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DIAGONAL FUNCTIONS OF THE k-PELL AND k-PELL-LUCAS POLYNOMIALS AND SOME IDENTITIES

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Abstract. In this paper, rising and descending diagonal functions of k-Pell and the k-Pell-Lucas polynomials are introduced and generating functions is provided. Some identities are also studied.

1. Introduction

Pell and Pell-Lucas sequences are integer sequences that have been studied by several authors and constructed with the same recurrence relation but different initial conditions. Namely, for $n \geq 2$, the Pell numbers $\{P_n\}$ satisfy the recurrence relation

$$P_n = 2P_{n-1} + P_{n-2}, \quad P_0 = 0, \quad P_1 = 1,$$

whereas the Pell-Lucas numbers $\{Q_n\}$ satisfy the recurrence relation

$$Q_n = 2Q_{n-1} + Q_{n-2}, \quad Q_0 = 2, \quad Q_1 = 2.$$

For more details about these sequences, see, for example, [**2**], [**6**], [**8**], [**10**], [**12**], [**15**], and [**16**], among others.

In the literature, we find several studies involving generalizations of these sequences, such as for positive real number k, the case of the k-Pell sequences and k-Pell-Lucas sequences, (see, for example, [**3**], [**4**], [**5**], [**18**], and [**17**], among others). Such sequences have the same recurrence relation but different initial conditions. The k-Pell sequences $\{P_{k,n}\}$ are defined by

$$P_{k,n} = 2P_{k,n-1} + kP_{k,n-2}, \quad P_{k,0} = 0, \quad P_{k,1} = 1, \tag{1}$$

whereas the k-Pell-Lucas sequences $\{Q_{k,n}\}$ satisfy the recurrence relation

$$Q_{k,n} = 2Q_{k,n-1} + kQ_{k,n-2}, \quad Q_{k,0} = 2, \quad Q_{k,1} = 2. \tag{2}$$

The polynomials that can be defined by Fibonacci-like recursion relations are called Fibonacci polynomials and they were studied in 1883 by E. C. Catalan and E. Jacobsthal. For example, E. C. Catalan studied the polynomials $F_n(x)$ defined

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by the recurrence relation

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \quad n \geq 3, \qquad F_1(x) = 1, \quad F_2(x) = x. \tag{3}$$

In 1965, V. E. Hoggatt [**11**], introduced Lucas polynomials defined by the recurrence relation

$$L_{n+1}(x) = xL_n(x) + L_{n-1}(x), \quad n \geq 1, \qquad L_0(x) = 2, \quad L_1(x) = x. \tag{4}$$

Several authors have dedicated their research to polynomials of different type of sequences such as the work of S. Arolkar and Y.S. Valaulikar in [**1**], where the authors study the $h(x)$-B-Tribonacci and $h(x)$-B-Tri Lucas Polynomials whenever $h(x)$ is a polynomial with real coefficients. Also in [**14**], some properties of the (p, q)-Fibonacci and (p, q)-Lucas polynomials are provided and in [**7**], the study of Pell polynomials modulo m is stated.

Pell numbers and PellLucas numbers are specific values of Pell polynomials $P_n(x)$ and Pell-Lucas polynomials $Q_n(x)$, respectively. Both families were studied extensively in 1985 by A.F. Horadam and J. M. Mahon in [**13**], where the authors recorded some properties of these polynomials. The first few terms of these sequences $\{P_n(x)\}$ and $\{Q_n(x)\}$ are

n	$P_n(x)$	$Q_n(x)$
$n=0$	0	2
$n=1$	1	$2x$
$n=2$	$2x$	$4x^2+2$
$n=3$	$4x^2+1$	$8x^3+6x$
$n=4$	$8x^3+4x$	$16x^4+16x^2+2$
$n=5$	$16x^4+12x^2+1$	$32x^5+40x^3+10x$
$n=6$	$32x^5+32x^3+6x$	$64x^6+96x^4+36x^2+2$
$n=7$	$64x^6+80x^4+24x^2+1$	$128x^7+224x^5+112x^3+14x$

Table 1. The Pell and Pell-Lucas polynomials for $0 \leq n \leq 7$.

