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BETA TYPE INTEGRAL FORMULA ASSOCIATED WITH WRIGHT GENERALIZED BESSEL FUNCTION

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Abstract. The object of the present paper is to establish an integral formula involving Wright generalized Bessel function (or generalized Bessel-Maitland function) $J_{\nu,q}^{\mu,\gamma}(z)$ defined by Singh et al. [**21**], which is expressed in the terms of generalized (Wright) hypergeometric functions. Some interesting special cases involving Bessel functions, generalized Bessel functions, generalized Mittag-Leffler functions are deduced.

1. Introduction

In recent years, many integral formulas involving a variety of special functions have been developed by many authors (see [**1, 2, 3, 4, 5**], [**7, 8**], [**10, 11, 12**]). Several integral formulas involving product of Bessel functions have been developed and play an important role in several physical problems. In fact, Bessel functions are associated with a wide range of problems in diverse areas of mathematical physics. Here, we aim at presenting two generalized integral formulas involving the generalized Bessel-Maitland function, which are expressed in terms of generalized (Wright) hypergeometric functions. Some interesting special cases of our main results are also considered.

The Bessel-Maitland function (or the Wright-generalized Bessel function) is defined by (see [**14**]):

$$J_{\nu}^{\mu}(z)=\sum_{m=0}^{\infty}\frac{(-z)^m}{\Gamma(\nu+\mu m+1)},\qquad(\mu>0;z\in\mathbb{C}). \tag{1.1}$$

An interesting generalization of the Bessel-Maitland function $J_{\nu,\sigma}^{\mu}(z)$ is defined by (see [**9**]):

$$J_{\nu,\sigma}^{\mu}(z)=\sum_{m=0}^{\infty}\frac{(-1)^m(\frac{z}{2})^{\nu+2\sigma+2m}}{\Gamma(\sigma+m+1)\Gamma(\nu+\sigma+\mu m+1)}, \tag{1.2}$$

where $\mu>0$; $z,\nu,\sigma\in\mathbb{C}$. Here and in the following, let $\mathbb{C}$ and $\mathbb{N}$ be the sets of all complex numbers and positive integers, respectively.

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Further, another generalization of the generalized Bessel-Maitland function $J_{\nu,q}^{\mu,\gamma}(z)$ is defined by (See [**21**]):

$$J_{\nu,q}^{\mu,\gamma}(z) = \sum_{m=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\nu+\mu m+1)} \frac{(-z)^m}{m!}, \tag{1.3}$$

where $\mu, \nu, \gamma \in \mathbb{C}$; $\Re(\nu) > 0$, $\Re(\mu) > 0, \Re(\gamma) > 0$ and $q \in (0,1) \bigcup \mathbb{N}$ and $(\gamma)_0 = 1$, $(\gamma)_{qn} = \frac{\Gamma(\gamma+qm)}{\Gamma(\gamma)}$, denote the generalized Pochhammer symbol.

We investigate some special cases of the generalized Besssl-Maitland function (1.3) by given particular values to the parameters μ, ν, γ, q.

(i) Setting $q = \gamma = 1$ and replacing ν by $\nu + \sigma$ and z by $\frac{z^2}{4}$, to get

$$J_{\nu+\sigma,1}^{\mu,1}\left(\frac{z^2}{4}\right) = \Gamma(\sigma+m+1)\left(\frac{z}{2}\right)^{-\nu-2\sigma} J_{\nu,\sigma}^{\mu}(z), \tag{1.4}$$

where $J_{\nu,\sigma}^{\mu}(z)$ denotes the Bessel-Maitland function defined by (1.2).

(ii) Letting $q = 0$, equation (1.3) reduces to

$$J_{\nu,0}^{\mu,\gamma}(z) = J_{\nu,}^{\mu}(z), \tag{1.5}$$

where $J_{\nu}^{\mu}(z)$ is the generalized Bessel function defined by (1.1).

