

THE CLASS OF ORDER-ALMOST LIMITED OPERATORS ON BANACH LATTICES

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ABSTRACT. We introduce and study the class of order-almost limited operators and derive the following interesting consequence the domination property of this class of operators, a characterization of the property (d). Then, we characterize Banach lattices E and F on which each operator from E into F that is order-almost limited and weak almost limited is an almost limited operator.

1. INTRODUCTION

Recently, J. X. Chen et al. introduced and studied the class of almost limited sets in Banach lattices [3]. Based on this concept of sets, A. Elbour et al. gave a new class of operators on Banach lattices which are called almost limited operators [7]. Following, the authors introduced the class of weak almost limited operators, that is, operators which send relatively weakly compact sets from Banach spaces onto almost limited sets in Banach lattices [5].

The aim of this paper is to introduce a new class of operators that we call order-almost limited operators and give some interesting applications of this class of operators. This class is bigger than the class of order limited (resp., limited) operators introduced in [6] (resp., in [2]).

The article is organized as follows after the introduction section, we give all common notations and definitions of Banach lattice theory in preliminaries section. Then in the first section, we give a characterization of a Banach lattice which have the property (d). In the second section, we show that if E and F are two Banach lattices then each order-almost limited and weak almost limited operator $T: E \rightarrow F$ is an almost limited operator if and only if the norm of E' is order continuous or F has the dual Schur property (see Theorem 4.2).

2. PRELIMINARIES

Throughout this paper, E, F denote Banach lattices. The positive cone of E is denoted by E^+ . B_X is the closed unit ball of the Banach space X .

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Let us recall from [3] that a norm bounded subset A of E is said to be almost limited if every disjoint weak* null sequence (f_n) in E' converges uniformly to zero on A . It is clear that all relatively compact sets and all limited sets in a Banach lattice are almost limited. The converse does not hold in general. For example, the closed unit ball of the Banach lattice B_{ℓ^∞} is almost limited, but it is not either compact or limited. A subset A of a vector lattice E is called order bounded if it includes in an order interval in E . A linear mapping T from a vector lattice E into another F is order bounded if it carries order bounded set of E into order bounded set of F .

An operator $T: X \rightarrow E$ is called weak almost limited if T carries each relatively weakly compact set in X to an almost limited set in E , equivalently, for every weakly null sequence $(x_n) \subset X$, and every disjoint weak* null sequence $(f_n) \subset E'$ we have $f_n(Tx_n) \rightarrow 0$ [5, Theorem 2.2]. An operator $T: X \rightarrow E$ is called almost limited whenever $T(B_X)$ is an almost limited set in E , equivalently, whenever $\|T'(f_n)\| \rightarrow 0$ for every disjoint weak* null sequence $(f_n) \subset E'$ [7, Definition 4.1]. The lattice operations in E' are called weak* sequentially continuous if the sequence $(|f_n|)$ converges to 0 in the weak* topology whenever the sequence (f_n) converges weak* to 0 in E' .

To show our results, we need to recall some definitions that will be used in this paper. E has the following properties

- the property (d) if $f_n \perp f_m$ in E' and $f_n \rightarrow 0$ in the $\sigma(E', E)$ -topology of E' implies $|f_n| \rightarrow 0$ in the $\sigma(E', E)$ -topology of E' . It is clear that if E is Dedekind σ -complete, then E has property (d). But the converse is false. In fact, the Banach lattice ℓ^∞/c_0 has the property (d), but is not Dedekind σ -complete [9, Remark 1.5].

- the Dunford-Pettis* property (DP* property) if every relatively weakly compact subset of E is limited. It turns out that E has the DP* property if and only if $\lim f_n(x_n) = 0$ for every weakly null sequence (x_n) in E and every weak* convergent sequence (f_n) in E' .

- the weak Dunford-Pettis* property (wDP* property) if every relatively weakly compact set in E is almost limited [3, Definition 3.1].

- the dual Schur property if $\|f_n\| \rightarrow 0$ for every weak* null sequence $(f_n) \subset E'$ consisting of pairwise disjoint terms [7, Definition 3.2].

- the Schur property, if $\|x_n\| \rightarrow 0$ for every weak null sequence $(x_n) \subset E$.

For terminologies concerning Banach lattice theory and positive operators, we refer the reader to the excellent book of Aliprantis-Burkinshaw [1].