Also in [**15**], we have more details about these polynomials. Recall that the Pell and Pell-Lucas polynomials, respectively, are defined by

$$P_{n+2}(x) = 2xP_{n+1}(x) + P_n(x), \quad P_0(x) = 0, \quad P_1(x) = 1 \tag{5}$$

and

$$Q_{n+2}(x) = 2xQ_{n+1}(x) + Q_n(x), \quad Q_0(x) = 2, \quad Q_1(x) = 2x. \tag{6}$$

Note that $P_n(1) = P_n,\ Q_n(1) = Q_n, P_n(\frac{1}{2}) = F_n, Q_n(\frac{1}{2}) = L_n$, where P_n, Q_n, F_n, and L_n are the nth Pell, Pell-Lucas, Fibonacci and Lucas numbers, respectively.

Furthermore, $P_n(\frac{1}{2}x) = F_n(x)$, $Q_n(\frac{1}{2}x) = L_n(x)$, where $F_n(x)$ and $L_n(x)$, respectively, are the nth Fibonacci and Lucas polynomials (see, [**13**]).

The diagonal function of k-Lucas polynomials and Lucas polynomials were introduced in [**9**]. Also, in the same paper, the authors an established rising and descending diagonal function and generating matrix of k-Lucas polynomials and Lucas polynomials.

In this paper, we introduce the k-Pell and k-Pell-Lucas polynomials and consider rising and descending diagonal functions of these polynomials. Furthermore, some properties and generating functions of these polynomials are provided.

2. The k-Pell and k-Pell-Lucas polynomials

In this section, we consider a new class of polynomials and provide some of its properties. For any positive real number k, k-Pell numbers and k-Pell-Lucas numbers are specific values of the k-Pell polynomials $P_{k,n}(x)$ and k-Pell-Lucas polynomials $Q_{k,n}(x)$, respectively. Both families are defined recursively by

$$P_{k,n+2}(x) = 2xP_{k,n+1}(x) + kP_{k,n}(x), \quad P_{k,0}(x) = 0, \quad P_{k,1}(x) = 1 \tag{7}$$

and

$$Q_{k,n+2}(x) = 2xQ_{k,n+1}(x) + kQ_{k,n}(x), \quad Q_{k,0}(x) = 2, \quad Q_{k,1}(x) = 2x. \tag{8}$$

The first few terms of these sequences $\{P_{k,n}(x)\}$ and $\{Q_{k,n}(x)\}$ are as follows

n	$P_{k,n}(x)$	$Q_{k,n}(x)$
$n=0$	0	2
$n=1$	1	$2x$
$n=2$	$2x$	$4x^2+2k$
$n=3$	$4x^2+k$	$8x^3+6xk$
$n=4$	$8x^3+4xk$	$16x^4+16x^2k+2k^2$
$n=5$	$16x^4+12x^2k+k^2$	$32x^5+40x^3k+10xk^2$
$n=6$	$32x^5+32x^3k+6xk^2$	$64x^6+96x^4k+36x^2k^2+2k^3$
$n=7$	$64x^6+80x^4k+24x^2k^2+k^3$	$128x^7+224x^5k+112x^3k^2+14xk^3$

Table 2. The k-Pell and k-Pell-Lucas polynomials for $0 \leq n \leq 7$.

For the particular case of $k = 1$, we get the Pell and Pell-Lucas polynomials referred before.

The next result states the Binet-style formula for each one of these polynomial sequences.

