Acta Math. Univ. Comenianae
Vol. LXXXVII, 1 (2018), pp. 141–146

n-JORDAN HOMOMORPHISMS ON COMMUTATIVE ALGEBRAS

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Abstract. It is proved that every n-Jordan homomorphism between two commutative algebras is an n-ring homomorphism when n is an arbitrary and fixed positive integer number. We employ this result to show that every involutive n-Jordan homomorphism between two commutative C^*-algebras is automatically norm continuous.

1. Introduction and Preliminaries

Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras. An *n-ring homomorphism* from $\mathcal{A}$ to $\mathcal{B}$ is a map $h\colon \mathcal{A} \to \mathcal{B}$ that is additive, (i.e., $h(a+b) = h(a)+h(b)$ for all $a, b \in \mathcal{A}$) and n-multiplicative (i.e., $h(a_1a_2 \cdots a_n) = h(a_1)h(a_1)\cdots h(a_n)$ for all $a_1, a_2, \cdots , a_n \in \mathcal{A}$). If $h\colon \mathcal{A} \to \mathcal{B}$ is a linear n-ring homomorphism, we say that h is an *n-homomorphism*. A map $h\colon \mathcal{A} \to \mathcal{B}$ is called an *n-Jordan homomorphism* if it is additive and $h(a^n) = (h(a))^n$ for all $a \in \mathcal{A}$.

The concept of n-homomorphisms was introduced by Hejazian et al. in [**8**]. Furthermore, the notion of n-Jordan homomorphisms was firstly dealt with by Herstein in [**9**]. Obviously, every n-homomorphism is an n-Jordan homomorphism. Further, there are some examples of n-Jordan homomorphisms which are not n-ring homomorphisms. Also, each homomorphism is an n-homomorphism for every $n \geq 2$, but the converse does not hold in general. For instance, if $h\colon \mathcal{A} \to \mathcal{B}$ is a homomorphism, then $g := -h$ is a 3-homomorphism which is not a homomorphism [**4**]. For certain properties of 3-homomorphisms, one may refer to [**4**]. However, it is easily verified that if $\mathcal{A}$ is unital and h is a 3-homomorphism, then $g(a) := h(1)h(a)$ is a homomorphism. Further, it was proved in [**2**, Theorem 2.3] that for an arbitrary natural number n, if $\phi\colon \mathcal{A} \to \mathbb{C}$ is an n-ring homomorphism, then $\psi(a) := \phi(u^n a)\, (a \in \mathcal{A})$ is a homomorphism in which $\phi(u) = 1$. It was G. Ancochea [**1**] who first studied the connection of Jordan homomorphisms and homomorphisms. The results of Ancochea were generalized and extended in several ways in [**11**], [**12**] and [**15**]. Later, in 1956, Herstein [**9**] proved the following result.

Received May 22, 2017.
2010 *Mathematics Subject Classification.* Primary 46H25, 47B48.
Key words and phrases. Banach algebra; commutative algebra; n-homomorphism; n-Joran homomorphism; n-ring.

Theorem 1.1. *If φ is a Jordan homomorphism of a ring R onto a prime ring R' of characteristic different from* 2 *and* 3*, then either φ is a homomorphism or an anti-homomorphism.*

Also, in [**9**], the n-Jordan mappings were considered. Concerning this topic, the next statement was verified.

Theorem 1.2 ([**9**])**.** *Let φ be an n-Jordan homomorphism from a ring R onto a prime ring R' of characteristic larger than n. Suppose further that R has a unit element. Then, $\varphi = \epsilon\tau$ where τ is either a homomorphism or an anti-homomorphism and ϵ is an $(n-1)$st root of unity lying in the center of R'.*

In [**9**], the author suggested: *One might conjecture that an appropriate variant of* Theorem 1.2 *would hold even if R does not have a unit element.* This problem was solved by Brešar, Martindale and Miers [**5**]. In other words, in [**5**] they proved the upcoming result.

Theorem 1.3. *Let $n \geq 3$ and let φ be an n-Jordan homomorphism of the ring R onto the prime ring R'. Suppose further that the characteristic of R' is zero or bigger than $2m(m+1)$ with $m = 4n-8$. Then, there exists $\epsilon \in C'$ (the extended centroid of R') such that $\epsilon^{n-1} = 1$ and a homomorphism or an anti-homomorphism $\tau\colon R \to R'C'$ such that $\varphi(x) = \epsilon\tau(x)$ for all $x \in R$.*

On the score of the above theorems, we notice the fact that the mapping in question is surjective and its range is a prime ring, is essential. However, there have been proved statements in which the surjectivity is not assumed. At the expense of this, we can suppose more about the domain and also about the range.

