β^S of the part of their spectra inside the unit circle are defined by:

$$E_\beta^S = \{(y, \mu) \in \mathbb{R}^n \times \mathbb{R}^n \mid \mu = L_\beta y\} \tag{64}$$

with $L_\beta \to L_1$ for $\beta \to 1$. Assume that the neighborhood O has been chosen so small that for $\beta \in O$ the representation (64) is valid.

For $\beta \in O$ let $\underline{\lambda}_\beta > 1$ be smaller than the minimum of absolute values of the eigenvalues of M_β outside the unit circle. Then, if O has been chosen sufficiently small, one has

$$\underline{\lambda}_\beta > \frac{1}{\beta} > 1 \tag{65}$$

for $\beta \in O$.

Theorem 3.2. *For $\beta \in O$ there exists a unique optimal control/response pair $\{(\hat{y}_t, \hat{v}_t)\}$ for the problem $(\mathbf{U}^{lq}_{\beta,y_0})$. The control $\{\hat{v}_t\}$ being generated by the feedback law*

$$\hat{v}_t = Z_\beta \hat{y}_t \tag{66}$$

with

$$Z_\beta = -[R_\beta^{-1} Q_\beta + R_\beta^{-1} B_\beta^{\mathrm{T}} (\boldsymbol{A}_\beta^{\mathrm{T}})^{-1} (L_\beta - \boldsymbol{P}_\beta)]. \tag{67}$$

Consequently, one has

$$\hat{y}_{t+1} = (A_\beta + B_\beta Z_\beta) \hat{y}_t, \tag{68}$$

There exists a unique sequence $\{\hat{\mu}_t\}$ such that $(\{\hat{y}_t\}, \{\hat{v}_t\}, \{\hat{\mu}_t\})$ is a normal extremal triple and one has

$$\hat{\mu}_t = L_\beta \hat{y}_t,$$

$\{(\hat{y}_t, \hat{v}_t, \hat{\mu}_t)\} \to (0, 0, 0)$ *for $t \to \infty$.*

Proof. For $\beta = 1$ uniqueness of optimal control/response pair and its convergence to $(0, 0)$ follows from the previous theorem.

Let now $1 > \beta \in O$. From Proposition 3.2 we know that for every y_0 there is an optimal control/response pair $\{(\hat{y}_t, \hat{v}_t)\}$ for the problem $(\mathbf{U}^{lq}_{\beta,y_0})$. The response $\{\hat{y}_t\}$ is the y-component of a solution $\{(\hat{y}_t, \hat{\mu}_t)\}$ of the system of equations (41), (42) with the matrix M_β.

First, we derive the formulas (66) for optimal control and (68) for the optimal response of the problem $(\mathbf{U}_{\beta,x_0})$ for both $\beta < 1$ and $\beta = 1$. Since $\hat{\mu}_t = L_\beta \hat{y}_t$, from (42) we obtain

$$\hat{\mu}_{t+1} = \frac{1}{\beta}[-(\mathbf{A}_\beta^{\mathrm{T}})^{-1}\mathbf{P}_\beta + (\mathbf{A}_\beta^{\mathrm{T}})^{-1} L_\beta]\hat{y}_t.$$

From (43) we obtain (66). Substituting (66) to (34) we obtain (68).

Now we claim that $\hat{y}_t \to 0$ for $\beta < 1$. Suppose the contrary. Then $(y_0, \mu_0) \notin E_\beta^S$. Therefore there exists a constant $c > 0$ such that:

$$\|(\hat{y}_t, \hat{\mu}_t)\| \geq c\underline{\lambda}_\beta^t \|(y_0, \mu_0)\|.$$

Because for $\beta \in O$, $\underline{\lambda}_\beta$ satisfies (65), the series $\sum_{t=0}^\infty \beta^t f_\beta^{lq}(y_t, v_t)$ diverges, contradicting our hypothesis. This proves our claim.

The convergence of $(\{\hat{v}_t\}, \{\hat{\mu}_t\})$ to $(0,0)$ follows trivially. □

Denote

$$V_\beta^{lq}(y_0) = \sup_{\{v_t\} \in \phi(\beta, y_0)} J^{lq}(y_0, \{v_t\}) \tag{69}$$

the value functions of the problem $(\mathbf{U}_\beta^{lq})$.