(iii) Setting $q = 0$ and replacing ν by $\nu - 1$, equation (1.3) reduces to

$$J_{\nu-1,0}^{\mu,\gamma}(-z) = \Phi(\mu, \nu; z), \tag{1.6}$$

where $\Phi(\mu, \nu; z)$ know as Wright function (see [**23, 24, 25**])

(iv) Replacing ν by $\nu - 1$, (1.3) reduces to

$$J_{\nu-1,q}^{\mu,\gamma}(-z) = E_{\mu,\nu}^{\gamma;q}(z), \tag{1.7}$$

where $\mu, \nu, \gamma \in \mathbb{C}$, $\Re(\mu) > 0, \Re(\nu), \Re(\gamma) > 0, q \in (0,1) \bigcup \mathbb{N}$ and $E_{\mu,\nu}^{\gamma;q}(z)$ denotes the generalized Mittag-Leffler function defined by Shukla and Prajapati [**19**].

(v) Setting $q = 1$ and replacing ν by $\nu - 1$, (1.3) reduces to

$$J_{\nu-1,1}^{\mu,\gamma}(-z) = E_{\mu,\nu}^{\gamma}(z), \tag{1.8}$$

where $\alpha, \beta, \gamma \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0$ and $E_{\mu,\nu}^{\gamma}(z)$ is the Mittag-Leffler function defined by Prabhakar [**16**].

(vi) For $\gamma = q = 1$ and replacing ν by $\nu - 1$, (1.3) reduces to

$$J_{\nu-1,1}^{\mu,1}(-z) = E_{\mu,\nu}(z), \tag{1.9}$$

where $\nu \in \mathbb{C}$; $\Re(\mu) > 0, \Re(\nu) > 0$ and $E_{\mu,\nu}(z)$ is the Mittag Leffler function defined by Wiman [**22**].

(vii) Setting $\nu = 0$, $q = \gamma = 1$, (1.3) reduces to

$$J_{0,1}^{\mu,1}(-z) = E_{\mu}(z), \tag{1.10}$$

where $z \in \mathbb{C}$ and $\Gamma(s)$ is the Gamma function; $\alpha \geq 0$ and $E_{\mu}(z)$ is the Mittag-Leffler function defined by Ghosta Mittag-Leffler [**15**].

The generalization of the generalized hypergeometric series ${}_pF_q$ is due to Fox [**6**] and Wright ([**23, 24, 25**]) who studied the asymptotic expansion of the generalized (Wright) hypergeometric function defined by (see [**20**, p.21]; see also [**18**]):

$$(1.11)\qquad {}_p\Psi_q\left[\begin{array}{c}(\alpha_1,A_1),\ldots,(\alpha_p,A_p);\\ (\beta_1\ ,\ B_1),\ldots,(\beta_q,B_q);\end{array} z\right]=\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(\alpha_j+A_jk)}{\prod_{j=1}^{q}\Gamma(\beta_j+B_jk)}\,\frac{z^k}{k!},$$

where the coefficients $A_1,\ldots,A_p$ and $B_1,\ldots,B_q$ are positive real numbers such that

(1.12)

$$\text{(i) } 1+\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j>0 \ \text{ and } \ 0<|z|<\infty;\ z\neq 0.$$

(1.13)

$$\text{(ii) } 1+\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j=0 \ \text{ and } \ 0<|z|<{A_1}^{-A_1}\ldots{A_p}^{-A_p}{B_1}^{B_1}\ldots{B_q}^{B_q}.$$

A special case of (1.11) is

$$(1.14)\qquad {}_p\Psi_q\left[\begin{array}{c}(\alpha_1,1),\ldots,(\alpha_p,1);\\ (\beta_1,1),\ldots,(\beta_q,1);\end{array} z\right]=\frac{\prod_{j=1}^{p}\Gamma(\alpha_j)}{\prod_{j=1}^{q}\Gamma(\beta_j)}\,{}_pF_q\left[\begin{array}{c}\alpha_1,\ldots,\alpha_p;\\ \beta_1,\ldots,\beta_q;\end{array} z\right],$$

where ${}_pF_q$ is the generalized hypergeometric series defined by [**17**]

$$(1.15)\qquad \begin{aligned}{}_pF_q\left[\begin{array}{c}\alpha_1,\ldots,\alpha_p;\\ \beta_1,\ldots,\beta_q;\end{array} z\right]&=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\ldots(\alpha_p)_n}{(\beta_1)_n\ldots(\beta_q)_n}\,\frac{z^n}{n!}\\ &={}_pF_q(\alpha_1,\ldots,\alpha_p;\ \beta_1,\ldots\beta_q;\ z),\end{aligned}$$

where $(\lambda)_n$ is the Pochhammer's symbol [**17**]

