Acta Math. Univ. Comenianae
Vol. LXXXVII, 1 (2018), pp. 127–140

A LOWER BOUND OF NORMALIZED SCALAR CURVATURE FOR BI-SLANT SUBMANIFOLDS IN GENERALIZED SASAKIAN SPACE FORMS USING CASORATI CURVATURES

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Abstract. In this paper, we prove two optimal inequalities between the normalized δ-Casorati curvature and the normalized scalar curvature for bi-slant submanifolds in a generalized Sasakian space form. Moreover, we show that the equality at all points characterizes the invariantly quasi-umbilical submanifolds in both cases.

1. Introduction

In 1993, Chen [**5**] initiated the theory of δ-invariants. Chen established a sharp inequality for a submanifold in a real space form using the scalar curvature, the sectional curvature, both being intrinsic invariants, and squared mean curvature, the main extrinsic invariant. That is, in [**4**] he established simple relationships between the main intrinsic invariants and the main extrinsic invariants of a submanifold in real space forms with any codimension. Now it has become one of the most interesting research topics in differential geometry of submanifolds. Instead of concentrating on the sectional curvature with the extrinsic squared mean curvature, the Casorati curvature of a submanifold in a Riemannian manifold was considered as an extrinsic invariant defined as the normalized square of the length of the second fundamental form. The notion of Casorati curvature extends the concept of the principal direction of a hypersurface of a Riemannian manifold. It was preferred by Casorati over the traditional Gauss curvature. Several geometers in [**6, 7, 10, 18, 19**] found geometrical meaning and the importance of the Casorati curvature. Therefore, it attracts the geometers to obtain optimal inequalities for the Casorati curvatures of submanifolds in different ambient spaces. Decu, Haesen and Verstraelen introduced the normalized δ-Casorati curvatures $\delta_c(n-1)$ and $\hat{\delta}_c(n-1)$, and established inequalities involving $\delta_c(n-1)$ and $\hat{\delta}_c(n-1)$ for submanifolds in real space forms [**6**]. Moreover, the same authors proved in [**7**] an inequality in which the scalar curvature is estimated from above by the normalized Casorati curvatures, while Ghisoiu [**8**] obtained some inequalities for the Casorati curvatures of slant submanifolds in complex space forms. Recently, Lee et al. [**13**]

Received May 8, 2017; revised July 7, 2017.
2010 *Mathematics Subject Classification.* Primary 53B05, 53C25, 53B20, 53C40.
Key words and phrases. Casorati curvature; bi-slant submanifold; generalized Sasakian space form; quasi-umbilical submanifold.

obtained optimal inequalities for submanifolds in real space forms, endowed with a semi-symmetric metric connection. Many authors obtained the optimal inequalities for the Casorati curvatures of the submanifolds of different ambient spaces [**12, 14, 17, 11, 22, 16, 15**].

The paper is structured as follows: Section 2 is devoted to preliminaries. In Section 3, we establish two sharp inequalities that relate the normalized scalar curvature with Casorati curvature for a bi-slant submanifold in a generalized Sasakian space form. In Section 4, we develop these inequalities for invariant, anti-invariant, CR, slant, semi-slant, hemi-slant submanifolds in the same ambient. Moreover, we also see the glimpse of these inequalities in different structures such as Sasakian space form, Kenmotsu space form and cosymplectic space form.

2. Preliminaries

A (2m + 1)-dimensional differentiable manifold $\overline{\mathcal{M}}$ is said to have an almost contact structure (ϕ, ξ, η, g) if on $\overline{\mathcal{M}}$ there exists a tensor field ϕ of type (1, 1), a vector field ξ, a 1-form η and a Riemannian metric g such that [**21**]

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \quad \eta(\phi) = 0, \quad \eta(X) = g(X, \xi) \tag{1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(\phi X, Y) + g(X, \phi Y) = 0. \tag{2}$$

Here X, Y, Z denote arbitrary vector fields on $\overline{\mathcal{M}}$. The fundamental 2-form φ on $\overline{\mathcal{M}}$ is defined by

$$\varphi(X, Y) = g(\phi X, Y).$$

Alegre et al. [**1**] introduced and studied the generalized Sasakian space forms. An almost contact metric manifold $(\overline{\mathcal{M}}, \phi, \xi, \eta, g)$ is said to be a *generalized Sasakian space form* if there exist differentiable functions f_1, f_2, f_3 such that curvature tensor R of $\overline{\mathcal{M}}$ is given by

$$\begin{aligned} \overline{R}(X, Y)Z = {} & f_1\big[g(Y, Z)X - g(X, Z)Y\big] + f_2\big[g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X \\ & + 2g(X, \phi Y)\phi Z\big] + f_3\big[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi \\ & - g(Y, Z)\eta(X)\xi\big] \end{aligned} \tag{3}$$

for all vector fields $X, Y, Z \in T\overline{\mathcal{M}}$.

The generalized Sasakian space form generalizes the concept of Sasakian space form, Kenmotsu space form and cosymplectic space form.

(i) A Sasakian space form is the generalized Sasakian space form with $f_1 = \frac{c+3}{4}$ and $f_2 = f_3 = \frac{c-1}{4}$.

(ii) A Kenmotsu space form is the generalized Sasakian space form with $f_1 = \frac{c-3}{4}$ and $f_2 = f_3 = \frac{c+1}{4}$.

(iii) A cosymplectic space form is the generalized Sasakian space form with $f_1 = f_2 = f_3 = \frac{c}{4}$.