Theorem 2.1 (The Binet-style formula)**.** *For $n \geq 0$ and any positive real number k, we have*
1. $P_{k,n}(x) = \frac{(r_1)^n - (r_2)^n}{r_1 - r_2}$,
2. $Q_{k,n}(x) = (r_1)^n + (r_2)^n$,

where $r_1 = x + \sqrt{x^2 + k}$, $r_2 = x - \sqrt{x^2 + k}$ are the roots of the characteristic equation $\lambda^2 - 2x\lambda - k = 0$ associated with the recurrence relations (7) *and* (8)*.*

Proof. 1. Since the characteristic equation has two distinct roots, the sequence

$$P_{k,n}(x) = c(r_1)^n + d(r_2)^n \tag{9}$$

is the solution of the equation (7). For $n = 0,\ 1$ in (9) and solving this system of linear equations, we obtain a unique value for c and d, which are, in this case, $c = \frac{1}{r_1 - r_2}$ and $d = \frac{1}{r_2 - r_1}$. So, using these values in (9) and performing some calculations, we get the required result.
2. In a similar way, easily we get the second Binet-style formula. □

Next, we present some identities related with this type of polynomial sequences. As a consequence of the Binet formula of Theorem 2.1 for these sequences, we get the following interesting identities.

Proposition 1 (Catalan's identities)**.** *For n and s, nonnegative integer numbers, such that $s \leq n$, and for any positive real number k, the Catalan identities for the k-Pell and k-Pell-Lucas polynomials are given by*

$$P_{k,n-s}(x)P_{k,n+s}(x) - (P_{k,n}(x))^2 = (-1)^{n+1-s}k^{n-s}\left(P_{k,s}(x)\right)^2$$

and

$$Q_{k,n-s}(x)Q_{k,n+s}(x) - (Q_{k,n}(x))^2 = (-k)^{n-s}\left((Q_{k,s}(x))^2 - 4(-k)^s\right),$$

respectively.

Proof. Using the Binet formulae in Theorem 2.1 and the fact that $r_1 r_2 = -k$, we get the required results for both cases. □

Note that considering Catalan's identity, in particular, the case of $s = 1$, we get the Cassini identity for the k-Pell and k-Pell-Lucas polynomial sequences. In fact, for $s = 1$, the identity stated in Proposition 1 yields for both cases

$$P_{k,n-1}(x)P_{k,n+1}(x) - (P_{k,n}(x))^2 = (-1)^n k^{n-1}\left(P_{k,1}(x)\right)^2,$$

$$Q_{k,n-1}(x)Q_{k,n+1}(x) - (Q_{k,n}(x))^2 = (-k)^{n-1}\left((Q_{k,1}(x))^2 - 4(-k)\right).$$

Using one of the initial conditions of each polynomial sequence, we obtain the following result.

Proposition 2 (Cassini's identities)**.** *For a natural number n and any positive real number k, if $P_{k,n}(x)$ and $Q_{k,n}(x)$, respectively, are the nth k-Pell and k-Pell-Lucas polynomials, then the following identities are true:*

1. $P_{k,n-1}(x)P_{k,n+1}(x) - (P_{k,n}(x))^2 = (-1)^n k^{n-1}$,
2. $Q_{k,n-1}(x)Q_{k,n+1}(x) - (Q_{k,n}(x))^2 = 4(-k)^{n-1}\left(x^2+k\right)$.

The d'Ocagne identities can also be obtained using the Binet formulae of Theorem 2.1, and in this case, we obtain the following propositions.

Proposition 3 (d'Ocagne's identities)**.** *For some natural numbers m, n and any positive real number k, if $m > n$ and $P_{k,n}(x)$, and $Q_{k,n}(x)$, respectively, are the nth k-Pell and k-Pell-Lucas polynomials, then the d'Ocagne identities for $P_{k,n}(x)$ and for $Q_{k,n}(x)$, are given by*

$$P_{k,m+1}(x)P_{k,n}(x) - P_{k,m}(x)P_{k,n+1}(x) = (-1)^{n+1}k^n P_{k,m-n}(x)$$

and

$$\begin{aligned}&Q_{k,m+1}(x)Q_{k,n}(x) - Q_{k,m}(x)Q_{k,n+1}(x)\\&= (-1)^n k^n 2\sqrt{x^2+k}\left(Q_{k,m-n}(x) - 2\left(x-\sqrt{x^2+k}\right)^{m-n}\right),\end{aligned}$$

respectively.