For our present investigation, the following interesting and useful result due to MacRobert [**13**] will be required:

$$(1.16)\qquad \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta}\mathrm{d}x=\frac{\Gamma(\alpha)\,\Gamma(\beta)}{a^{\alpha}b^{\beta}\Gamma(\alpha+\beta)},$$

provided $\Re(\alpha)>0$ and $\Re(\beta)>0$.

2. Integral formula involving generalized Bessel-Maitland function

Theorem 2.1. *If* $\alpha, \beta, \gamma, \mu, \nu, \lambda \in C$, $1 + \mu - \lambda > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\nu) > 0$, *then*

$$\begin{aligned} &\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta} J_{\nu,\lambda}^{\mu,\gamma}\left[\frac{2abx(1-x)}{[ax+b(1-x)]^2}\right] \mathrm{d}x \\ &= \frac{1}{a^\alpha b^\beta \Gamma(\gamma)} {}_3\Psi_2 \left[\begin{array}{cccc} (\gamma,\lambda), & (\beta,1), & (\alpha,1); & -2 \\ (\nu+1,\mu), & (\alpha+\beta,2) & ; & \end{array} \right]. \end{aligned} \tag{2.1}$$

Proof. To establish our main result (2.1), we denote the left-hand side of (2.1) by I and then using(1.7), we have

$$I = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta} J_{\nu,\lambda}^{\mu,\gamma}\left[\frac{2abx(1-x)}{[ax+b(1-x)]}\right] \mathrm{d}x. \tag{2.2}$$

Now changing the order of integration and summation, which is justified by the uniform convergence of the series in the interval (0,1), we arrive

$$I = \frac{1}{a^\alpha b^\beta \Gamma(\gamma)} \sum_{m=0}^{\infty} \frac{\Gamma(\gamma+\lambda m)\Gamma(\alpha+m)\Gamma(\beta+m)}{\Gamma(\nu+1+\mu m)\Gamma(\alpha+\beta+2m)} \frac{-2^m}{m!}. \tag{2.3}$$

Finally, summing up the above series with the help of (1.11), we easily arrive at the right-hand side of (2.1). This completes the proof. □

Next, we consider other variation of (2.1). In fact, we establish an integral formula for the Bessel-Maitland function $J_{\nu,\lambda}^{\mu,\gamma}(z)$, which is expressed in terms of the generalized hypergeometric function ${}_pF_q$.

Variation of (2.1). Let the conditions of our main result be satisfied. Then the following integral formula holds true:

$$\begin{aligned} &\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta} J_{\nu,\lambda}^{\mu,\gamma}\left[\frac{2abx(1-x)}{[ax+b(1-xy)]^2}\right] \mathrm{d}x \\ &= \frac{\Gamma(\alpha)\Gamma(\beta)}{a^\alpha b^\beta \Gamma(\nu+1)\Gamma(\alpha+\beta)} {}_{\lambda+2}F_{\mu+2} \left[\begin{array}{cccc} \Delta(\lambda;\ \gamma), & \alpha, & \beta; & \frac{\lambda^\lambda q^q}{2\mu^\mu} \\ \Delta(\mu;\nu+1), & & \Delta(2;\alpha+\beta); & \end{array} \right], \end{aligned} \tag{2.4}$$

where $\Delta(m;\ l)$ abbreviates the array of m parameters $\frac{l}{m}, \frac{l+1}{m}, \ldots, \frac{l+m-1}{m}$, $m \in \mathbb{N}$.