It was shown in [**6**] that every n-Jordan homomorphism between commutative Banach algebras is also an n-ring homomorphism when $n \in \{3, 4\}$. For the case $n \in \{5, 6, 7\}$, the same was proved in [**3**]. Note that for $n = 2$, the proof is simple and routine. In [**3**], the first author and Shojaee asked the following: Is every n-Jordan homomorphism between two commutative algebras also a n-ring homomorphism when $n \in \mathbb{N}$? Lee [**13**] and Gselmann [**7**] answered their question in the affirmative. Also, Gselmann [**7**] proved a more general result than the main result of the current paper (c.f. Theorem 2.2). In fact, he proved the following theorem.

Theorem 1.4. *Let $n \in \mathbb{N}, n \geq 2$, R, R' be commutative rings such that $\operatorname{char}(R') > n$, and assume that the mapping $\varphi\colon R \to R'$ is an n-Jordan homomorphism. Then, φ is an n-homomorphism. Moreover, if R is unitary, then $\varphi(1) = \varphi(1)^n$ and the mapping ψ defined by*

$$\psi(x) = \varphi^{n-2}(1)\varphi(x) \qquad (x \in R)$$

is a homomorphism between R and R'.

Furthermore, in [**9**], there are some results which are proved for topological rings that could also be used while proving automatic continuity.

Theorem 1.5 ([**9**])**.** *Let $n \in \mathbb{N}$, $n \geq 2$, $\mathbb{F}$ be a field of characteristic zero, R be a commutative topological ring and R' be a commutative topological algebra over the* *field* $\mathbb{F}$. *Furthermore, let us consider the additive mapping* $\varphi\colon R \to R'$ *and suppose that for the map* ϕ *defined on* R *by*

$$\phi(x) = \varphi(x^n) - \varphi(x)^n \qquad (x \in R).$$

one of the following statements hold:

(i) *the mapping* ϕ *is continuous at a point;*

(ii) *assuming that* R' *is locally convex, the mapping* ϕ *is bounded on a nonvoid open set of* B*;*

(iii) *assuming that* R *is locally compact,* R' *is locally convex, the mapping* ϕ *is bounded on a measurable set of positive measure;*

(iv) *assuming that* R *is locally compact and* R' *is locally bounded and locally convex, the mapping* ϕ *is measurable on a measurable set of positive measure.*

Then and only then the mapping φ *is a continuous mapping or it is an* n*-homomorphism.*

In this paper, we prove that for a fixed and arbitrary positive integer n, every n-Jordan homomorphism between two commutative algebras is an n-ring homomorphism. Indeed, our way which is based on the property of the Vandermonde matrix, is different from the methods which are used in [**7**] and [**13**]. Moreover, as some applications, we study the automatic continuity of linear n-Jordan homomorphisms on C^*-algebras.

2. Main results

To achieve our aim in this section, we need the following lemma.

Lemma 2.1. *If* $x_1, x_2, \cdots x_n \in \mathbb{R}$*, then*

$$\det\left(\begin{bmatrix} x_1 & x_1^2 & \cdots & x_1^n \\ x_2 & x_2^2 & \cdots & x_2^n \\ \cdots & \cdots & \cdots & \cdots \\ x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}\right) = (-1)^{\frac{n(n-1)}{2}} \prod_{k=1}^{n} x_k \prod_{i<j}(x_i - x_j).$$

Proof. The result follows from determinant of the well-known Vandermonde matrix. Indeed,

$$\det\left(\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{bmatrix}\right) = (-1)^{\frac{n(n-1)}{2}} \prod_{i<j}(x_i - x_j).$$

□

The following result is the main key to prove some results related to automatic continuity of n-Jordan homomorphism on C^*-algebras.