Proof. Once more, using the Binet Formulae of Theorem 2.1, and the fact that $r_1 r_2 = -k$, $r_1 - r_2 = 2\sqrt{x^2+k}$, we get the required results. □

The next identity is a consequence of Catalan's identity established in Proposition 1.

Proposition 4. *For a natural number n and any positive real number k, the Gelin-Cesàro identities for $P_{k,n}(x)$ and for $Q_{k,n}(x)$, are given by*

$$\begin{aligned}&P_{k,n-2}(x)P_{k,n-1}(x)P_{k,n+1}(x)P_{k,n+2}(x) - (P_{k,n}(x))^4\\&= (-1)^n k^{n-2}\left(4x^2(-1)^{n-1}k^{n-1} + (k-4x^2)(P_{k,n}(x))^2\right)\end{aligned}$$

and

$$\begin{aligned}&Q_{k,n-2}(x)Q_{k,n-1}(x)Q_{k,n+1}(x)Q_{k,n+2}(x) - (Q_{k,n}(x))^4\\&= 4(-k)^{n-1}(x^2+k)\left(16x^2(-k)^{n-2}(x^2+k) + \left(1+(-k)^{-1}4x^2\right)(Q_{k,n}(x))^2\right),\end{aligned}$$

respectively.

Proof. Using the first identity of Proposition 1, we have for $s=1$ and $s=2$,

$$P_{k,n-1}(x)P_{k,n+1}(x) - (P_{k,n}(x))^2 = (-1)^n k^{n-1}(P_{k,1}(x))^2$$

and

$$P_{k,n-2}(x)P_{k,n+2}(x) - (P_{k,n}(x))^2 = (-1)^{n-1}k^{n-2}(P_{k,2}(x))^2.$$

Now, by the initial conditions, we have

$$P_{k,n-1}(x)P_{k,n+1}(x) - (P_{k,n}(x))^2 = (-1)^n k^{n-1}$$

and

$$P_{k,n-2}(x)P_{k,n+2}(x) - (P_{k,n}(x))^2 = (-1)^{n-1} k^{n-2} (2x)^2 .$$

These identities are equivalent to

$$P_{k,n-1}(x)P_{k,n+1}(x) = (-1)^n k^{n-1} + (P_{k,n}(x))^2$$

and

$$P_{k,n-2}(x)P_{k,n+2}(x) = (-1)^{n-1} k^{n-2} 4x^2 + (P_{k,n}(x))^2 ,$$

respectively.

Therefore, multiplying both sides of these identities, we obtain

$$\begin{aligned} &P_{k,n-1}(x)P_{k,n+1}(x)P_{k,n-2}(x)P_{k,n+2}(x) \\ &= \left((-1)^n k^{n-1} + (P_{k,n}(x))^2\right)\left((-1)^{n-1} k^{n-2} 4x^2 + (P_{k,n}(x))^2\right) \\ &= 4x^2(-1)^{2n-1}k^{2n-3} + \left((-1)^n k^{n-1} + 4x^2(-1)^{n-1}k^{n-2}\right)(P_{k,n}(x))^2 + (P_{k,n}(x))^4 , \end{aligned}$$

and then the result follows.

For the Gelin-Cesàro identity for $Q_{k,n}(x)$, a similar way can be used in the proof. □

The next result establishes relation between both sequences of polynomials.