3. ORDER-ALMOST LIMITED OPERATORS

Definition 3.1. An operator T from E into F is said to be order-almost limited if it carries each order bounded subset of E into an almost limited set of F , i.e., for each $x \in E^+$, the subset $T([-x, x])$ is almost limited in F .

Note that there exist operators which are order-almost limited, but fail to be order limited. Indeed, $Id_{\ell^\infty}: \ell^\infty \rightarrow \ell^\infty$ is order-almost limited (because ℓ^∞ has the property (d)), but it is not order limited (because the lattice operations of $(\ell^\infty)'$ are not weak* sequentially continuous).

By a simple proof we can investigate that each almost limited operator $T: E \rightarrow F$ is order-almost limited and weak almost limited. On the other hand, E has the property (d) if and only if for each $x \in E^+$, $[-x, x]$ is almost limited. Also, an operator $T: E \rightarrow F$ is order-almost limited if and only if for every disjoint weak* null sequence (f_n) of F' , we have $|T'(f_n)| \rightarrow 0$ for $\sigma(E', E)$.

Proposition 3.2. *Let E, F and G be three Banach lattices. Then*

- (1) *the class of order-almost limited operators is a norm closed vector subspace of the space $L(E, F)$ of all operators from E into F .*
- (2) *if $T: E \rightarrow F$ is an order bounded operator, then for each order-almost limited operator $S: F \rightarrow G$, the composed operator $S \circ T$ is order-almost limited.*

Proof. (1) Clearly the class of order-almost limited operators is a vector subspace of $L(E, F)$. To see that the class of order-almost limited operators is also norm closed, let S be in norm closure of the class of order-almost limited operators. To this end, let x be a nonzero in E^+ and $\varepsilon > 0$. Choose T which is order-almost limited and satisfying $\|S - T\| \leq \frac{\varepsilon}{\|x\|}$. Observe that $S([-x, x]) \subset T([-x, x]) + \varepsilon B_F$ holds. Since T is order-almost limited, $T([-x, x])$ is almost limited set and hence by [7, Lemma 2.2], $S([-x, x])$ is an almost limited set. This shows that S is order-almost limited.

(2) Let $T: E \rightarrow F$ be an order bounded operator. Then for each $x \in E^+$, $T([-x, x])$ is an order interval, and since S is order-almost limited, then $S(T([-x, x]))$ is an almost limited set in F , and hence $S \circ T$ is order-almost limited. \square

An order interval of E is not necessary almost limited set. In fact, the order interval $[-\mathbf{1}, \mathbf{1}]$ of the Banach lattice c is not almost limited, where $\mathbf{1} := (1, 1, \dots) \in c$ [3, Example 2.1].

The next Proposition characterizes Banach lattices on which every order interval is an almost limited set

Proposition 3.3. *The following statements are equivalent:*

- (1) *each positive operator from E into E is order-almost limited;*
- (2) *the identity operator of E is order-almost limited;*
- (3) *E has the property (d).*

Proof. (1) \implies (2) Obvious.

(2) \implies (3) Let $x \in E^+$. Since the identity operator of E is order-almost limited then $[-x, x]$ is an almost limited set in E , and hence E has the property (d).

(3) \implies (1) Let $T: E \rightarrow E$ be a positive operator and $x \in E^+$. Then $T[-x, x] \subset [-T(x), T(x)]$. Since E has the property (d), then $[-T(x), T(x)]$ is an almost limited set, and hence $T[-x, x]$ is an almost limited set. Finally we conclude that T is an order-almost limited operator. \square

The next result gives some equivalent conditions for T to be an order-almost limited operator, where $T: E \rightarrow F$ is an order bounded operator.

Theorem 3.4. *Let $T: E \rightarrow F$ be an order bounded operator such that the lattice operations of E' are weak* sequentially continuous or F has the property (d). Then, the following assertions are equivalent*

- (1) T is an order-almost limited operator;
- (2) $f_n(T(x_n)) \rightarrow 0$ for every order bounded disjoint sequence $(x_n) \subset E^+$ and every disjoint weak* null sequence $(f_n) \subset F'$;
If F has the property (d), we may add:
- (3) $f_n(T(x_n)) \rightarrow 0$ for every order bounded disjoint sequence $(x_n) \subset E^+$ and every disjoint weak* null sequence $(f_n) \subset (F')^+$.

Proof. It is a simple consequence of [7, Theorem 2.7]. \square

The domination property for order-almost limited operators can also be derived from Theorem 3.4.