Proof. In order to prove the result (2.4), using the results

$$\Gamma(\alpha+n) = \Gamma(\alpha)(\alpha)_n$$

and

$$(l)_{kn} = k^{kn}\Big(\frac{l}{k}\Big)_n \Big(\frac{l+1}{k}\Big)_n \cdots \Big(\frac{l+k-1}{k}\Big)_n,$$

(Gauss multiplication theorem) in (2.3) and summing up the given series with the help of (1.15), we easily arrive at our required result (2.4). □

3. Special Cases

(i). On setting $\lambda=\gamma=1$ and replacing ν by $\nu+\sigma$ and z by $(\frac{z^2}{4})$ in (2.1) and then by using (1.4), we get

(3.1)
$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta} J^{\mu}_{\nu,\sigma}\left[\frac{2abx(1-x)}{[ax+b(1-xy)]^2}\right]\mathrm{d}x$$
$$=\frac{1}{a^{\alpha}b^{\beta}}\, {}_3\Psi_2\left[\begin{matrix}(1,1), & (\alpha,\ 2), & (\beta,\ 2); \\ & (\nu+\sigma+1,\mu), & (\alpha+\beta,4);\end{matrix}\ -1\right],$$

where $\alpha,\beta,\nu,\mu,\sigma\in\mathbb{C}$; $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\nu)>0$, $\Re(\sigma)>0$.

(ii). On setting $\lambda=\gamma=1$ and replacing ν by $\nu+\sigma$ and z by $(\frac{z^2}{4})$ in (2.4) and then by using (1.4), we find:

(3.2)
$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta} J^{\mu}_{\nu,\sigma}\left[\frac{2abx(1-x)}{[ax+b(1-xy)]^2}\right]\mathrm{d}x$$
$$=\frac{\Gamma(\alpha)\Gamma(\beta)}{a^{\alpha}b^{\beta}\Gamma(\nu+\sigma)\Gamma(\alpha+\beta)}\, {}_5F_{\mu+4}\left[\begin{matrix}1, & \Delta(2;\alpha), & \Delta(2;\beta); \\ & \Delta(\mu;\nu+\sigma+1), & \Delta(4;\alpha+\beta);\end{matrix}\ \frac{-1}{\mu^{\mu}}\right],$$

where $\alpha,\beta,\nu,\mu,\sigma\in\mathbb{C}$; $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\nu)>0$, $\Re(\sigma)>0$.

(iii). On setting $\lambda=0$ in (2.1) and then by using (1.5), we attain:

(3.3)
$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta} J^{\mu}_{\nu}\left[\frac{2abx(1-x)}{[ax+b(1-xy)]^2}\right]\mathrm{d}x$$
$$=\frac{1}{a^{\alpha}b^{\beta}}\, {}_2\Psi_2\left[\begin{matrix}(\alpha,1), & (\beta,1); \\ (\nu+1,\mu), & (\alpha+\beta,2);\end{matrix}\ -2\right],$$

where $\alpha,\beta,\nu,\mu,\in\mathbb{C}$; $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\nu)>0$.

(iv) On setting $\lambda=0$ in (2.4), and then by using (1.5), we acquire:

(3.4)
$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta} J^{\mu}_{\nu}\left[\frac{2abx(1-x)}{[ax+b(1-xy)]^2}\right]\mathrm{d}x$$
$$=\frac{\Gamma(\alpha)\Gamma(\beta)}{a^{\alpha}b^{\beta}\Gamma(\nu+1)\Gamma(\alpha+\beta)}\, {}_2F_{\mu+2}\left[\begin{matrix}\alpha, & \beta; \\ \Delta(\mu;\nu+1), & \Delta(2;\alpha+\beta);\end{matrix}\ \frac{-1}{2\mu^{\mu}}\right],$$

where $\alpha,\beta,\nu,\mu,\in\mathbb{C}$; $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\nu)>0$.