In the following, we consider $\overline{\mathcal{M}}$ as a generalized Sasakian space form $\overline{\mathcal{M}}(f_1, f_2, f_3)$ of dimension $(2m + 1)$ and let $\mathcal{M}$ be an $(n + 1)$-dimensional submanifold of $\overline{\mathcal{M}}(f_1, f_2, f_3)$. Let $T\mathcal{M}$ and $T^{\perp}\mathcal{M}$ denote the Lie algebra of vector fields and the set of all normal vector fields on $\mathcal{M}$, respectively. The operator of covariant differentiation with respect to the Levi-Civita connection in $\mathcal{M}$ and $\overline{\mathcal{M}}$ is denoted by ∇ and $\overline{\nabla}$, respectively. Let $\overline{R}$ and R be the curvature tensor of $\overline{\mathcal{M}}(f_1, f_2, f_3)$ and $\mathcal{M}$, respectively. The Gauss equation is given by [**21**]

$$\begin{aligned}\overline{R}(X,Y,Z,W) = R(X,Y,Z,W) &- g(h(X,W),h(Y,Z)) \\ &+ g(h(X,Z),h(Y,W))\end{aligned} \tag{4}$$

for all vector fields $X, Y, Z \in T\overline{\mathcal{M}}$.

For any vector field $X \in T\mathcal{M}$, we put [**21**]

$$\phi X = PX + QX, \tag{5}$$

where PX and QX denote the tangential and normal components of ϕX, respectively. Then P is an endomorphism of $T\mathcal{M}$ and Q is the normal bundle valued 1-form on $T\mathcal{M}$.

In the same way, for any vector field $V \in T^{\perp}\mathcal{M}$, we put [**21**]

$$\phi V = BV + CV, \tag{6}$$

where BV and CV denote tangential and normal components of ϕV, respectively.

It is easy to see that F and B are skew-symmetric and

$$g(QX, V) = -g(X, BV) \tag{7}$$

for any vector fields $X \in T\mathcal{M}$ and $V \in T^{\perp}\mathcal{M}$.

The structural vector field ξ can be decomposed as

$$\xi = \xi_1 + \xi_2, \tag{8}$$

where ξ_1 and ξ_2 are the tangential and the normal components of ξ.

A submanifold $\mathcal{M}$ of an almost contact metric manifold $\overline{\mathcal{M}}$ is said to be *invariant* if $Q \equiv 0$, that is, $\phi X \in T\mathcal{M}$, and *anti-invariant* if $P \equiv 0$, that is, $\phi X \in T^{\perp}\mathcal{M}$, for any vector field $X \in T\mathcal{M}$.

There are some other important classes of submanifolds which are determined by the behavior of tangent bundle of the submanifold under the action of an almost contact metric structure ϕ of $\overline{\mathcal{M}}$:

(i) A submanifold $\mathcal{M}$ of $\overline{\mathcal{M}}$ is called a *contact CR-submanifold* [**20**] of $\overline{\mathcal{M}}$ if there exists a differentiable distribution D on $\mathcal{M}$ whose orthogonal complementary distribution $D^{\perp}$ is anti-invariant.

(ii) A submanifold $\mathcal{M}$ of $\overline{\mathcal{M}}$ is called a *slant submanifold* [**3**] of $\overline{\mathcal{M}}$ if the angle between ϕX and $T_x\mathcal{M}$ is constant for all $X \in T\mathcal{M} - \{\xi_x\}$ and $x \in \mathcal{M}$.

(iii) A submanifold $\mathcal{M}$ of $\overline{\mathcal{M}}$ is called *semi-slant submanifold* [**2**] of $\overline{\mathcal{M}}$ if there exists a pair of orthogonal distributions D and D_θ such that D is invariant and D_θ is proper slant.

(iv) A submanifold $\mathcal{M}$ of $\overline{\mathcal{M}}$ is called *hemi-slant submanifold* (or *pseudo-slant*) [**9**] of $\overline{\mathcal{M}}$ if there exists a pair of orthogonal distributions $D^{\perp}$ and D_θ such that $D^{\perp}$ is anti-invariant and D_θ is proper slant.

Bi-slant submanifolds were first defined by A. Cariazo et al. in [**2**] as a generalization of CR and semi-slant submanifolds. Such submanifolds generalize complex, totally real, slant and hemi-slant submanifolds as well. Here we define a bi-slant submanifold of an almost contact metric manifold as follows.

Definition 2.1. A submanifold $\mathcal{M}$ of an almost contact metric manifold $\overline{\mathcal{M}}$ is said to be a *bi-slant submanifold* if there exists a pair of orthogonal distributions D_{θ_1} and D_{θ_2} of $\mathcal{M}$ such that
(i) $T\mathcal{M} = D_{\theta_1} \oplus D_{\theta_2} \oplus \{\xi\}$;
(ii) $\phi D_{\theta_1} \perp D_{\theta_2}$ and $\phi D_{\theta_2} \perp D_{\theta_1}$;
(iii) Each distribution D_{θ_i} is slant with the slant angle θ_i for $i = 1, 2$.

A bi-slant submanifold of an almost contact metric manifold $\overline{\mathcal{M}}$ is called *proper* if the slant distributions D_{θ_1} and D_{θ_2} are of the slant angles $\theta_1, \theta_2 \neq 0, \frac{\pi}{2}$.

