ON SOME CLASSES OF WEAKLY M-PROJECTIVELY SYMMETRIC MANIFOLDS

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Abstract. The object of the present paper is to study weakly M-projectively symmetric manifolds. At first some geometric properties of \((\text{WMPS})_n\) \((n > 2)\) studied. Finally, we consider the decomposability of \((\text{WMPS})_n\).

1. Introduction

A non-flat Riemannian manifold \((M^n,g)(n > 2)\) is called weakly symmetric [30] if the curvature tensor \(R\) of type \((0,4)\) satisfies the condition

\[
+ C(Z)R(Y, X, U, V) + D(U)R(Y, Z, X, V)
+ E(V)R(Y, Z, U, X)
\]

for all vector fields \(X, Y, Z, U, V \in \chi(M^n)\), where \(R(Y, Z, U, V) = g(\mathcal{R}(Y, Z)U, V)\), \(\mathcal{R}\) is the curvature tensor of type \((1,3)\) and \(A, B, C, D\) and \(E\) are 1-forms and non-zero simultaneously and \(\nabla\) is the operator of covariant differentiation with respect to the Riemannian metric \(g\). The 1-forms are called the associated 1-forms of the manifold and an \(n\)-dimensional manifold of this kind is denoted by \((WS)_n\).

Tamássy and Binh [31] further studied weakly symmetric Sasakian manifolds and proved that such a manifold does not always exist. De and Bandyopadhyay [8] established the existence of a \((WS)_n\) by an example and proved that in a \((WS)_n\), the associated 1-forms \(B = C\) and \(D = E\).

Hence (1.1) reduces to the following form

\[
+ B(Z)R(Y, X, U, V) + D(U)R(Y, Z, X, V)
+ D(V)R(Y, Z, U, X).
\]

If in (1.2) the 1-form \(A\) is replaced by 2\(A\), \(B\) and \(D\) are replaced by \(A\), then the manifold \((M^n,g)\) reduces to a pseudo symmetric manifold in the sense of Chaki [3].
Again if \( A = B = D = 0 \), the manifold defined by (1.2) reduces to a symmetric manifold in the sense of Cartan[2]. The existence of a \((WS)_{n}\) was proved by Prvanović [25] and a concrete example was given by De and Bandyopadhyay ([18],[9]).

Weakly symmetric manifolds have been studied by several authors ([1], [6], [7], [10], [11], [12], [13],[16], [17], [18], [19], [20], [21], [22], [23], [25]) and many others.

Let \( \rho_{1}, \rho_{2} \) and \( \rho_{3} \) are the basic vectors corresponding to the 1-forms \( A \), \( B \) and \( D \) respectively, that is

\[
(1.3) \quad g(X, \rho_{1}) = A(X), \quad g(X, \rho_{2}) = B(X) \quad \text{and} \quad g(X, \rho_{3}) = D(X).
\]

A. Gray [15] introduced the notion of cyclic parallel Ricci tensor and Codazzi type of Ricci tensor. A Riemannian or semi-Riemannian manifold is said to satisfy cyclic parallel Ricci tensor if its Ricci tensor \( S \) of type \((0,2)\) is non-zero and satisfies the condition

\[
(1.4) \quad (\nabla_{X}S)(Y, Z) + (\nabla_{Y}S)(Z, X) + (\nabla_{Z}S)(X, Y) = 0.
\]

Again a Riemannian or semi-Riemannian manifold is said to satisfy Codazzi type of Ricci tensor if its Ricci tensor \( S \) of type \((0,2)\) is non-zero and satisfies the following condition

\[
(1.5) \quad (\nabla_{X}S)(Y, Z) = (\nabla_{Y}S)(X, Z).
\]

In 1971, Pokhariyal and Mishra [24] introduced a new curvature tensor of type \((1,3)\) in an \( n \)-dimensional Riemannian or semi-Riemannian manifold \((M^{n}, g)\) \((n > 2)\) denoted by \( \mathcal{M} \) and defined by

\[
(1.6) \quad \mathcal{M}(Y, Z)U = R(Y, Z)U - \frac{1}{2(n - 1)} [S(Z, U)Y - S(Y, U)Z + g(Y, U)LZ],
\]

where \( L \) is the Ricci operator defined by \( g(LX, Y) = S(X, Y) \). Such a tensor \( \mathcal{M} \) is known as M-projective curvature tensor. In the same paper the authors studied relativistic significance of such a tensor \( \mathcal{M} \). Recently M-projective curvature tensor has been studied by J. P. Singh [27], S. K. Chaubey and R. H. Ojha [4], S. K. Chaubey [5], R. N. Singh and S. K. Pandey [28] and many others. On the other hand, De and Mallick [14] studied spacetimes admitting M-projective curvature tensor.