(v) On setting $\lambda = 0$ and replacing ν by $\nu - 1$ and then by using (1.6), we find:

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta}\Phi\left(\mu;\nu;\frac{2abx(1-x)}{[ax+b(1-xy)]^2}\right)\mathrm{d}x \tag{3.5}$$

$$= \frac{1}{a^{\alpha}b^{\beta}}\ {}_2\Psi_2\left[\begin{array}{ll} (\alpha,1), & (\beta,1); \\ (\nu,\mu), & (\alpha+\beta,2); \end{array} 2\right],$$

where $\alpha,\beta,\nu,\mu \in \mathbb{C}$; $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\nu) > 0$.

(vi). On setting $\lambda = 0$ and replacing ν by $\nu - 1$ and then by using (1.6), we get:

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta}\Phi(\mu;\nu;\frac{2abx(1-x)}{[ax+b(1-xy)]^2})\mathrm{d}x \tag{3.6}$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{a^{\alpha}b^{\beta}\Gamma(\nu)\Gamma(\alpha+\beta)}\ {}_2F_{\mu+2}\left[\begin{array}{ll} \alpha, & \beta; \\ (\mu;\nu), & (2;\alpha+\beta); \end{array} \frac{1}{2\mu^{\mu}}\right],$$

where $\alpha,\beta,\nu,\mu \in \mathbb{C}$; $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\nu) > 0$.

(vii) On replacing ν by $\nu - 1$ and then by using (1.7), we obtain:

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta}E^{\gamma,\lambda}_{\mu,\nu}\left[\frac{-2abx(1-x)}{[ax+b(1-xy)]^2}\right]\mathrm{d}x \tag{3.7}$$

$$= \frac{1}{a^{\alpha}b^{\beta}\Gamma(\gamma)}\ {}_3\Psi_2\left[\begin{array}{lll} (\gamma,\lambda), & (\alpha,\ 1), & (\beta,\ 1); \\ & (\nu,\mu), & (\alpha+\beta+2); \end{array} 2\right],$$

where $\alpha,\beta,\gamma,\mu,\lambda \in \mathbb{C}$; $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\lambda) > 0$, $\Re(\gamma) > 0$, $\Re(\nu) > 0$.

(viii) Replacing ν by $\nu - 1$ and then by using (1.7), we get
(3.8)

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta}E^{\gamma,\lambda}_{\mu,\nu}\left[\frac{-2abx(1-x)}{[ax+b(1-xy)]^2}\right]\mathrm{d}x$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{a^{\alpha}b^{\beta}\Gamma(\nu)\Gamma(\alpha+\beta)}\ {}_{\lambda+2}F_{\mu+2}\left[\begin{array}{lll} \Delta(\lambda;\gamma), & \alpha, & \beta; \\ & \Delta(\mu;\nu), & \Delta(2;\alpha+\beta); \end{array} \frac{\lambda^{\lambda}}{2\mu^{\mu}}\right],$$

where $\alpha,\beta,\gamma,\mu \in \mathbb{C}$; $\Re(\mu) > 05$, $\Re(\lambda) > 0$, $\Re(\gamma) > 0$, $\Re(\nu) > 0$.

(ix) Setting $\lambda = 1$ and replacing ν by $\nu - 1$ in (2.1) and then by using (1.8), we get

$$\begin{aligned}&\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta}E_{\mu,\nu}^{\gamma}\left[\frac{2abx(1-x)}{[ax+b(1-xy)]^2}\right]\mathrm{d}x \\ &= \frac{1}{a^\alpha b^\beta\Gamma(\gamma)}\ {}_3\Psi_2\left[\begin{matrix}(\gamma,1), & (\alpha,1), & (\beta,1); \\ (\nu,\mu), & (\alpha+\beta+2) & ;\end{matrix}\ 2\right],\end{aligned} \tag{3.9}$$

where $\alpha,\beta,\gamma,\mu\in\mathbb{C}$; $\Re(\mu)>0$, $\Re(\lambda)>0$, $\Re(\gamma)>0$, $\Re(\nu)>0$.