Suppose that $\mathcal{M}$ is a bi-slant submanifold of dimension $n+1 = 2n_1 + 2n_2 + 1$ in $\overline{\mathcal{M}}(f_1, f_2, f_3)$. Let us assume the orthonormal basis of $\mathcal{M}$ as follows

$$\begin{aligned} E_1, E_2 &= \sec\theta_1 Pe_1, \dots, E_{2n_1-1}, E_{2n_1} = \sec\theta_1 Pe_{2n_1-1}, E_{2n_1+1}, E_{2n_1+2} \\ &= \sec\theta_2 Pe_{2n_1+1}, \dots, E_{2n_1+2n_2-1}, E_{2n_1+2n_2} = \sec\theta_2 Pe_{2n_1+2n_2-1}, E_{2n_1+2n_2+1} \\ &= \xi. \end{aligned}$$

Also,

$$g^2(\phi E_{i+1}, E_i) = \begin{cases} \cos^2\theta_1 & \text{for } i = 1, \dots, 2n_1 - 1 \\ \cos^2\theta_2 & \text{for } i = 2n_1+1, \dots, 2n_1+2n_2-1. \end{cases} \tag{9}$$

Hence, we have

$$\sum_{i,j=1}^{n+1} g^2(\phi E_j, E_i) = 2\{n_1 \cos^2\theta_1 + n_2 \cos^2\theta_2\}.$$

Remark. If we assume:
(i) $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{2}$, then $\mathcal{M}$ is a *CR-submanifold.*
(ii) $\theta_1 = 0$ and $\theta_2 \neq 0, \frac{\pi}{2}$, then $\mathcal{M}$ is a *semi-slant submanifold.*
(iii) $\theta_1 = \frac{\pi}{2}$ and $\theta_2 \neq 0, \frac{\pi}{2}$, then $\mathcal{M}$ is a *hemi-slant submanifold.*

3. A Lower Bound of Normalized Scalar Curvature

In this section, we study the Casorati curvature of bi-slant submanifolds $\mathcal{M}$ of dimension $(n+1)$ in a generalized Sasakian space form $\overline{\mathcal{M}}(f_1, f_2, f_3)$ of dimension $(2m+1)$. Then we establish two optimal inequalities for $\mathcal{M}$ in $\overline{\mathcal{M}}(f_1, f_2, f_3)$. For this, let us consider a local orthonormal tangent frame $\{E_1, \dots, E_{n+1}\}$ of the tangent bundle $T\mathcal{M}$ of $\mathcal{M}$ and a local orthonormal normal frame $\{E_{n+2}, \dots, E_{2m+1}\}$ of the normal bundle $T^\perp\mathcal{M}$ of $\mathcal{M}$ in $\overline{\mathcal{M}}(f_1, f_2, f_3)$. At any $p \in \mathcal{M}$, the scalar curvature τ at that point is given by

$$\tau = \sum_{i \leq i < j \leq n+1} R(E_i, E_j, E_j, E_i)$$

and the normalized scalar curvature ρ of M is defined as

$$\rho = \frac{2\tau}{n(n+1)}.$$

The mean curvature vector denoted by $\mathcal{H}$ of $\mathcal{M}$ is given by

$$\mathcal{H} = \sum_{i=1}^{n+1} \frac{1}{n+1} h(E_i, E_i).$$

Conveniently, let us put

$$h_{ij}^r = g(h(E_i, E_j), E_r)$$

for $i, j = \{1, \ldots, n+1\}$ and $r = \{n+2, \ldots, 2m+1\}$. Then the squared norm of a mean curvature vector of $\mathcal{M}$ is defined as

$$\|\mathcal{H}\|^2 = \frac{1}{(n+1)^2} \sum_{r=n+2}^{2m+1} \left\{ \sum_{i=1}^{n+1} h_{ii}^r \right\}^2.$$

and the squared norm of second fundamental form h is denoted by

$$\mathcal{C} = \frac{1}{n+1} \|h\|^2 \tag{10}$$

where

$$\|h\|^2 = \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n+1} \left(h_{ij}^r\right)^2.$$

It is known as the *Casorati curvature* $\mathcal{C}$ of $\mathcal{M}$.

If we suppose that $\mathcal{L}$ is an s-dimensional subspace of $T\mathcal{M}$, $s \geq 2$, and $\{E_1, \ldots, E_s\}$ is an orthonormal basis of $\mathcal{L}$, then the scalar curvature of the s-plane section $\mathcal{L}$ is given by

$$\tau(\mathcal{L}) = \sum_{i \leq i < j \leq s} R(E_i, E_j, E_j, E_i)$$

and the Casorati curvature of the subspace $\mathcal{L}$ is as follows:

$$\mathcal{C}(\mathcal{L}) = \frac{1}{s} \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{s} \left(h_{ij}^r\right)^2.$$

The normalized Casorati curvatures $\delta_c(n)$ and $\widehat{\delta}_c(n)$ are defined as

$$[\delta_c(n)]_p = \frac{1}{2}\mathcal{C}_p + \frac{n+2}{2(n+1)} \inf\{\mathcal{C}(\mathcal{L}) | \mathcal{L} : \text{a hyperplane of } T_p\mathcal{M}\}$$

and

$$[\widehat{\delta}_c(n)]_p = 2\mathcal{C}_p - \frac{2n+1}{2(n+1)} \sup\{\mathcal{C}(\mathcal{L}) | \mathcal{L} : \text{a hyperplane of } T_p\mathcal{M}\}.$$

Throughout this paper, we use the above notations.

A point $p \in \mathcal{M}$ is said to be an *invariantly quasi-umbilical point* if there exists a $2m - n$ orthogonal unit normal vector $\{E_{n+2}, \ldots, E_{2m+1}\}$ such that the shape operator with respect to all directions E_r have an eigenvalue of multiplicity n and that for each E_r the distinguished eigendirection is the same. The submanifold

$\mathcal{M}$ is said to be an *invariantly quasi-umbilical submanifold* if each of its points is an invariantly quasi-umbilical point.

Here we construct some optimal inequalities consisting of the normalized scalar curvature and the normalized δ-Casorati curvatures for bi-slant submanifolds $\mathcal{M}$ in $\overline{\mathcal{M}}(f_1, f_2, f_3)$.