Now (1.6) can be expressed as

\[
(1.7) \quad M(Y, Z, U, V) = R(Y, Z, U, V) - \frac{1}{2(n - 1)} [S(Z, U)g(Y, V) - S(Y, U)g(Z, V) + g(Z, U)S(Y, V) - g(Y, U)S(Z, V)],
\]
where $M(Y,Z,U,V) = g(M(Y,Z)U,V)$. Since the M-projective curvature tensor satisfies the properties of the Riemannian curvature tensor, therefore weakly M-projective symmetric manifold is characterized by the condition

\begin{equation}
+ B(Z)M(Y,X,U,V)
+ D(U)M(Y,Z,X,V)
+ D(V)M(Y,Z,U,X),
\end{equation}

where the 1-forms $A, B$ and $D$ are non-zero simultaneously. Such a manifold is denoted by $(WMPS)_n$.

We also have a very useful lemma as follows.

**Theorem 1.1 (Walker’s Lemma [32]).** If $a_{ij}, b_i$ are numbers satisfying $a_{ij} = a_{ji}$, and $a_{ij}b_k + a_{jk}b_i + a_{ki}b_j = 0$ for $i,j,k = 1,2,\ldots,n$, then either all $a_{ij}$ or all $b_i$ are zero.

In 2013 the author [22] characterize weakly projective symmetric manifolds. Weyl projective curvature tensor $\mathcal{P}$ was introduced by H. Weyl in 1918 and defined by [29]

\begin{equation}
\mathcal{P}(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}[S(Y,Z)X - S(X,Z)Y].
\end{equation}

Weyl projective curvature tensor defined by (1.9) and M-projective curvature tensor defined by (1.6) are two completely different curvature tensors. Therefore the results of the paper [22] are different from the present paper.

In the present paper, we are interested in studying weakly M-projectively symmetric manifolds.

The paper is organized as follows.

After preliminaries, in Section 3, some curvature properties of $(WMPS)_n$ are studied. Section 4 deals with the study of Einstein $(WMPS)_n(n > 2)$. Finally, we consider decomposable $(WMPS)_n(n > 2)$.

2. Preliminaries

Let $S$ and $r$ denote the Ricci tensor of type $(0,2)$ and the scalar curvature respectively. $L$ denotes the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor $S$, that is,

\begin{equation}
g(LX,Y) = S(X,Y).
\end{equation}

In this section, some formulas are derived, which will be useful to the study of $(WMPS)_n$. Let \{e_i\} be an orthonormal basis of the tangent space at each point of the manifold where $1 \leq i \leq n$.

From (1.6) we can easily verify that the tensor $M$ satisfies the following properties:

(i) $M(Y,Z)U = -M(Z,Y)U$,

(ii) \begin{equation}
M(Y,Z)U + M(Z,U)Y + M(U,Y)Z = 0.
\end{equation}
Also from (1.7) we have

\[(2.3)\quad \sum_{i=1}^{n} M(Y, Z, e_i, e_i) = 0 = \sum_{i=1}^{n} M(e_i, e_i, U, V)\]

and

\[(2.4)\quad \sum_{i=1}^{n} M(e_i, Z, U, e_i) = \sum_{i=1}^{n} M(Z, e_i, e_i, U) = \frac{n}{2(n-1)} [S(Z, U) - \frac{r}{n} g(Z, U)],\]

where \(r = \sum_{i=1}^{n} S(e_i, e_i)\) is the scalar curvature.

Let

\[(2.5)\quad \tilde{M}(Z, U) = S(Z, U) - \frac{r}{n} g(Z, U).\]

Therefore,

\[(2.6)\quad \tilde{M}(e_i, e_i) = 0.\]

From (1.7) and (2.2) it follows:

(i) \(M(Y, Z, U, V) = -M(Z, Y, U, V)\),

(ii) \(M(Y, Z, U, V) = -M(Y, Z, V, U)\),

(iii) \(M(Y, Z, U, V) = M(U, V, Y, Z)\),

(iv) \(2.7\) \(M(Y, Z, U, V) + M(Z, U, Y, V) + M(U, Y, Z, V) = 0\).