Theorem 2.2. *Let* n *be a fixed positive integer number. If* $h\colon \mathcal{A} \to \mathcal{B}$ *is an* n*-Jordan homomorphism between two commutative algebras, then* h *is an* n*-ring homomorphism.*

Proof. By assumption, we have $h((x+ky)^n) = (h(x+ky))^n$ for all $x, y \in \mathcal{A}$, where k is an integer with $2 \leq k \leq n$. Thus

$$h\left(\sum_{j=0}^{n} \binom{n}{j} k^j x^{n-j} y^j\right) = \sum_{j=0}^{n} \binom{n}{j} k^j h(x)^{n-j} h(y)^j \tag{2.1}$$

for all $x, y \in \mathcal{A}$, where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. The equality (2.1) implies that

$$\sum_{j=1}^{n-1} k^j F_j(x,y) = 0 \tag{2.2}$$

for all $x, y \in \mathcal{A}$, in which $F_j(x,y) = \binom{n}{j}\left[h\left(x^{n-j}y^j\right) - (h(x))^{n-j}(h(y))^j\right]$, where $1 \leq j \leq n-1$. We can rewrite the equality (2.2) as follows

$$\begin{bmatrix} 2 & 2^2 & \cdots & 2^{n-1} \\ 3 & 3^2 & \cdots & 3^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ n & n^2 & \cdots & n^{n-1} \end{bmatrix} \begin{bmatrix} F_1(x,y) \\ F_2(x,y) \\ \cdots \\ F_{n-1}(x,y) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix} \tag{2.3}$$

for all $x, y \in \mathcal{A}$. The invertibility of the above square matrix (Lemma 2.1) shows that $F_j(x,y) = 0$ for all $1 \leq j \leq n-1$ and all $x, y \in \mathcal{A}$. In particular, $F_{n-1}(x,y) = 0$ for all $x, y \in \mathcal{A}$. Hence,

$$h(xy^{n-1}) = h(x)(h(y))^{n-1} \qquad (x, y \in \mathcal{A}) \tag{2.4}$$

for all $x, y \in \mathcal{A}$. We claim that

$$h(x_1x_2\dots x_m x_{m+1}^{n-m}) = h(x_1)h(x_2)\dots h(x_m)h(x_{m+1}^{n-m}) \tag{2.5}$$

for all $1 \leq m \leq n-1$ and all $x_1, x_2, \cdots, x_m, x_{m+1} \in \mathcal{A}$. We argue by induction on m. For $m=1$, the result holds by (2.4). Suppose that (2.5) is true for $m=k$. We desire to show that (2.5) holds for $m=k+1$. For each $2 \leq p \leq n-k+2$, we have

$$\begin{aligned} &h\left(x_1x_2\dots x_k(x_{k+1}+px_{k+2})^{n-k}\right) \\ &= h(x_1)h(x_2)\dots h(x_k)\left(h(x_{k+1}+px_{k+2})\right)^{n-k} \end{aligned} \tag{2.6}$$

for all $x_1, x_2, \dots, x_k, x_{k+1}, x_{k+2} \in \mathcal{A}$. The equality (2.6) necessitates that

$$\sum_{j=0}^{n-k} p^j G_j(x_1, x_2, \dots, x_k, x_{k+1}, x_{k+2}) = 0, \tag{2.7}$$

where

$$\begin{aligned} G_j(x_1, x_2, \dots, x_k, x_{k+1}, x_{k+2}) = \binom{n-k}{j}\Big[&h\left(x_1x_2\dots x_k x_{k+1}^{n-k-j}x_{k+2}^j\right) \\ &-h(x_1)h(x_2)\dots h(x_k)(h(x_{k+1}))^{n-k-j}(h(x_{k+2}))^j\Big]. \end{aligned}$$

The relation (2.7) can be represented in the following form

$$\begin{bmatrix} 1 & 2 & 2^2 & \cdots & 2^{n-k} \\ 1 & 3 & 3^2 & \cdots & 3^{n-k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & n-k+2 & (n-k+2)^2 & \cdots & (n-k+2)^{n-k} \end{bmatrix}$$

$$\times \begin{bmatrix} G_0(x_1, x_2, \cdots, x_k, x_{k+1}, x_{k+2}) \\ G_1(x_1, x_2, \cdots, x_k, x_{k+1}, x_{k+2}) \\ \cdots \\ G_{n-k}(x_1, x_2, \cdots, x_k, x_{k+1}, x_{k+2}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix}$$

for all $x_1, x_2, \cdots, x_k, x_{k+1}, x_{k+2} \in \mathcal{A}$. Since the Vandermonde matrix is invertible, $G_j(x_1, x_2, \cdots, x_k, x_{k+1}, x_{k+2}) = 0$, where $1 \leq j \leq n-k$. In particular, $G_{n-k-1}(x_1, x_2, \cdots, x_k, x_{k+1}, x_{k+2}) = 0$. Therefore,

$$h\left(x_1 x_2 \cdots x_k x_{k+1} x_{k+2}^{n-k-1}\right) = h(x_1)h(x_2)\cdots h(x_k)h(x_{k+1})(h(x_{k+2}))^{n-k-1}$$

for all $x_1, x_2, \cdots, x_k, x_{k+1}, x_{k+2} \in \mathcal{A}$. This finishes the proof. □