Theorem 3.1. *Let M be an $(n+1)$-dimensional bi-slant submanifold $\mathcal{M}$ of a generalized Sasakian space form $\overline{\mathcal{M}}(f_1, f_2, f_3)$ of dimension $(2m+1)$ and dimensions of D_{θ_1} and D_{θ_2} are $2n_1$ and $2n_2$, respectively. Then it yields:*

(i) *The normalized Casorati curvature $\delta_c(n)$ satisfies*

$$\rho \leq \delta_c(n) + f_1 + \frac{6f_2}{n(n+1)}(n_1 \cos^2\theta_1 + n_2 \cos^2\theta_2) - \frac{2f_3}{n+1}\|\xi_1\|^2. \tag{11}$$

(ii) *The normalized Casorati curvature $\widehat{\delta}_c(n)$ satisfies*

$$\rho \leq \widehat{\delta}_c(n) + f_1 + \frac{6f_2}{n(n+1)}(n_1 \cos^2\theta_1 + n_2 \cos^2\theta_2) - \frac{2f_3}{n+1}\|\xi_1\|^2. \tag{12}$$

Moreover, the equalities hold in the relations (11) and (12) if and only if $\mathcal{M}$ is an invariantly quasi-umbilical submanifold with the flat normal connection in $\overline{\mathcal{M}}(f_1, f_2, f_3)$ such that with some orthonormal tangent frame $\{E_1, \dots, E_{n+1}\}$ of $T\mathcal{M}$ and orthonormal normal frame $\{E_{n+2}, \dots, E_{2m+1}\}$ of $T^{\perp}\mathcal{M}$, the shape operator $S_r, r \in \{n+2, \dots, 2m+1\}$, respectively, take the following form:

$$S_{n+2} = \begin{pmatrix} b & 0 & 0 & \dots & 0 & 0 \\ 0 & b & 0 & \dots & 0 & 0 \\ 0 & 0 & b & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b & 0 \\ 0 & 0 & 0 & \dots & 0 & 2b \end{pmatrix}, \qquad S_{n+3} = \dots = S_{2m+1} = 0 \tag{13}$$

and

$$S_{n+2} = \begin{pmatrix} 2b & 0 & 0 & \dots & 0 & 0 \\ 0 & 2b & 0 & \dots & 0 & 0 \\ 0 & 0 & 2b & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2b & 0 \\ 0 & 0 & 0 & \dots & 0 & b \end{pmatrix}, \qquad S_{n+3} = \dots = S_{2m+1} = 0. \tag{14}$$

Proof. Let $\{E_1, \dots, E_{n+1}\}$ and $\{E_{n+2}, \dots, E_{2m+1}\}$ be the orthonormal basis of $T\mathcal{M}$ and and $T^{\perp}\mathcal{M}$, respectively, at any point $p \in M$. Putting $X = W = E_i$,

$Y = Z = E_j$ into (3) and considering $i \neq j$, we have

$$\sum_{i,j=1}^{n+1} R(E_i, E_j, E_j, E_i) = \sum_{i,j=1}^{n+1} \Big\{ f_1 \{ g(E_j, E_j) g(E_i, E_i) - g(E_i, E_j) g(E_j, E_i) \} + f_2 \{ g(E_i, \phi E_j) g(\phi E_j, E_i) - g(\phi E_i, E_i) g(E_j, \phi E_j) + 2 g(E_i, \phi E_j) g(E_i, \phi E_j) \} + f_3 \{ \eta(E_i)\eta(E_j) g(E_i, E_i) - \eta(E_j)\eta(E_j) g(E_i, E_i) + \eta(E_i)\eta(E_j) g(E_i, E_j) - \eta(E_i)\eta(E_i) g(E_j, E_j) \} \Big\}.$$

From this and together with Gauss equation, we get

$$2\tau(p) = n(n+1)f_1 + 6f_2(n_1 \cos^2\theta_1 + n_2 \cos^2\theta_2) - 2nf_3\|\xi_1\|^2 + (n+1)^2\|\mathcal{H}\|^2 - (n+1)\mathcal{C}, \tag{15}$$

where we have used (10).

Now, we define the following function, denoted by $\mathcal{Q}$, which is a quadratic polynomial in the components of the second fundamental form

$$\mathcal{Q} = \frac{1}{2} n(n+1)\mathcal{C} + \frac{1}{2}(n+2)\mathcal{C}(\mathcal{L}) - 2\tau(p) + n(n+1)f_1 + 6f_2(n_1\cos^2\theta_1 + n_2\cos^2\theta_2) - 2nf_3\|\xi_1\|^2, \tag{16}$$

where $\mathcal{L}$ is a hyperplane of $T_p\mathcal{M}$. We can assume without loss of generality that $\mathcal{L}$ is spanned by $\{E_1, \ldots, E_n\}$. Then we have

$$\mathcal{Q} = \frac{n}{2} \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n+1} (h_{ij}^r)^2 + \frac{n+1}{2n} \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n} (h_{ij}^r)^2 - 2\tau(p) + n(n+1)f_1 + 6f_2(n_1\cos^2\theta_1 + n_2\cos^2\theta_2) - 2nf_3\|\xi_1\|^2. \tag{17}$$

From (15) and (17), we obtain

$$\mathcal{Q} = \frac{n+2}{2} \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n+1} (h_{ij}^r)^2 + \frac{n+1}{2n} \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n} (h_{ij}^r)^2 - \sum_{r=n+2}^{2m+1} \Big(\sum_{i,j=1}^{n+1} h_{ij}^r \Big)^2.$$

Now we can easily derive that

$$\mathcal{Q} = \sum_{r=n+2}^{2m+1} \sum_{i=1}^{n} \Big[\frac{n^2+n+2}{2n} (h_{ii}^r)^2 + (n+2)(h_{in}^r)^2 \Big] + \sum_{r=n+2}^{2m+1} \Big[\frac{(n+1)(n+2)}{n} \sum_{i<j=1}^{n} (h_{ij}^r)^2 - 2 \sum_{i<j=1}^{n} h_{ii}^r h_{jj}^r + \frac{n}{2}(h_{nn}^r)^2 \Big]. \tag{18}$$