3. Some curvature properties of \((WMPS)_n (n > 2)\)

Contracting (1.8) over \(Y\) and \(V\), we get

\[(3.1)\frac{n}{2(n-1)}(\nabla X \tilde{M})(Z, U) = \frac{n}{2(n-1)} A(X) [S(Z, U) - \frac{r}{n} g(Z, U)] + B(M(X, Z) U) + \frac{n}{2(n-1)} B(Z) [S(X, U) - \frac{r}{n} g(X, U)] + \frac{n}{2(n-1)} D(U) [S(Z, X) - \frac{r}{n} g(Z, X)] - D(M(U, X) Z).\]

Again contracting (3.1) over \(Z\) and \(U\), we get

\[(3.2)\frac{n}{2(n-1)} [A(X) \{r - r\} + \{B(LX) - \frac{r}{n} B(X)\} + \{B(LX) - \frac{r}{n} B(X)\}] + [D(LX) - \frac{r}{n} D(X)] + \{D(LX) - \frac{r}{n} D(X)\} = 0,\]

which implies

\[(3.3) B(LX) + D(LX) = \frac{r}{n} [B(X) + D(X)].\]
The above equation can be written as
\[(3.4)\]
\[F(LX) = \frac{r}{n} F(X),\]
where
\[(3.5)\]
\[F(X) = B(X) + D(X).\]
Hence
\[(3.6)\]
\[S(X, \rho_2 + \rho_3) = \frac{r}{n} g(X, \rho_2 + \rho_3),\]
which implies
\[(3.7)\]
\[S(X, \rho) = \frac{r}{n} g(X, \rho),\]
where \(\rho = \rho_2 + \rho_3, \rho_2\) and \(\rho_3\) are defined by (1.3).

Hence we have the following theorem.

**Theorem 3.1.** In a \((WMPS)\), \(\frac{r}{n}\) is an eigenvalue of the Ricci tensor \(S\) corresponding to the eigen vector \(\rho\) defined by \(\rho = \rho_2 + \rho_3\).

Now writing (3.1) cyclically and adding we obtain
\[(3.8)\]
\[\begin{align*}
&\left(\nabla_X \tilde{M}(Z, U) + (\nabla_Z \tilde{M})(U, X) + (\nabla_U \tilde{M})(X, Z)\right) \\
&= \{A(X) + B(X) + D(X)\} \{S(Z, U) - \frac{r}{n} g(Z, U)\} \\
&\quad + \{A(Z) + B(Z) + D(Z)\} \{S(X, U) - \frac{r}{n} g(X, U)\} \\
&\quad + \{A(U) + B(U) + D(U)\} \{S(Z, X) - \frac{r}{n} g(Z, X)\} \\
&\quad + \frac{2(n-1)}{n} \left\{B(\mathcal{M}(X, Z)U) + B(\mathcal{M}(Z, U)X) + B(\mathcal{M}(U, X)Z)\right\} \\
&\quad - \frac{2(n-1)}{n} \left\{D(\mathcal{M}(X, Z)U) + D(\mathcal{M}(Z, U)X) + D(\mathcal{M}(U, X)Z)\right\}.
\end{align*}\]

Now using (2.2) and (2.7) in (3.8) we get
\[(3.9)\]
\[\begin{align*}
&\left(\nabla_X \tilde{M}(Z, U) + (\nabla_Z \tilde{M})(U, X) + (\nabla_U \tilde{M})(X, Z)\right) \\
&= E(X) \tilde{M}(Z, U) + E(Z) \tilde{M}(U, X) + E(U) \tilde{M}(X, Z),
\end{align*}\]
where
\[(3.10)\]
\[E(X) = A(X) + B(X) + D(X).\]

If the \((WMPS)\) has \(\tilde{M}\)-cyclic parallel tensor, then we have
\[(3.11)\]
\[\left(\nabla_X \tilde{M}(Z, U) + (\nabla_Z \tilde{M})(U, X) + (\nabla_U \tilde{M})(X, Z)\right) = 0.

Also we have from (2.5)
\[(3.12)\]
\[\tilde{M}(X, Y) = \tilde{M}(Y, X).
\]

Using (3.11) in (3.9) we get
\[(3.13)\]
\[E(X) \tilde{M}(Z, U) + E(Z) \tilde{M}(U, X) + E(U) \tilde{M}(X, Z) = 0.
\]
Then by Walker’s lemma we can see that either $E(X) = 0$ or $\tilde{M}(Z,U) = 0$ for all $X, Z, U$. Thus we have either

\begin{equation}
A(X) + B(X) + D(X) = 0
\end{equation}

or the manifold is an Einstein manifold. Thus we can state the following theorem.

**Theorem 3.2.** If in a $(WMPS)_n(n > 2)$, the tensor $\tilde{M}$ defined by (2.5) is cyclic parallel then the manifold is either an Einstein manifold or the sum of the associated 1-forms is zero.