Proposition 5. *For any positive real number k and a non negative integer n, if $P_{k,n}(x)$ and $Q_{k,n}(x)$ are the nth term of the k-Pell and k-Pell-Lucas polynomials, then*

$$P_{k,n+1}(x) + kP_{k,n-1}(x) = Q_{k,n}(x).$$

Proof. Using the Binet formulas of the k-Pell and k-Pell-Lucas polynomials and the fact that $\frac{1}{r_1} = -\frac{r_2}{k}$ and $\frac{1}{r_2} = -\frac{r_1}{k}$, we have

$$\begin{aligned} P_{k,n+1}(x) + kP_{k,n-1}(x) &= \frac{(r_1)^{n+1} - (r_2)^{n+1} + k(r_1)^{n-1} - k(r_2)^{n-1}}{r_1 - r_2} \\ &= \frac{(r_1)^n\left(r_1 + \frac{k}{r_1}\right) + (r_2)^n\left(-r_2 - \frac{k}{r_2}\right)}{r_1 - r_2}, \end{aligned}$$

and finally, performing some calculations, we get the required result. □

3. Rising Diagonal function for the k-Pell and k-Pell-Lucas polynomials

This section aims to set out the Rising Diagonal function for k-Pell and k-Pell-Lucas polynomials and some elementary results.

Consider the Rising Diagonal functions $R_i(x,k)$ and $r_i(x,k)$, $0 \leq i$, for the k-Pell and k-Pell-Lucas polynomials, respectively, stated in Table 3.

n	$R_n(x,k)$	$r_n(x,k)$
$n=0$	0	2
$n=1$	1	$2x$
$n=2$	$2x$	$4x^2$
$n=3$	$4x^2$	$8x^3+2k$
$n=4$	$8x^3+k$	$16x^4+6xk$
$n=5$	$16x^4+4xk$	$32x^5+16x^2k$
$n=6$	$32x^5+12x^2k$	$64x^6+40x^3k+2k^2$
$n=7$	$64x^6+32x^3k+k^2$	$128x^7+96x^4k+10xk^2$

Table 3. The Rising Diagonal functions for the k-Pell and k-Pell-Lucas polynomials for $0 \leq n \leq 7$.

Observe that in Table 3, for $n \geq 3$,

$$r_{n-1}(x,k) = R_n(x,k) + kR_{n-3}(x,k), \tag{10}$$

$$r_n(x,k) = 2xr_{n-1}(x,k) + kr_{n-3}(x,k), \tag{11}$$

$$R_n(x,k) = 2xR_{n-1}(x,k) + kR_{n-3}(x,k). \tag{12}$$

Next, we give the generating functions for these Rising Diagonal polynomials. We write each type of sequences of polynomials as a power series where each term of the sequence correspond to coefficients of the series, and from that fact, we find the generating function. Let $f(x,k,t)$ and $g(x,k,t)$ be the generating functions of the Rising Diagonal polynomials corresponding to the k-Pell and k-Pell-Lucas polynomials, respectively. By definition of the generating function of some sequences, we have

$$f(x,k,t) = \sum_{i=1}^{\infty} R_i(x,k)t^i \tag{13}$$

and

$$g(x,k,t) = \sum_{i=1}^{\infty} r_i(x,k)t^i. \tag{14}$$

Theorem 3.1. *The generating functions for the Rising Diagonal polynomials* $\{R_i(x,k)\}$ *and* $\{r_i(x,k)\}$*, respectively, are*

1. $f(x,k,t) = \dfrac{t}{1-2xt-kt^3}$,
2. $g(x,k,t) = \dfrac{2(1-xt)}{1-2xt-kt^3}$.

Proof. 1. For the $\{R_i(x,k)\}$ sequences, by taking into account (12) and the values of $R_1(x,k)$ and $R_2(x,k)$, we have

$$\begin{aligned} f(x,k,t) &= \sum_{i=1}^{\infty} R_i(x,k)t^i = R_1(x,k)t + R_2(x,k)t^2 + \sum_{i=3}^{\infty} R_i(x,k)t^i \\ &= R_1(x,k)t + R_2(x,k)t^2 + \sum_{i=3}^{\infty} \left(2xR_{i-1}(x,k) + kR_{i-3}(x,k)\right)t^i \\ &= R_1(x,k)t + R_2(x,k)t^2 + 2x\sum_{i=3}^{\infty} R_{i-1}(x,k)t^i + k\sum_{i=3}^{\infty} R_{i-3}(x,k)t^i \\ &= t + 2xt^2 + 2xt\Big(\sum_{i=1}^{\infty} R_i(x,k)t^i - R_1(x,k)t\Big) + kt^3\Big(\sum_{i=1}^{\infty} R_i(x,k)t^i - R_0(x,k)\Big). \end{aligned}$$