From (18), we can find that the critical points

$$h^c = (h_{11}^{n+2}, h_{12}^{n+2}, \ldots, h_{n+1n+1}^{n+2}, \ldots, h_{11}^{2m+1}, \ldots, h_{n+1n+1}^{2m+1})$$

of $\mathcal{Q}$ are the solutions of the following system of linear homogeneous equations:

$$\begin{aligned} \frac{\partial \mathcal{Q}}{\partial h^r_{ii}} &= \frac{(n+1)(n+2)}{n} h^r_{ii} - 2\sum_{l=1}^{n+1} h^r_{ll} = 0,\\ \frac{\partial \mathcal{Q}}{\partial h^r_{n+1n+1}} &= n h_{n+1n+1} - 2\sum_{l=1}^{n} h^r_{ll} = 0,\\ \frac{\partial \mathcal{Q}}{\partial h^r_{ij}} &= \frac{2(n+1)(n+2)}{n} h^r_{ij} = 0,\\ \frac{\partial \mathcal{Q}}{\partial h_{in+1}} &= 2(n+2) h^r_{in+1} = 0, \end{aligned} \tag{19}$$

where $i,j = \{1,2,\ldots,n\}, i \neq j$, and $r \in \{n+2,\ldots,2m+1\}$.

Hence, every solution h^c has $h^r_{ij} = 0$ for $i \neq j$ and the corresponding determinant to the first two sets of equations of the above system (19) is zero (there exist solutions for non-totally geodesic submanifolds). Moreover, we find that the Hessian matrix $H(\mathcal{Q})$ has the following eigenvalues:

$$\begin{aligned} &\lambda_{11} = 0, \qquad \lambda_{22} = \frac{2n^2 - n + 2}{n},\\ &\lambda_{33} = \cdots = \lambda_{n+1n+1} = \frac{(n+1)(n+2)}{n},\\ &\lambda_{ij} = \frac{2(n+1)(n+2)}{n}, \qquad \lambda_{in+1} = 2(n+2),\\ &\text{for all } i,j \in \{1,2,\ldots,n\},\ i \neq j. \end{aligned} \tag{20}$$

Thus, we know that $\mathcal{Q}$ is parabolic and reaches a minimum $\mathcal{Q}(h^c) = 0$ for the solution h^c of the system (19). It follows that $\mathcal{Q} \geq 0$, and hence we have

$$\begin{aligned} 2\tau(p) \leq \frac{1}{2} n(n+1)\mathcal{C} + \frac{1}{2}(n+2)\mathcal{C}(\mathcal{L}) + n(n+1) f_1\\ + 6 f_2 (n_1 \cos^2\theta_1 + n_2 \cos^2\theta_2) - 2n f_3 \|\xi_1\|^2, \end{aligned}$$

whereby we obtain

$$\begin{aligned} \rho \leq \frac{1}{2}\mathcal{C} + \frac{1}{2}\left(\frac{(n+2)}{n(n+1)}\right)\mathcal{C}(\mathcal{L}) + f_1\\ + \frac{6 f_2}{n(n+1)}(n_1\cos^2\theta_1 + n_2\cos^2\theta_2) - \frac{2f_3}{n+1}\|\xi_1\|^2 \end{aligned}$$

for every tangent hyperplane $\mathcal{L}$ of $T_p\mathcal{M}$. If we take the infimum over all tangent hyperplanes $\mathcal{L}$, the result trivially follows. Moreover, the equality sign holds if and only if

$$h^r_{ij} = 0, \qquad \text{for all } i,j \in \{1,\ldots,n+1\},\ i \neq j,\ r \in \{n+2,\ldots,2m+1\} \tag{21}$$

and

$$h_{n+1n+1} = 2h^r_{11} = \cdots = 2h^r_{nn}, \qquad \text{for all } r \in \{n+2,\ldots,2m+1\}. \tag{22}$$

From (21) and (22), we conclude that the equality sign holds in the inequality (11) if and only if the submanifold $\mathcal{M}$ is invariantly quasi-umbilical with trivial normal connection in $\mathcal{M}$, such that with respect to suitable orthonormal tangent and normal orthonormal frames, the shape operators take the form of (13).

In the same manner, we can establish an inequality in the second part of the theorem. □

4. Some Applications of the Theorem 3.1 for Different Kinds of Submanifolds

In this section, we quote the developed optimal inequalities for hemi-slant, semi-slant, slant, CR, anti-invariant and invariant submanifolds in the same ambient space, that is, generalized Sasakian space form.

Theorem 4.1. *Let M be an $(n+1)$-dimensional hemi-slant submanifold $\mathcal{M}$ of a generalized Sasakian space form $\overline{\mathcal{M}}(f_1, f_2, f_3)$ of dimension $(2m+1)$ and dimensions of D_{θ_1} and D_{θ_2} are $2n_1$ and $2n_2$, respectively. Then it holds:*
(i) *The normalized Casorati curvature $\delta_c(n)$ satisfies*

$$\rho \leq \delta_c(n) + f_1 + \frac{6f_2}{n(n+1)}(n_1 \cos^2 \theta_1) - \frac{2f_3}{n+1}\|\xi_1\|^2. \tag{23}$$

(ii) *The normalized Casorati curvature $\widehat{\delta}_c(n)$ satisfies*

$$\rho \leq \widehat{\delta}_c(n) + f_1 + \frac{6f_2}{n(n+1)}(n_1 \cos^2 \theta_1) - \frac{2f_3}{n+1}\|\xi_1\|^2. \tag{24}$$

Moreover, the equalities hold in the relations (23) and (24) if and only if $\mathcal{M}$ is an invariantly quasi-umbilical submanifold with the flat normal connection in $\overline{\mathcal{M}}(f_1, f_2, f_3)$ such that with an orthonormal tangent frame $\{E_1, \ldots, E_{n+1}\}$ of $T\mathcal{M}$ and orthonormal normal frame $\{E_{n+2}, \ldots, E_{2m+1}\}$ of $T^{\perp}\mathcal{M}$, the shape operator $S_r, r \in \{n+2, \ldots, 2m+1\}$, are given by (13) and (14), respectively.