Corollary 3.3. *One has*

$$V_\beta^{lq}(y_0) = \sum_{t=0}^\infty \beta^t f^{lq}(\hat{y}_t, Z_\beta \hat{y}_t) = \frac{1}{2} y_0^{\mathrm{T}} W_\beta y_0 \tag{70}$$

with

$$W_\beta = \sum_{t=0}^\infty \beta^t (A_\beta^{\mathrm{T}} + Z_\beta^{\mathrm{T}} B_\beta^{\mathrm{T}})^t (P_\beta + 2Q_\beta^{\mathrm{T}} Z_\beta + Z_\beta^{\mathrm{T}} R_\beta Z_\beta)(A_\beta + B_\beta Z_\beta)^t. \tag{71}$$

Proof. Because optimal response $\hat{y}_t$ satisfies (68) for every t, one has

$$\hat{y}_t = (A + BZ)^t y_0. \tag{72}$$

Substituting (66) to (38), we obtain:

$$f^{lq}(\hat{y}_t, \hat{v}_t) = \frac{1}{2}\hat{y}_t^{\mathrm{T}}(P + 2Q^{\mathrm{T}} Z + Z^{\mathrm{T}} RZ)\hat{y}_t. \tag{73}$$

The substitution (72) to (73) leads to:

$$f^{lq}(\hat{y}_t, \hat{v}_t) = \frac{1}{2} y_0^{\mathrm{T}} [(A^{\mathrm{T}} + Z^{\mathrm{T}} B^{\mathrm{T}})^t (P + 2Q^{\mathrm{T}} Z + Z^{\mathrm{T}} RZ)(A + BZ)^t] y_0. \tag{74}$$

By (74) we have:

$$V^{lq}\beta(y_0) = \lim_{N \to \infty} \sum_{t=0}^N \beta^t f^{lq}(\hat{y}_t, \hat{v}_t) = \frac{1}{2} y_0^{\mathrm{T}} W_\beta y_0 \tag{75}$$

with W_β defined in (71). □

4. The Full Problem

In this section we deal with the full problem for fixed $1 > \beta \in O$, where O has been chosen so small that all the conclusions of Sections 2, 3 hold. We prove that the optimal feedback law (66) and the value function (75) are linear resp. quadratic approximations of their local at $\overline{x}_\beta$ counterparts for the full nonlinear problem. To this end, in the optimal problem $(\mathbf{U}_\beta)$ we choose $X = \{x \in \mathbb{R}^n : \|y\| < \eta\}$, $U = \{u \in \mathbb{R}^m : \|v\| < \eta\}$ with η sufficiently small. Since β is fixed in this section we will drop it as a subscript.

Proposition 4.1. *There exists a $\delta > 0$ such that, for $y_0 = x_0 - \overline{x}_\beta$, $\|y_0\| < \delta$, $\Phi(\beta, x_0) \neq \emptyset$ for problem $(\mathbf{U}_{x_0})$ with $1 > \beta \in O$; for every $(\{x_t\}, \{u_t\}) \in \Phi(\beta, x_0)$ one has $\beta^t(\|y_t\| + \|v_t\|) \to 0$ for $t \to \infty$.*

Proof. Because the pair of matrices (A, B) is stabilizable, for $1 > \beta \in O$ there exists a matrix Z such that the spectrum of $A + BZ = D_xF(\overline{x}, \overline{u}) + D_uF(\overline{x}, \overline{u})Z$ is inside the unit circle. For $g(y) = F(\overline{x} + y, \overline{u} + Zy) - F(\overline{x}, \overline{u}) = (A + BZ)y + o(\|y\|)$ one has $Dg(0) = A + BZ$. Let $0 < \eta_1 < \frac{\eta}{1+\|Z\|}$. By [**8**], for sufficiently small $\delta > 0$ one has $g^t(y_0) < \eta_1$ for $\|y_0\| < \delta$ and all t. However, one has $g^t(y_0) = y_t = x_t - \overline{x}$, where $\{x_t\}$ is the response of the control $\{u_t\}$ with $u_t = v_t + \overline{u}$ generated by the feedback rule $u_t = \overline{u} + Z(x_t - \overline{x})$, or equivalently $v_t = Zy_t$. Hence $\|u_t - \overline{u}\| = \|v_t\| < \eta$ and, consequently, $\Phi(x_0, \beta) \neq \emptyset$ as soon as $\|y_0\| < \delta$.