Therefore,

$$f(x,k,t) = t + 2xtf(x,k,t) + kt^3 f(x,k,t)$$

which is equivalent to

$$f(x,k,t)\left(1-2xt-kt^3\right) = t,$$

and then the generating function of $\{R_i(x,k)\}$ can be written as

$$f(x,k,t) = \frac{t}{1-2xt-kt^3},$$

as required.
2. Similarly, we can show the generating functions of $\{r_i(x,k)\}$ by taking account into (11) and the values of $r_1(x,k),\ r_2(x,k)$. ☐

In what follows, we present the summation formulae related to these polynomial sequences and standard techniques are used to generate them.

Proposition 6. *For any positive real number* k *and natural numbers* $n,\ j$*, if* $\{R_j(x,k)\}$ *and* $\{r_j(x,k)\}$ *are the* j*th term of the Rising Diagonal polynomials for the* k*-Pell and* k*-Pell-Lucas polynomials, then for* $(2x+k-1 \neq 0,\ x \neq 0)$*, the following identities are valid:*

1. $\sum\limits_{j=0}^{n} R_j(x,k) = \dfrac{R_{n+1}(x,k) + k\left(R_{n-1}(x,k) + R_n(x,k)\right) - 1}{2x+k-1}$,
2. $\sum\limits_{j=0}^{n} r_j(x,k) = \dfrac{r_{n+1}(x,k) + k\left(r_{n-1}(x,k) + r_n(x,k)\right) + 2(x-1)}{2x+k-1}$,
3. $\sum\limits_{j=0}^{n} \left(R_j(x,k) + r_j(x,k)\right)$
$$= \left(\frac{(3-2x)\left(R_{n+1}(x,k) + k\left(R_{n-1}(x,k) + R_n(x,k)\right) - 1\right)}{2x+k-1}\right) + 2R_{n+1}(x,k).$$

Proof. 1. We have

$$\begin{aligned} &R_0(x,k)=0, && R_3(x,k)=\frac{1}{2x}\left(R_4(x,k)-kR_1(x,k)\right),\\ &R_1(x,k)=1, && \vdots\\ &R_2(x,k)=\frac{1}{2x}R_3(x,k), && R_n(x,k)=\frac{1}{2x}\left(R_{n+1}(x,k)-kR_{n-2}(x,k)\right)\end{aligned}$$

by the use of (12), and then

$$\begin{aligned} &R_0(x,k)+R_1(x,k)+R_2(x,k)+R_3(x,k)+\cdots+R_n(x,k)\\ &=R_0(x,k)+R_1(x,k)+\frac{1}{2x}R_3(x,k)+\frac{1}{2x}\left(R_4(x,k)-kR_1(x,k)\right)+\dots\\ &\quad+\frac{1}{2x}\left(R_{n+1}(x,k)-kR_{n-2}(x,k)\right).\end{aligned}$$

Hence

$$\begin{aligned} 2x\sum_{j=0}^{n}R_j(x,k)&=2xR_0(x,k)+2xR_1(x,k)+\sum_{j=0}^{n}R_j(x,k)\\ &\quad+R_{n+1}(x,k)-R_0(x,k)-R_1(x,k)-R_2(x,k)\\ &\quad-k\sum_{j=0}^{n}R_j(x,k)+kR_{n-1}(x,k)+kR_n(x,k).\end{aligned}$$