Theorem 4.2. *Let M be an $(n+1)$-dimensional semi-slant submanifold $\mathcal{M}$ of a generalized Sasakian space form $\overline{\mathcal{M}}(f_1, f_2, f_3)$ of dimension $(2m+1)$ and dimensions of D_{θ_1} and D_{θ_2} are $2n_1$ and $2n_2$, respectively. Then it holds:*
(i) *The normalized Casorati curvature $\delta_c(n)$ satisfies*

$$\rho \leq \delta_c(n) + f_1 + \frac{6f_2}{n(n+1)}(n_1 + n_2 \cos^2 \theta_2) - \frac{2f_3}{n+1}\|\xi_1\|^2. \tag{25}$$

(ii) *The normalized Casorati curvature $\widehat{\delta}_c(n)$ satisfies*

$$\rho \leq \widehat{\delta}_c(n) + f_1 + \frac{6f_2}{n(n+1)}(n_1 + n_2 \cos^2 \theta_2) - \frac{2f_3}{n+1}\|\xi_1\|^2. \tag{26}$$

Moreover, the equalities hold in the relations (25) and (26) if and only if $\mathcal{M}$ is an invariantly quasi-umbilical submanifold with the flat normal connection in $\overline{\mathcal{M}}(f_1, f_2, f_3)$ such that with an orthonormal tangent frame $\{E_1, \ldots, E_{n+1}\}$ of $T\mathcal{M}$ and orthonormal normal frame $\{E_{n+2}, \ldots, E_{2m+1}\}$ of $T^{\perp}\mathcal{M}$, the shape operator $S_r, r \in \{n+2, \ldots, 2m+1\}$, are given by (13) and (14), respectively.

Theorem 4.3. *Let M be an $(n+1)$-dimensional slant submanifold $\mathcal{M}$ of a generalized Sasakian space form $\overline{\mathcal{M}}(f_1, f_2, f_3)$ of dimension $(2m+1)$ and dimensions of D_{θ_1} and D_{θ_2} are $2n_1$ and $2n_2$, respectively. Then it holds:*

(i) *The normalized Casorati curvature $\delta_c(n)$ satisfies*

$$\rho \leq \delta_c(n) + f_1 + \frac{3f_2}{n+1}\cos^2\theta - \frac{2f_3}{n+1}\|\xi_1\|^2. \tag{27}$$

(ii) *The normalized Casorati curvature $\widehat{\delta}_c(n)$ satisfies*

$$\rho \leq \widehat{\delta}_c(n) + f_1 + \frac{3f_2}{n+1}\cos^2\theta - \frac{2f_3}{n+1}\|\xi_1\|^2. \tag{28}$$

Moreover, the equalities hold in the relations (27) *and* (28) *if and only if $\mathcal{M}$ is an invariantly quasi-umbilical submanifold with the flat normal connection in $\overline{\mathcal{M}}(f_1, f_2, f_3)$ such that with orthonormal tangent frame $\{E_1, \ldots, E_{n+1}\}$ of $T\mathcal{M}$ and orthonormal normal frame $\{E_{n+2}, \ldots, E_{2m+1}\}$ of $T^{\perp}\mathcal{M}$, the shape operator $S_r, r \in \{n+2, \ldots, 2m+1\}$, are given by* (13) *and* (14)*, respectively.*

Theorem 4.4. *Let M be an $(n+1)$-dimensional CR submanifold $\mathcal{M}$ of a generalized Sasakian space form $\overline{\mathcal{M}}(f_1, f_2, f_3)$ of dimension $(2m+1)$ and dimensions of D_{θ_1} and D_{θ_2} are $2n_1$ and $2n_2$, respectively. Then the following statements hold:*

(i) *The normalized Casorati curvature $\delta_c(n)$ satisfies*

$$\rho \leq \delta_c(n) + f_1 + \frac{6f_2}{n(n+1)}n_1 - \frac{2f_3}{n+1}\|\xi_1\|^2. \tag{29}$$

(ii) *The normalized Casorati curvature $\widehat{\delta}_c(n)$ satisfies*

$$\rho \leq \widehat{\delta}_c(n) + f_1 + \frac{6f_2}{n(n+1)}n_1 - \frac{2f_3}{n+1}\|\xi_1\|^2. \tag{30}$$

Moreover, the equalities hold in the relations (29) *and* (30) *if and only if $\mathcal{M}$ is an invariantly quasi-umbilical submanifold with the flat normal connection in $\overline{\mathcal{M}}(f_1, f_2, f_3)$ such that with orthonormal tangent frame $\{E_1, \ldots, E_{n+1}\}$ of $T\mathcal{M}$ and orthonormal normal frame $\{E_{n+2}, \ldots, E_{2m+1}\}$ of $T^{\perp}\mathcal{M}$, the shape operator $S_r, r \in \{n+2, \ldots, 2m+1\}$, are given by* (13) *and* (14)*, respectively.*

Theorem 4.5. *Let M be an $(n+1)$-dimensional anti-invariant submanifold $\mathcal{M}$ of a generalized Sasakian space form $\overline{\mathcal{M}}(f_1, f_2, f_3)$ of dimension $(2m+1)$ and dimensions of D_{θ_1} and D_{θ_2} are $2n_1$ and $2n_2$, respectively. Then it holds:*