Because X and U are taken bounded, boundedness of feasible pairs is trivial and convergence $\beta^t(\|y_t\| + \|v_t\|) \to 0$ follows. $\square$

Recall that for $1 > \beta \in O$ the deviations of the normal extremal responses of $(\mathbf{U})$ from $\overline{x}$ are the y-components of solutions of the system (32), (33), estimate (65) holds and that E^S is expressed by (62).

For $\mu_t = \psi_t - \overline{\psi}$ denote $z_t = (y_t, \mu_t)$ and write (32), (33) as

$$z_{t+1} = G(z_t) = Mz_t + o(\|z_t\|), \tag{76}$$

G being the vector of the right-hand sides of the system of equations (32), (33), with $M = M_\beta$ defined by (58).

Let $\Omega = \{z = (y, \mu) : \|y\| + \|\mu\| < \eta\}$,

$$W^S(0) = \{z \in \Omega : G^t(z) \to 0 \text{ for } t \to \infty \text{ and } G^t(z) \in \Omega \text{ for every } t \geq 0\}, \tag{77}$$

the local stable manifold of the fixed point 0 of the system (32), (33).

By [**8**, Theorem D.1], for sufficiently small Ω, $W^S(0)$ is a C^2 manifold tangent to E^S at the origin 0; trajectories of points off $W^S(0)$ leave Ω for $t \to \infty$. Thus we have

Corollary 4.1. *If O and $\eta > 0$ are sufficiently small then*

$$W^S(0) = \{z = (y, \mu) \in \Omega : \mu = h(y)\}, \tag{78}$$

where

$$h(y) = Ly + o(\|y\|) \tag{79}$$

is C^2 differentiable.

Corollary 4.2. *For $\eta > 0$ and $\delta > 0$ sufficiently small there is a unique $\mu_0 = h(y_0)$ such that the solution (y_t, μ_t) of* (32), (33) *through (y_0, μ_0) satisfies $\|y_t\| + \|v_t\| + \|\mu_t\| < \eta$ for $t > 0$ with v_t given by the formula* (29).

For fixed y_0 denote this solution $(\hat{y}_t, \hat{v}_t, \hat{\mu}_t)$. As a consequence, for $\|x_0 - \overline{x}\| < \delta$ sufficiently small the full problem $(\mathbf{U}_{x_0})$ with $1 > \beta \in O$ has a unique solution $(\hat{x}_t, \hat{\psi}_t)$, where $\hat{x}_t = \hat{y}_t + \overline{x}$, $\hat{\psi}_t = \hat{\mu}_t + \overline{\psi}$, of (32), (33) through (x_0, ψ_0) such that $(\hat{x}_t, \hat{u}_t, \hat{\psi}_t)$ with $\hat{u}_t = \hat{v}_t + \overline{u}$ stay close to the steady state extremal triple $(\overline{x}, \overline{u}, \overline{\psi})$ for all t.

To distinguish the unique solutions of (32), (33) through (y_0, μ_0) from those of the equations (41), (42) for the truncated problem we label the latter ones by the superscript lq.

The following proposition is a consequence of the fact that $(\{\hat{y}_t\}, \{\hat{\mu}_t\})$ resp. $(\{\hat{y}_t^{lq}\}, \{\hat{\mu}_t^{lq}\})$ lie in $W^S(0)$ resp. E^S well known in the field of dynamical systems. Nevertheless, for the convenience of the reader we present an outline of its proof.