Corollary 3.5. *Let E and F be two Banach lattices such that F has the property (d). If S and T are two positive operators from E into F such that $0 \leq S \leq T$ and T is order-almost limited, then S itself is order-almost limited.*

Proof. Let $S, T: E \rightarrow F$ be two operators such that $0 \leq S \leq T$ and T is order-almost limited. Let $(f_n) \subset (F')^+$ be a disjoint weak* null sequence and $(x_n) \subset E^+$ be an order bounded disjoint sequence. As T is order-almost limited, $f_n(T(x_n)) \rightarrow 0$. Using the inequalities $0 \leq f_n(S(x_n)) \leq f_n(T(x_n))$ for all n , we get $f_n(S(x_n)) \rightarrow 0$. Finally, it follows from Theorem 3.4 that S is well order-almost limited. \square

Proposition 3.6. *If the norm of E is order continuous and F has the wDP* property then, each operator T from E into F is order-almost limited.*

Proof. Let $T: E \rightarrow F$ be an operator. Since the norm of E is order continuous then from [8, Theorem 2.4.3], it follows that for each $x \in E^+$, the order interval $[-x; x]$ is weakly compact, and hence $T([-x; x])$ is weakly compact in F . Since F has the wDP* property, then $T([-x; x])$ is an almost limited set in F , and hence T is order-almost limited. \square

4. MAIN RESULTS

To establish our main result, we will need the following Lemma.

Lemma 4.1. *Let E be a Banach lattice. If E does not have the dual Schur property, then there exists a disjoint weak* null sequence (f_n) of E' , $(y_n) \subset B_E^+$, and some $\varepsilon > 0$ such that $|f_n(y_n)| \geq \varepsilon$ for all n .*

Proof. If E does not have the dual Schur property, then there exists a disjoint weak* null sequence (f_n) such that $\|f_n\| > 2\varepsilon$ for $\varepsilon > 0$ and for all n . As $\|f_n\| = \sup\{|f_n(y)| : y \in B_F\}$ for all n , then there exists $y_n \in B_F$ such that $\|y_n\| \leq 1$ and $|f_n(y_n)| \geq 2\varepsilon$. Note that $|f_n(y_n^+)| \geq \varepsilon$ or $|f_n(y_n^-)| \geq \varepsilon$. Replacing y_n with y_n^+ or by y_n^- , we can assume that for all n , there exists $y_n \in B_E^+$ such that $|f_n(y_n)| \geq \varepsilon$. \square

Note that there exist operators which are order-almost limited and weak almost limited, but fail to be almost limited. Indeed, $Id_{\ell^1} : \ell^1 \rightarrow \ell^1$ is order-almost limited (resp., weak almost limited) because ℓ^1 has the property (d) (resp., because ℓ^1 has

the wDP* property), but it is not almost limited (because ℓ^1 does not have the dual Schur property).

In the following major result, we characterize Banach lattices E and F on which each operator from E into F which is order-almost limited and weak almost limited, is almost limited.

Theorem 4.2. *The following assertions are equivalent:*

- (1) *each order-almost limited and weak almost limited operator $T: E \rightarrow F$ is almost limited;*
- (2) *one of the following is valid:*
 - (a) *the norm of E' is order continuous;*
 - (b) *F has the dual Schur property.*

Proof. (1) \implies (2) Assume that (2) is false, i.e, the norm of E' is not order continuous and F does not have the dual Schur property. We will construct an operator $T: E \rightarrow F$ which is order-almost limited and weak almost limited but is not almost limited. Indeed, suppose that the norm of E' is not order continuous. By [8, Theorem 2.4.14] we may assume that ℓ^1 is a closed sub-lattice of E and from [8, Proposition 2.3.11], it follows that there is a positive projection P from E into ℓ^1 .

On the other hand, F does not have the dual Schur property, from Lemma 4.1, it follows that there exist disjoint weak* null sequence (f_n) of F' , $(y_n) \subset B_F^+$ and $\varepsilon > 0$ such that $|f_n(y_n)| \geq \varepsilon$ for all n .