(x) Setting $\lambda = 1$ and replacing ν by $\nu - 1$ in (2.4) and then by using (1.8) , we get

$$\begin{aligned}&\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta}E_{\mu,\nu}^{\gamma}\left[\frac{2abx(1-x)}{[ax+b(1-xy)]^2}\right]\mathrm{d}x \\ &= \frac{\Gamma(\alpha)\Gamma(\beta)}{a^\alpha b^\beta\Gamma(\nu)\Gamma(\alpha+\beta)}\ {}_3F_{\mu+2}\left[\begin{matrix}\gamma, & \alpha, & \beta; \\ \Delta(\mu;\nu,), & (\alpha+\beta+2), & ;\end{matrix}\ \frac{1}{2\mu^\mu}\right]\end{aligned} \tag{3.10}$$

where $\alpha,\beta,\gamma,\mu\in\mathbb{C}$; $\Re(\mu)>0$, $\Re(\lambda)>0$, $\Re(\gamma)>0$, $\Re(\nu)>0$.

(xi) Setting $\lambda = \gamma = 1$ and replacing ν by $\nu - 1$ in (2.1) and then by using (1.9), we get

$$\begin{aligned}&\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta}E_{\mu}^{\nu}\left[\frac{2abx(1-x)}{[ax+b(1-xy)]^2}\right]\mathrm{d}x \\ &= \frac{1}{a^\alpha b^\beta}\ {}_3\Psi_2\left[\begin{matrix}(1,1), & (\alpha,1), & (\beta,1); \\ (\nu,\mu), & (\alpha+\beta+2) & ;\end{matrix}\ 2\right],\end{aligned} \tag{3.11}$$

where $\alpha,\beta,\nu,\mu\in\mathbb{C}$; $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\nu)>0$, $\Re(\mu)>0$.

(xii) Setting $\lambda = \gamma = 1$ and replacing ν by $\nu - 1$ in (2.4) and then by using (1.9), we get

$$\begin{aligned}&\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta}E_{\mu}^{\nu}\left[\frac{2abx(1-x)}{[ax+b(1-xy)]^2}\right]\mathrm{d}x \\ &= \frac{\Gamma(\alpha)\Gamma(\beta)}{a^\alpha b^\beta\Gamma(\nu)\Gamma(\alpha+\beta)}\ {}_3F_{\mu+2}\left[\begin{matrix}1, & \alpha, & \beta; \\ \Delta(\mu;\nu), & \Delta(2;\alpha+\beta) & ;\end{matrix}\ \frac{1}{2\mu^\mu}\right],\end{aligned} \tag{3.12}$$

where $\alpha,\beta,\nu,\mu\in\mathbb{C}$; $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\nu)>0$, $\Re(\mu)>0$.

(xiii) On setting $\nu = 0, \lambda = \gamma = 1$ in (2.1) and then by using (1.10), we get

$$\begin{aligned}&\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta}E_\mu\left[\frac{2abx(1-x)}{[ax+b(1-xy)]^2}\right]\mathrm{d}x\\ &= \frac{1}{a^\alpha b^\beta}\,{}_3\Psi_2\left[\begin{matrix}(1,1), & (\alpha,1), & (\beta,1); \\ (1,\mu), & (\alpha+\beta,2) & ;\end{matrix}\; 2\right],\end{aligned} \tag{3.13}$$

where $\alpha,\beta,\mu\in\mathbb{C}$; $\Re(\mu)>0$, $\Re(\alpha)>0$, $\Re(\beta)>0$.

(xiv). On setting $\nu = o, \lambda = \gamma = 1$ in (2.4) and then by using (1.10), we get

$$\begin{aligned}&\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta}E_\mu\left[\frac{2abx(1-x)}{[ax+b(1-xy)]^2}\right]\mathrm{d}x\\ &= \frac{\Gamma(\alpha)\Gamma(\beta)}{a^\alpha b^\beta\Gamma(\alpha+\beta)}\,{}_3F_{\mu+2}\left[\begin{matrix}1, & \alpha, & \beta; \\ \Delta(\mu;1), & \Delta(2;\alpha+\beta) & ;\end{matrix}\; 2\right],\end{aligned} \tag{3.14}$$

where $\alpha,\beta,\mu\in\mathbb{C}$; $\Re(\mu)>0$, $\Re(\alpha)>0$, $\Re(\beta)>0$.

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