Proposition 4.2. *For $\eta > 0$ and $\delta > 0$ sufficiently small there are $\kappa > 0$, $\omega > 0$ and $0 < \lambda < 1$ such that*

$$\|\hat{y}_t\| + \|\hat{v}_t\| + \|\hat{\mu}_t\| \leq \kappa \lambda^t \|y_0\|, \tag{80}$$

$$\|\hat{y}_t - \hat{y}_t^{lq}\| + \|\hat{v}_t - \hat{v}_t^{lq}\| + \|\hat{\mu}_t - \hat{\mu}_t^{lq}\| \leq \omega \|y_0\|^2. \tag{81}$$

Proof. Denote Π the projection of $\mathbb{R}^n$ to E^S annihilating E^U and $\zeta = \Pi z$. The spectrum of $M^S = \Pi M|_{E^S} : E^S \to E^S$ is the part of the spectrum of M inside the unit circle. Therefore, there is a $\kappa_1 > 0$ and a $0 < \lambda_1 < 1$ such that

$$\|(M^S)^t\| \leq \kappa_1 \lambda_1^t. \tag{82}$$

Since $W^S(0)$ is tangent to E^S at 0 and C^2, one has

$$\|z - \Pi z\| = O(\|z\|^2) \tag{83}$$

and the restriction of G to $W^S(0)$ can locally be represented by its projection to E^S as

$$\zeta_{t+1} = M^S \zeta_t + \gamma(\zeta_t) \tag{84}$$

with $\|\gamma(\zeta)\| \leq \kappa_2 \|\zeta\|^2$ for some $\kappa_2 > 0$. By [**8**, Theorem 4.33], 0 is a locally exponentially stable fixed point of (84), i.e., there is a $\kappa_3 > 0$ and a $\lambda_1 < \lambda < 1$ such that for ζ_0 from a sufficiently small neighborhood of 0 one has

$$\|\zeta_t\| \leq \kappa_3 \lambda^t \|\zeta_0\|. \tag{85}$$

The inequality (80) with a suitable $\kappa > 0$ follows from (85) because of (79), (29), (33).

To prove (81) denote $\xi_t = \zeta_t - \zeta_t^{lq}$. One has $\xi_0 = 0$ and, because of $\zeta_t \in E^S$,

$$\xi_{t+1} = M^S \xi_t + \gamma(\zeta_t),$$

hence,

$$\xi_t = \sum_{s=0}^{t-1} (M^S)^{t-1-s} \gamma(\zeta_s),$$

$$\|\xi_t\| \le \kappa_1 \lambda^{t-1-s} \kappa \lambda^s (\|\zeta_0\|)^2 = \kappa\kappa_1 (\|\zeta_0\|)^2, \tag{86}$$

from which (81) follows because of (83). □

Theorem 4.1. *Let O, η, δ are sufficiently small. Then for $\|y_0\| < \delta$, $\{(\hat{x}_t, \hat{u}_t)\}$ is the optimal control/response pair.*

Proof. We have

$$H(x,u,\psi) = H(x,u,\overline{\psi}+\mu) = H(x,u,\overline{\psi}) + \beta\mu^{\mathrm{T}} F(x,u). \tag{87}$$

Because $D^2_{(x,u)} H(\overline{x}, \overline{u}, \overline{\psi})$ is negative definite, $H(x,u,\overline{\psi}+\mu)$ is strictly concave in (x,u) and has a negative definite Hessian in $X \times U$, provided O, η have been chosen sufficiently small. Since H is strictly concave in (x,u), so is $\hat{H}(x, \overline{\psi}+\mu) = \sup_{u\in U} H(x,u,\overline{\psi}+\mu)$ in x.

To prove that $\hat{u}_t$ is an optimal control we show that for any other u_t admissible $J(x_0, \{u_t\}) - J(x_0, \{\hat{u}_t\}) \le 0$.