Using the values of the first three elements of the sequence $\{R_j(x,k)\}$, we obtain

$$(2x+k-1)\sum_{j=0}^{n}R_j(x,k)=R_{n+1}(x,k)-1+kR_{n-1}(x,k)+kR_n(x,k),$$

which is equivalent to the required result.
2. In a similar way, we show the summation formula for the sequence $\{r_j(x,k)\}$.
3. By the identities (10) and (12), and the value of initial conditions, we have

$$\begin{aligned} \sum_{j=0}^{n}\left(R_j(x,k)+r_j(x,k)\right)&=\sum_{j=0}^{n}\left(R_j(x,k)+R_{j+1}(x,k)+kR_{j-2}(x,k)\right)\\ =\sum_{j=0}^{n}&\left(R_j(x,k)+R_{j+1}(x,k)+\left(R_{j+1}(x,k)-2xR_j(x,k)\right)\right)\\ =(1-2x)&\sum_{j=0}^{n}R_j(x,k)+2\sum_{j=0}^{n}R_{j+1}(x,k)\\ =(1-2x)&\sum_{j=0}^{n}R_j(x,k)+2\sum_{j=0}^{n}R_j(x,k)-2R_0(x,k)+2R_{n+1}(x,k)\\ =(3-2x)&\sum_{j=0}^{n}R_j(x,k)+2R_{n+1}(x,k).\end{aligned}$$

Now, the use of the identity 1. gives the required result. □

4. Descending Diagonal function for the k-Pell and k-Pell-Lucas polynomials

In this section, we present the Descending Diagonal function for the k-Pell and k-Pell- -Lucas polynomials and some properties are provided.

Consider the Descending Diagonal functions $D_i(x,k)$ and $d_i(x,k)$, $0 \le i$, for k-Pell and k-Pell-Lucas polynomials, respectively, stated in Table 4.

n	$D_n(x,k)$	$d_n(x,k)$
$n=0$	0	2
$n=1$	1	$2x+k$
$n=2$	$2x+k$	$(2x+k)(2x+2k)$
$n=3$	$(2x+k)^2$	$(2x+k)^2(2x+2k)$
$n=4$	$(2x+k)^3$	$(2x+k)^3(2x+2k)$
$n=5$	$(2x+k)^4$	$(2x+k)^4(2x+2k)$
$n=6$	$(2x+k)^5$	$(2x+k)^5(2x+2k)$
$n=7$	$(2x+k)^6$	$(2x+k)^6(2x+2k)$

Table 4. The Descending Diagonal functions for the k-Pell and k-Pell-Lucas polynomials for $0 \le n \le 7$.

Observe that in Table 4, for $n \ge 2$,

$$D_n(x,k) = (2x+k)D_{n-1}(x,k), \tag{15}$$

$$d_n(x,k) = (2x+k)d_{n-1}(x,k), \tag{16}$$

$$d_n(x,k) = (2x+2k)D_n(x,k), \tag{17}$$

$$\frac{D_n(x,k)}{D_{n-1}(x,k)} = \frac{d_n(x,k)}{d_{n-1}(x,k)} = 2x+k, \tag{18}$$

$$\frac{d_n(x,k)}{D_n(x,k)} = 2x+2k. \tag{19}$$

Let

$$h(x,k,t) = \sum_{i=1}^{\infty} D_i(x,k)t^i \tag{20}$$

and

$$l(x,k,t) = \sum_{i=1}^{\infty} d_i(x,k)t^i \tag{21}$$

be the generating functions for the Descending Diagonal polynomials for the k-Pell and k-Pell-Lucas polynomials, respectively.

Theorem 4.1. *The generating functions for the Descending Diagonal polynomials* $\{D_i(x,k)\}$ *and* $\{d_i(x,k)\}$ *are*

1. $h(x,k,t) = \dfrac{t}{1-2xt-kt}$,
2. $l(x,k,t) = \dfrac{(2x+k)t}{1-2xt-kt}$, *respectively.*