(i) *The normalized Casorati curvature $\delta_c(n)$ satisfies*

$$\rho \leq \delta_c(n) + f_1 - \frac{2f_3}{n+1}\|\xi_1\|^2. \tag{31}$$

(ii) *The normalized Casorati curvature $\widehat{\delta}_c(n)$ satisfies*

$$\rho \leq \widehat{\delta}_c(n) + f_1 - \frac{2f_3}{n+1}\|\xi_1\|^2. \tag{32}$$

Moreover, the equalities hold in the relations (31) *and* (32) *if and only if* $\mathcal{M}$ *is an invariantly quasi-umbilical submanifold with the flat normal connection in* $\overline{\mathcal{M}}(f_1, f_2, f_3)$ *such that with orthonormal tangent frame* $\{E_1, \dots, E_{n+1}\}$ *of* $T\mathcal{M}$ *and orthonormal normal frame* $\{E_{n+2}, \dots, E_{2m+1}\}$ *of* $T^\perp\mathcal{M}$*, the shape operator* $S_r, r \in \{n+2, \dots, 2m+1\}$*, are given by* (13) *and* (14)*, respectively.*

Theorem 4.6. *Let* M *be an* $(n+1)$*-dimensional invariant submanifold* $\mathcal{M}$ *of a generalized Sasakian space form* $\overline{\mathcal{M}}(f_1, f_2, f_3)$ *of dimension* $(2m+1)$ *and dimensions of* D_{θ_1} *and* D_{θ_2} *are* $2n_1$ *and* $2n_2$*, respectively. Then*
(i) *The normalized Casorati curvature* $\delta_c(n)$ *satisfies*

$$\rho \leq \delta_c(n) + f_1 + \frac{6f_2}{n(n+1)} - \frac{2f_3}{n+1}\|\xi_1\|^2 \tag{33}$$

(ii) *The normalized Casorati curvature* $\widehat{\delta}_c(n)$ *satisfies*

$$\rho \leq \widehat{\delta}_c(n) + f_1 + \frac{6f_2}{n(n+1)} - \frac{2f_3}{n+1}\|\xi_1\|^2 \tag{34}$$

Moreover, the equalities hold in the relations (33) *and* (34) *if and only if* $\mathcal{M}$ *is an invariantly quasi-umbilical submanifold with the flat normal connection in* $\overline{\mathcal{M}}(f_1, f_2, f_3)$ *such that with some orthonormal tangent frame* $\{E_1, \dots, E_{n+1}\}$ *of* $T\mathcal{M}$ *and orthonormal normal frame* $\{E_{n+2}, \dots, E_{2m+1}\}$ *of* $T^\perp\mathcal{M}$*, the shape operator* $S_r, r \in \{n+2, \dots, 2m+1\}$*, are given by* (13) *and* (14)*, respectively.*

4.1. A glimpse of inequalities in different ambient spaces

Ambient Space: Sasakian Space forms	
Submanifolds	Optimal Inequalities
hemi-slant	(i) $\rho \leq \delta_c(n) + \frac{c+3}{4} + \frac{3(c-1)}{2n(n+1)}(n_1 \cos^2\theta_1) - \frac{(c-1)}{2(n+1)}\|\xi_1\|^2$ (ii) $\rho \leq \widehat{\delta}_c(n) + \frac{c+3}{4} + \frac{3(c-1)}{2n(n+1)}(n_1 \cos^2\theta_1) - \frac{(c-1)}{2(n+1)}\|\xi_1\|^2$
semi-slant	(i) $\rho \leq \delta_c(n) + \frac{c+3}{4} + \frac{3(c-1)}{2n(n+1)}(n_1 + n_2\cos^2\theta_2) - \frac{(c-1)}{2(n+1)}\|\xi_1\|^2$ (ii) $\rho \leq \widehat{\delta}_c(n) + \frac{c+3}{4} + \frac{3(c-1)}{2n(n+1)}(n_1 + n_2\cos^2\theta_2) - \frac{(c-1)}{2(n+1)}\|\xi_1\|^2$
slant	(i) $\rho \leq \delta_c(n) + \frac{c+3}{4} + \frac{3(c-1)}{4(n+1)}\cos^2\theta - \frac{(c-1)}{2(n+1)}\|\xi_1\|^2$ (ii) $\rho \leq \widehat{\delta}_c(n) + \frac{c+3}{4} + \frac{3(c-1)}{4(n+1)}\cos^2\theta - \frac{(c-1)}{2(n+1)}\|\xi_1\|^2$
CR	(i) $\rho \leq \delta_c(n) + \frac{c+3}{4} + \frac{3(c-1)}{2n(n+1)}n_1 - \frac{(c-1)}{2(n+1)}\|\xi_1\|^2$ (ii) $\rho \leq \widehat{\delta}_c(n) + \frac{c+3}{4} + \frac{3(c-1)}{2n(n+1)}n_1 - \frac{(c-1)}{2(n+1)}\|\xi_1\|^2$
anti-invariant	(i) $\rho \leq \delta_c(n) + \frac{c+3}{4} - \frac{(c-1)}{2(n+1)}\|\xi_1\|^2$ (ii) $\rho \leq \widehat{\delta}_c(n) + \frac{c+3}{4} - \frac{(c-1)}{2(n+1)}\|\xi_1\|^2$
invariant	(i) $\rho \leq \delta_c(n) + \frac{c+3}{4} + \frac{3(c-1)}{2n(n+1)} - \frac{(c-1)}{2(n+1)}\|\xi_1\|^2$ (ii) $\rho \leq \widehat{\delta}_c(n) + \frac{c+3}{4} + \frac{3(c-1)}{2n(n+1)} - \frac{(c-1)}{2(n+1)}\|\xi_1\|^2$