From the concavity of H we obtain

$$\begin{aligned}
H(x_t, u_t, \hat{\psi}_{t+1}) - H(\hat{x}_t, \hat{u}_t, \hat{\psi}_{t+1}) &\le \sup_{u\in U} H(x_t, u, \hat{\psi}_{t+1}) - H(\hat{x}_t, \hat{u}_t, \hat{\psi}_{t+1}) \\
&= \hat{H}(x_t, \hat{\psi}_{t+1}) - \hat{H}(\hat{x}_t, \hat{\psi}_{t+1}) \\
&\le D_x \hat{H}(\hat{x}_t, \hat{\psi}_{t+1})(x_t - \hat{x}_t).
\end{aligned}$$

Taking into account that $x_0 = \hat{x}_0$ and that (15) holds one obtains

$$\begin{aligned}
&J_N(x_0, \{u_t\}) - J_N(x_0, \{\hat{u}_t\}) = \sum_{t=0}^{N} \beta^t [f(x_t, u_t) - f(\hat{x}_t, \hat{u}_t)] \\
&= \sum_{t=0}^{N} \beta^t [H(x_t, u_t, \hat{\psi}_{t+1}) - H(\hat{x}_t, \hat{u}_t, \hat{\psi}_{t+1}) - \beta \hat{\psi}^{\mathrm{T}}_{t+1} (F(x_t, u_t) - F(\hat{x}_t, \hat{u}_t))] \\
&\le \sum_{t=0}^{N} \beta^t [D_x \hat{H}(\hat{x}_t, \hat{\psi}_{t+1})(x_t - \hat{x}_t) - \beta \hat{\psi}^{\mathrm{T}}_{t+1} (x_{t+1} - \hat{x}_{t+1})] \\
&= \sum_{t=0}^{N} \beta^t [\hat{\psi}_t (x_t - \hat{x}_t) - \beta \hat{\psi}^{\mathrm{T}}_{t+1} (x_{t+1} - \hat{x}_{t+1})] \\
&= \hat{\psi}^{\mathrm{T}}_0 (x_0 - \hat{x}_0) - \beta^{N+1} \hat{\psi}^{\mathrm{T}}_{N+1} (x_{N+1} - \hat{x}_{N+1}) \\
&= -\beta^{N+1} \hat{\psi}^{\mathrm{T}}_{N+1} (x_{N+1} - \hat{x}_{N+1}) \to 0
\end{aligned}$$

for $N \to \infty$. Hence

$$J(x_0, \{u_t\}) - J(x_0, \{\hat{u}_t\}) = \lim_{N\to\infty} \big(J_N(x_0, \{u_t\}) - J_N(x_0, \{\hat{u}_t\})\big) \le 0.$$

□

Denote

$$V(x_0) = J(x_0, \{\hat{u}_t\}) = \sum_{t=0}^{\infty} \beta^t f(\hat{x}_t, \hat{u}_t). \tag{88}$$

the value function of the problem $(\mathbf{U}_{x_0})$.

Theorem 4.2. *For x_0 sufficiently near $\overline{x}$ the (locally) optimal control $\{\hat{u}_t\}$ is generated by the C^1 feedback law*

$$\hat{v}_t = Z\hat{y}_t + o(\|\hat{y}_t\|), \tag{89}$$

with $Z = Z_\beta$ defined in (67)*. Consequently, one has*

$$\hat{y}_{t+1} = (A + BZ)\hat{y}_t + o(\|\hat{y}_t\|). \tag{90}$$

The function V is C^2 in a sufficiently small neighborhood of $\overline{x}$ and satisfies

$$V(x) = \frac{f(\overline{x}, \overline{u})}{1-\beta} + \overline{\psi}^{\mathrm{T}}(x - \overline{x}) + V^{lq}(x - \overline{x}) + o(\|x - \overline{x}\|^2) \tag{91}$$

with V^{lq} defined in (70).