Proof. 1. For the $\{D_i(x,k)\}$ sequences, by taking into account (15) and the value of $D_1(x,k)$, we have

$$\begin{aligned} h(x,k,t) &= \sum_{i=1}^{\infty} D_i(x,k)t^i = D_1(x,k)t + \sum_{i=2}^{\infty} D_i(x,k)t^i \\ &= D_1(x,k)t + \sum_{i=2}^{\infty} \left((2x+k)D_{i-1}(x,k)\right)t^i \\ &= D_1(x,k)t + (2x+k)t\sum_{i=2}^{\infty} D_{i-1}(x,k)t^{i-1} \\ &= t + (2x+k)t\left(\sum_{i=2}^{\infty} D_i(x,k)t^i - 1\right). \end{aligned}$$

Therefore,

$$h(x,k,t) = t + (2x+k)th(x,k,t)$$

which is equivalent to

$$h(x,k,t)\,(1-2xt-kt) = t,$$

and then the generating function of $\{D_i(x,k)\}$ can be written as

$$h(x,k,t) = \frac{t}{1-2xt-kt},$$

as required.

2. Similarly, we can show the generating functions of $\{d_i(x,k)\}$ by taking into account (16) and the value of $d_1(x,k)$. $\square$

Next, we present the summation formulae related to these polynomial sequences and we use the identities (15) and (16) in the proof.

Proposition 7. *For any positive real number* k *and natural numbers* n, j, *if* $\{D_j(x,k)\}$ *and* $\{d_j(x,k)\}$ *are the* j*th term of the Descending Diagonal polynomials for the* k*-Pell and* k*-Pell-Lucas polynomials, then for* $(1-2x-k\neq 0,\ x\neq 0)$, *the following identities are valid:*

1. $\displaystyle\sum_{j=0}^{n} D_j(x,k) = \frac{1-D_{n+1}(x,k)}{1-2x-k}$,

2. $\displaystyle\sum_{j=0}^{n} d_j(x,k) = \frac{2-(2x+k)-d_{n+1}(x,k)}{1-2x-k}$,

3. $\displaystyle\sum_{j=0}^{n} (D_j(x,k)+d_j(x,k)) = (2x+2k+1)\left(\frac{1-D_{n+1}(x,k)}{1-2x-k}\right)$.

Proof. 1. We have

$$\begin{aligned}
&D_0(x,k) = 0, && D_3(x,k) = (2x+k)D_2(x,k),\\
&D_1(x,k) = 1, && \vdots\\
&D_2(x,k) = (2x+k)D_1(x,k), && D_n(x,k) = (2x+k)D_{n-1}(x,k)
\end{aligned}$$

by the use of (15), and then

$$\begin{aligned}
&D_0(x,k)+D_1(x,k)+D_2(x,k)+D_3(x,k)+\ldots+D_n(x,k)\\
&= 0+1+(2x+k)D_1(x,k)+(2x+k)D_2(x,k)+\ldots+(2x+k)D_{n-1}(x,k).
\end{aligned}$$

Hence

$$\sum_{j=0}^{n} D_j(x,k) = 1+(2x+k)\sum_{j=0}^{n} D_j(x,k) - (2x+k)D_n(x,k).$$

So, performing some calculations and using (15), we get the required result.
2. Using a similar way, we show the summation formula for the sequence $\{d_j(x,k)\}$.
3. By the identity (17), we have

$$\begin{aligned}
\sum_{j=0}^{n} (D_j(x,k)+d_j(x,k)) &= \sum_{j=0}^{n} (D_j(x,k)+(2x+2k)D_j(x,k))\\
&= (2x+2k+1)\sum_{j=0}^{n} D_j(x,k).
\end{aligned}$$

Now the use of identity 1. is sufficient to obtain the desired result. □

5. Conclusions

Sequences of polynomials have been studied for several years, with emphasis on the well known Fibonacci polynomial sequence, and consequently, on the Lucas polynomial sequence. In this paper, we have also contributed to the study of k-Pell and k-Pell-Lucas polynomial sequences and the respective Rising and Descending Diagonal functions. We have deduced some formulae for the sums of such sequences of polynomials, presenting the generating functions, the Binet formula and some identities. Our intention is to continue the study of this type of sequences, exploring the several types of its generating matrices.

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