Ambient Space: Kenmotsu Space forms	
Submanifolds	Optimal Inequalities
hemi-slant	(i) $\rho \leq \delta_c(n) + \frac{c-3}{4} + \frac{3(c+1)}{2n(n+1)}(n_1 \cos^2\theta_1) - \frac{(c+1)}{2(n+1)}\|\xi_1\|^2$
	(ii) $\rho \leq \widehat{\delta}_c(n) + \frac{c-3}{4} + \frac{3(c+1)}{2n(n+1)}(n_1 \cos^2\theta_1) - \frac{(c+1)}{2(n+1)}\|\xi_1\|^2$
semi-slant	(i) $\rho \leq \delta_c(n) + \frac{c-3}{4} + \frac{3(c+1)}{2n(n+1)}(n_1 + n_2\cos^2\theta_2) - \frac{(c+1)}{2(n+1)}\|\xi_1\|^2$
	(ii) $\rho \leq \widehat{\delta}_c(n) + \frac{c-3}{4} + \frac{3(c+1)}{2n(n+1)}(n_1 + n_2\cos^2\theta_2) - \frac{(c+1)}{2(n+1)}\|\xi_1\|^2$
slant	(i) $\rho \leq \delta_c(n) + \frac{c-3}{4} + \frac{3(c+1)}{4(n+1)}\cos^2\theta - \frac{(c+1)}{2(n+1)}\|\xi_1\|^2$
	(ii) $\rho \leq \widehat{\delta}_c(n) + \frac{c-3}{4} + \frac{3(c+1)}{4(n+1)}\cos^2\theta - \frac{(c+1)}{2(n+1)}\|\xi_1\|^2$
CR	(i) $\rho \leq \delta_c(n) + \frac{c-3}{4} + \frac{3(c+1)}{2n(n+1)}n_1 - \frac{(c+1)}{2(n+1)}\|\xi_1\|^2$
	(ii) $\rho \leq \widehat{\delta}_c(n) + \frac{c-3}{4} + \frac{3(c+1)}{2n(n+1)}n_1 - \frac{(c+1)}{2(n+1)}\|\xi_1\|^2$
anti-invariant	(i) $\rho \leq \delta_c(n) + \frac{c-3}{4} - \frac{(c+1)}{2(n+1)}\|\xi_1\|^2$
	(ii) $\rho \leq \widehat{\delta}_c(n) + \frac{c-3}{4} - \frac{(c+1)}{2(n+1)}\|\xi_1\|^2$
invariant	(i) $\rho \leq \delta_c(n) + \frac{c-3}{4} + \frac{3(c+1)}{2n(n+1)} - \frac{(c+1)}{2(n+1)}\|\xi_1\|^2$
	(ii) $\rho \leq \widehat{\delta}_c(n) + \frac{c-3}{4} + \frac{3(c+1)}{2n(n+1)} - \frac{(c+1)}{2(n+1)}\|\xi_1\|^2$

Ambient Space: Cosymplectic Space forms	
Submanifolds	Optimal Inequalities
hemi-slant	(i) $\rho \leq \delta_c(n) + \frac{c}{4} + \frac{3c}{2n(n+1)}(n_1 \cos^2\theta_1) - \frac{c}{2(n+1)}\|\xi_1\|^2$
	(ii) $\rho \leq \widehat{\delta}_c(n) + \frac{c}{4} + \frac{3c}{2n(n+1)}(n_1 \cos^2\theta_1) - \frac{c}{2(n+1)}\|\xi_1\|^2$
semi-slant	(i) $\rho \leq \delta_c(n) + \frac{c}{4} + \frac{3c}{2n(n+1)}(n_1 + n_2\cos^2\theta_2) - \frac{c}{2(n+1)}\|\xi_1\|^2$
	(ii) $\rho \leq \widehat{\delta}_c(n) + \frac{c}{4} + \frac{3c}{2n(n+1)}(n_1 + n_2\cos^2\theta_2) - \frac{(c}{2(n+1)}\|\xi_1\|^2$
slant	(i) $\rho \leq \delta_c(n) + \frac{c}{4} + \frac{3c}{4(n+1)}\cos^2\theta - \frac{c}{2(n+1)}\|\xi_1\|^2$
	(ii) $\rho \leq \widehat{\delta}_c(n) + \frac{c}{4} + \frac{3c}{4(n+1)}\cos^2\theta - \frac{c}{2(n+1)}\|\xi_1\|^2$
CR	(i) $\rho \leq \delta_c(n) + \frac{c}{4} + \frac{3c}{2n(n+1)}n_1 - \frac{c}{2(n+1)}\|\xi_1\|^2$
	(ii) $\rho \leq \widehat{\delta}_c(n) + \frac{c}{4} + \frac{3c}{2n(n+1)}n_1 - \frac{c}{2(n+1)}\|\xi_1\|^2$
anti-invariant	(i) $\rho \leq \delta_c(n) + \frac{c}{4} - \frac{c}{2(n+1)}\|\xi_1\|^2$
	(ii) $\rho \leq \widehat{\delta}_c(n) + \frac{c}{4} - \frac{c}{2(n+1)}\|\xi_1\|^2$
invariant	(i) $\rho \leq \delta_c(n) + \frac{c}{4} + \frac{3c}{2n(n+1)} - \frac{c}{2(n+1)}\|\xi_1\|^2$
	(ii) $\rho \leq \widehat{\delta}_c(n) + \frac{c}{4} + \frac{3c}{2n(n+1)} - \frac{c}{2(n+1)}\|\xi_1\|^2$

Acknowledgment. The authors are grateful to the referee for his/her valuable comments and suggestions.

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