Proof. By substituting (79) to (33) one obtains

$$\hat{\mu}_{t+1} = \frac{1}{\beta}(\mathbf{A}^{\mathrm{T}})^{-1}(L - \mathbf{P})\hat{y}_t + o(\|\hat{y}_t\|). \tag{92}$$

From (29) we obtain (89). Substituting (89) to (25) one obtains (90). Using (9)–(11), (25)–(27) we obtain (for simplicity arguments x, u of f, F dropped if $x = \overline{x}, u = \overline{u}$)

$$\begin{aligned}
J_N(x_0, \{\hat{u}_t\}) &= \sum_{t=0}^{N} \beta^t f(\overline{x} + \hat{y}_t, \overline{u} + \hat{v}_t) \\
&= \sum_{t=0}^{N} \beta^t \Big(f + D_x f \hat{y}_t + D_u f \hat{v}_t + \frac{1}{2}\hat{y}_t^{\mathrm{T}} D^2_{xx} f \hat{y}_t + \hat{v}_t^{\mathrm{T}} D^2_{xu} f \hat{y}_t \\
&\quad + \frac{1}{2}\hat{v}_t^{\mathrm{T}} D^2_{uu} f \hat{v}_t + o(\|\hat{y}_t\|^2 + \|\hat{v}_t\|^2)\Big) \\
&= \frac{1-\beta^{N+1}}{1-\beta} f + \sum_{t=0}^{N} \beta^t \Big(\overline{\psi}^{\mathrm{T}}(I - \beta D_x F)\hat{y}_t - \beta \psi^{\mathrm{T}} D_u F \hat{v}_t \\
&\quad + \frac{1}{2}\hat{y}_t^{\mathrm{T}} D^2_{xx} f \hat{y}_t + \hat{v}_t^{\mathrm{T}} D^2_{xu} f \hat{y}_t + \frac{1}{2}\hat{v}_t^{\mathrm{T}} D^2_{uu} f \hat{v}_t + o(\|\hat{y}_t\|^2 + \|\hat{v}_t\|^2)\Big) \\
&= \frac{1-\beta^{N+1}}{1-\beta} f + \sum_{t=0}^{N} \beta^t \Big(\overline{\psi}^{\mathrm{T}}(I - \beta D_x F)\hat{y}_t - \beta \psi^{\mathrm{T}} D_u F \hat{v}_t \\
&\quad + \frac{1}{2}\hat{y}_t^{\mathrm{T}} P \hat{y}_t + \hat{y}_t^{\mathrm{T}} Q \hat{v}_t + \frac{1}{2}\hat{v}_t^{\mathrm{T}} R \hat{v}_t - \beta \Big[\frac{1}{2}\hat{y}_t^{\mathrm{T}} D^2_{xx}\{\overline{\psi}^{\mathrm{T}} F\}\, \hat{y}_t \\
&\quad + \hat{y}_t^{\mathrm{T}} D^2_{xu}\{\overline{\psi}^{\mathrm{T}} F\}\hat{v}_t + \frac{1}{2}\hat{v}_t^{\mathrm{T}} D^2_{uu}\{\overline{\psi}^{\mathrm{T}} F\}\hat{v}_t\Big] + o(\|\hat{y}_t\|^2 + \|\hat{v}_t\|^2)\Big) \\
&= \frac{1-\beta^{N+1}}{1-\beta} f + \sum_{t=0}^{N} \beta^t \Big(\frac{1}{2}\hat{y}_t^{\mathrm{T}} P \hat{y}_t + \hat{y}_t^{\mathrm{T}} Q \hat{v}_t + \frac{1}{2}\hat{v}_t^{\mathrm{T}} R \hat{v}_t + \overline{\psi}^{\mathrm{T}} \hat{y}_t \\
&\quad - \beta[D_x\{\overline{\psi}^{\mathrm{T}} F\}\hat{y}_t + D_u\{\overline{\psi}^{\mathrm{T}} F\}\hat{v}_t + \frac{1}{2}\hat{y}_t^{\mathrm{T}} D^2_{xx}\{\overline{\psi}^{\mathrm{T}} F\}\hat{y}_t \\
&\quad + \hat{y}_t^{\mathrm{T}} D^2_{xu}\{\overline{\psi}^{\mathrm{T}} F\}\hat{v}_t + \frac{1}{2}\hat{v}_t^{\mathrm{T}} D^2_{uu}\{\overline{\psi}^{\mathrm{T}} F\}\hat{v}_t] + o(\|\hat{y}_t\|^2 + \|\hat{v}_t\|^2)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1-\beta^{N+1}}{1-\beta} f + \sum_{t=0}^{N} \beta^t \Big(\frac{1}{2} \hat{y}_t^{\mathrm{T}} P \hat{y}_t + \hat{y}_t^{\mathrm{T}} Q \hat{v}_t + \frac{1}{2} \hat{v}_t^{\mathrm{T}} R \hat{v}_t \\
&\quad + [\overline{\psi}^{\mathrm{T}} (\hat{y}_t - \beta[F(\hat{y}_t + \overline{x}, \hat{v}_t + \overline{u}) - F(\overline{x}, \overline{u})]) + o(\|\hat{y}_t\|^2 + \|\hat{v}_t\|^2)] \Big) \\
&= \frac{1-\beta^{N+1}}{1-\beta} f + \sum_{t=0}^{N} \beta^t \overline{\psi}^{\mathrm{T}} (\hat{y}_t - \beta \hat{y}_{t+1}) \\
&\quad + \sum_{t=0}^{N} \beta^t \left(f^{lq}(\hat{y}_t, \hat{v}_t) + o(\|\hat{y}_t\|^2 + \|\hat{v}_t\|^2) \right) \\
&= \frac{1-\beta^{N+1}}{1-\beta} f + \sum_{t=0}^{N} \beta^t \overline{\psi}^{\mathrm{T}} (\hat{y}_t - \beta \hat{y}_{t+1}) \\
&\quad + \sum_{t=0}^{N} \beta^t f^{lq}(\hat{y}_t, \hat{v}_t) + o(\|x_0 - \overline{x}\|^2). \qquad (93)
\end{aligned}
$$