Now, we consider the operator $T = S \circ P: E \rightarrow \ell^1 \rightarrow F$ where S is the operator defined by

$$S: \ell^1 \rightarrow F, \quad (\lambda_n) \rightarrow \sum_{n=1}^{\infty} \lambda_n y_n.$$

Since ℓ^1 has the Schur property, then T is Dunford-Pettis and hence T is weak almost limited. The operator T is also order-almost limited. In fact, since ℓ^1 is discrete and its norm is order continuous, we have $P([-x, x])$ is relatively compact in ℓ^1 , then $S \circ P([-x, x])$ is a relatively compact set in F , and hence $T([-x, x])$ is an almost limited set in F for each $x \in E^+$. Finally, we conclude that T is order-almost limited.

But the operator T is not almost limited. Indeed, (f_n) is a disjoint weak* null sequence in F' . As the operator $P: E \rightarrow \ell^1$ is surjective, there exists $\delta > 0$ such that $\delta \cdot B_{\ell^1} \subset P(B_E)$. Hence

$$\begin{aligned} \|T'(f_n)\| &= \sup_{x \in B_E} |T'(f_n)(x)| = \sup_{x \in B_E} |f_n(T(x))| = \sup_{x \in B_E} |f_n \circ S(P(x))| \\ &\geq \delta \cdot |f_n \circ S(e_n)| \geq \delta \cdot |f_n(y_n)| > \delta \cdot \varepsilon \end{aligned}$$

(where $(e_n)_{n=1}^{\infty}$ is the canonical basis of ℓ^1). Then $\|T'(f_n)\| > \delta \cdot \varepsilon$ for all n , and we conclude that T is not almost limited.

(2, a) \implies (1) Let (f_n) be a disjoint weak* null sequence of F' . We have to prove that $\|T'(f_n)\| \rightarrow 0$. By using Corollary 2.7 of Dodds-Fremlin [4], it suffices to prove that $|T'(f_n)| \rightarrow 0$ in $\sigma(E', E)$ and $T'(f_n)(x_n) \rightarrow 0$ for every norm bounded disjoint sequence $(x_n) \subset E^+$. Indeed, as (f_n) is a disjoint weak* null sequence of

F' and T is order-almost limited, we have $|T'(f_n)| \rightarrow 0$ for $\sigma(E', E)$.

On the other hand, since the norm of E' is order continuous, from Corollary 2.9 of Dodds-Fremlin [4], it follows that $x_n \rightarrow 0$ in the weak topology $\sigma(E, E')$. Hence, as T is a weak almost limited operator, we obtain $T'(f_n)(x_n) = f_n(T(x_n)) \rightarrow 0$. We conclude that T is an almost limited operator.

(2, b) \implies (1) Obvious. \square

As an immediate consequence of Theorem 4.2, we have the following characterizations.

Corollary 4.3. *The following assertions are equivalent:*

- (1) *each order-almost limited and weak almost limited operator $T: E \rightarrow E$ is almost limited;*
- (2) *the norm of E' is order continuous.*

Corollary 4.4. *Let E and F be two Banach lattices such that E or F has the DP^* property. Then the following assertions are equivalent:*

- (1) *each order-almost limited operator $T: E \rightarrow F$ is almost limited;*
- (2) *one of the following is valid:*
 - (a) *the norm of E' is order continuous;*
 - (b) *F has the dual Schur property.*

REFERENCES

1. Aliprantis C. D. and Burkinshaw O., *Positive operators*, Reprint of the 1985 original, Springer, Dordrecht, 2006.
 2. Bourgain J. and Diestel J., *Limited operators and strict cosingularity*, Math. Nachr. **119** (1984), 55–58.
 3. Chen J. X., Chen Z. L. and Ji G. X., *Almost limited sets in Banach lattices*, J. Math. Anal. Appl. **412** (2014), 547–553.
 4. Dodds, P. G. and Fremlin, D. H., *Compact operators on Banach lattices*, Israel J. Math. **34** (1979), 287–320.
 5. Elbour, A., Machrafi N. and Moussa M., *On the class of weak almost limited operators*, <http://arxiv.org/abs/1403.0136>.
 6. El Kaddouri A., and Moussa M., *About the class of order limited operator*, Acta Universitatis Carolinae **54(1)** (2013), 37–42.
 7. Machrafi N., Elbour A. and Moussa M., *Some characterizations of almost limited sets and applications*, <http://arxiv.org/abs/1312.2770>.
 8. Meyer-Nieberg, P., *Banach lattices*, Universitext. Springer-Verlag, Berlin, 1991.
 9. Wnuk W., *On the dual positive Schur property in Banach lattices*, Positivity **17** (2013), 759–773.
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