A left ideal $\mathcal{I}$ of an algebra $\mathcal{A}$ is a modular left ideal if there exists $u \in \mathcal{A}$ such that $\mathcal{A}(e_{\mathcal{A}} - u) \subseteq \mathcal{I}$, where $\mathcal{A}(e_{\mathcal{A}} - u) = \{x - xu : x \in \mathcal{A}\}$. The Jacobson radical Rad $(\mathcal{A})$ of $\mathcal{A}$ is the intersection of all maximal modular left ideals of $\mathcal{A}$. An algebra $\mathcal{A}$ is called *semisimple* whenever its Jacobson radical Rad$(\mathcal{A})$ is trivial. Also, an algebra $\mathcal{A}$ is called *factorizable* if for each $u \in \mathcal{A}$, there are $v, w \in \mathcal{A}$ such that $u = vw$.

Here, we bring some applications of Theorem 2.2.

Theorem 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be commutative Banach algebras such that $\mathcal{B}$ is semisimple and factorizable. Then, every surjective n-Jordan homomorphism $h\colon \mathcal{A} \to \mathcal{B}$ is automatically continuous.*

Proof. We can immediately obtain the result from [**10**, Theorem 2.2 and Theorem 2.7] and Theorem 2.2. □

It is well-known that evey C^*-algebra has a bounded approximate identity and thus it is factorizable. Also every C^*-algebra is semisimple. So, we have the next corollary.

Corollary 2.4. *If $\mathcal{A}$ and $\mathcal{B}$ are commutative C^*-algebras, then every surjective n-Jordan homomorphism $h : \mathcal{A} \to \mathcal{B}$ is automatically continuous.*

Let $\mathcal{A}$ be a C^*-algebra. An element a in $\mathcal{A}$ is *positive* if a is hermitian, that is, $a = a^*$, and $\sigma(a) \subseteq \mathbb{R}^+$, where $\sigma(a)$ is the spectrum of a. We write $a \geq 0$ to mean a is positive. Also a linear map $T : \mathcal{A} \to \mathcal{B}$ between two C^*-algebras is positive if $a \geq 0$ implies $T(a) \geq 0$ for all $a \in \mathcal{A}$. We say that the map T is *completely positive* if for any natural number k, the induced map $T_k\colon M_k(\mathcal{A}) \to M_k(\mathcal{B}); T_k((a_{ij})) \mapsto (T(a_{ij}))$ on $k \times k$ matrices is positive.

The following theorem was proved by Park and Trout in [**14**, Theorm 3.2].

Theorem 2.5. *Let $\phi\colon \mathcal{A} \to \mathcal{B}$ be an involutive (i.e., $*$-linear) n-homomorphism between two C^*-algebras. If $n \geq 3$ is odd, then $\|\phi\| \leq 1$, that is, ϕ is norm-contractive.*

Corollary 2.6. *Let $n \geq 3$ be an odd integer, and let $\mathcal{A}$ and $\mathcal{B}$ be commutative C^*-algebras. If $h\colon \mathcal{A} \to \mathcal{B}$ is an involutive n-Jordan homomorphism, then $\|h\| \leq 1$, i.e., h is norm contractive.*

Proof. The result follows from Theorem 2.2 and Theorem 2.5. □

For the even case, we need the following theorem which was proved in [**14**, Theorem 2.3].

Theorem 2.7. *Let $\phi\colon \mathcal{A} \to \mathcal{B}$ be an involutive n-homomorphism between two C^*-algebras. If $n \geq 2$ is even, then ϕ is completely positive. Thus, ϕ is bounded.*

Corollary 2.8. *Let n be an even positive integer. If $h\colon \mathcal{A} \to \mathcal{B}$ is an involutive n-Jordan homomorphism between two commutative C^*-algebras, then h is completely positive. Thus, h is bounded.*

Proof. Using Theorem 2.2 and Theorem 2.7, one can obtain the desired result. □

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