Using (81) we obtain

$$
\sum_{t=0}^{\infty} \beta^t f^{lq}(\hat{y}_t, \hat{v}_t) = \sum_{t=0}^{\infty} \beta^t f^{lq}(\hat{y}_t^{lq}, \hat{v}_t^{lq}) + o(\|x_0 - \overline{x}\|^2). \qquad (94)
$$

This, together with (93), yields (91) □

5. Concluding Remarks

Let us note that virtually all the hypotheses except **Q2**, **Q3** are of "generic" nature, they are satisfied "almost always" in a well defined measure theoretic or topological sense. This means that there is no reason to expect that they would not be satisfied unless the problem exhibits some symmetry. On the other hand, Hypotheses **Q2**, **Q3** are crucial: they say that the problem has an equilibrium solution representing maximum of an associated nonlinear programming problem.

One of our goals was to find a general condition the data of the problem should satisfy in order that the "saddle point property" of [**4**] is met. Such a condition is **Q3** for the limit case $\beta = 1$ which takes care of β sufficiently close to 1 as well. Unfortunately, in most of the interesting economic examples the Hessian of f at the steady state is merely negative semidefinite. In [**3**] a two dimensional example is worked out in detail. It is based on the discretization of the continuous time model of [**13**], the definiteness condition **Q3** being secured by a somewhat artificial quadratic term of utility loss associated with the effort to increase human capital.

A nearly complete analogy to the theory of this paper holds for continuous time systems. Because of the invertibility of the dynamics of a continuous time system, hypothesis **Q1** is satisfied automatically. On the other hand, the proofs of the analogies of Propositions 3.1 and 3.2 become more technical. Those technicalities can be taken care of similarly as in the proofs of Theorems 2,3 of [**6**].

In addition to the quoted paper there is a series of papers studying problems similar to ours, the engineering optimal stabilization problem serving as motivation(cf. [**11**], [**10**], [**12**], [**15**]). In those papers linear terms in the Taylor expansion of the cost function at equilibrium are missing. The presence of linear terms in our problem makes it necessary to derive the linear-quadratic approximation from an associated problem involving second derivatives at equilibrium of the dynamics and the equilibrium adjoint vector. Let us note that in the one-dimensional problem [**14**], [**7**] this can be done in an explicit elementary way by placing all the nonlinearity to the objective function,

Acknowledgment. PB gratefully acknowledges support by the grants VEGA 1/0711/12 and 1/2429/12. MZ gratefully acknowledges support by the grant VEGA 1/0426/12. Special thanks from both authors go to M.Halická for discussions which lead to substantial improvement of the paper.

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