Conversely, suppose the manifold $(WMPS)_n(n > 2)$ is not an Einstein manifold. Then $A(X) + B(X) + D(X) = 0$ for all $X$.

Hence from (3.9) it follows that

\begin{equation}
(\nabla_X \tilde{M})(Z,U) + (\nabla_Z \tilde{M})(U,X) + (\nabla_U \tilde{M})(X,Z) = 0,
\end{equation}

which implies that the $\tilde{M}$ tensor is cyclic parallel. Thus we can state the following theorem.

**Theorem 3.3.** In a $(WMPS)_n(n > 2)$, the tensor $\tilde{M}$ is cyclic parallel if and only if sum of the associated 1-forms is zero, provided the manifold is not an Einstein manifold.

4. Einstein $(WMPS)_n$

This section deals with a $(WMPS)_n$ which is an Einstein manifold. Then the Ricci tensor satisfies

\begin{equation}
S(Y,Z) = \frac{r}{n}g(Y,Z).
\end{equation}

From which it follows that

$$dr(X) = 0$$

and

\begin{equation}
(\nabla_X S)(Y,Z) = 0 \quad \text{for all } X,Y,Z.
\end{equation}

Using (4.1) and (4.2) we get from (1.7)

\begin{equation}
\end{equation}

Using (4.3) in (1.8) we get

\begin{equation}
\end{equation}
Using (1.7) and (4.1) in (4.4) we get
\( (\nabla X R)(Y, Z, U, V) \)
\[ = A(X)[R(Y, Z, U, V) - \frac{2r}{2n(n-1)} \{ g(Z, U)g(Y, V) - g(Y, U)g(Z, V) \}] \]
\[ + B(Y)[R(Y, Z, U, V) - \frac{2r}{2n(n-1)} \{ g(Z, U)g(X, V) - g(X, U)g(Z, V) \}] \]
\[ + B(Z)[R(Y, X, U, V) - \frac{2r}{2n(n-1)} \{ g(X, U)g(Y, V) - g(Y, U)g(X, V) \}] \]
\[ + D(U)[R(Y, Z, X, V) - \frac{2r}{2n(n-1)} \{ g(Z, X)g(Y, V) - g(Y, X)g(Z, V) \}] \]
\[ + D(V)[R(Y, Z, X, U) - \frac{2r}{2n(n-1)} \{ g(Z, U)g(Y, X) - g(Y, U)g(Z, X) \}] \].

Now if the Einstein (WMPS)_n is a (WS)_n, then from (1.2) and (4.5) we get
\[ r[A(X)\{g(Z, U)g(Y, V) - g(Y, U)g(Z, V)\}] \]
\[ + B(Y)\{g(Z, U)g(X, V) - g(X, U)g(Z, V)\} \]
\[ + B(Z)\{g(X, U)g(Y, V) - g(Y, U)g(X, V)\} \]
\[ + D(U)\{g(Z, X)g(Y, V) - g(Y, X)g(Z, V)\} \]
\[ + D(V)\{g(Z, U)g(Y, X) - g(Y, U)g(Z, X)\}] = 0. \]

Contracting (4.6) over Y and V, we get
\[ r[(n - 1)A(X)g(Z, U) + B(X)g(Z, U) + (n - 2)B(Z)g(X, U)] \]
\[ + D(X)g(Z, U) + (n - 2)D(U)g(Z, X)] = 0. \]

Again contracting (4.7) over Z and U, we get
\[ r[n(n - 1)A(X) + 2(n - 1)B(X) + 2(n - 1)D(X)] = 0. \]

Similarly, contracting (4.7) over X and Z, we get
\[ r[(n - 1)A(U) + (n - 1)B(U) + (n^2 - 2n + 1)D(U)] = 0. \]

Replacing U by X in (4.9) we get
\[ r[(n - 1)A(X) + (n - 1)B(X) + (n^2 - 2n + 1)D(X)] = 0. \]

Again contracting (4.7) over X and U, we get
\[ r[(n - 1)A(Z) + (n^2 - 2n + 1)B(Z) + (n - 1)D(Z)] = 0. \]

Replacing Z by X in (4.11), we get
\[ r[(n - 1)A(X) + (n^2 - 2n + 1)B(X) + (n - 1)D(X)] = 0. \]

Now adding (4.8), (4.10) and (4.12), we get
\[ r(n - 1)(n + 2)[A(X) + B(X) + D(X)] = 0. \]

Since \( n > 2 \), so from (4.13) we get either \( r = 0 \) or \( A(X) + B(X) + D(X) = 0 \).

Hence we have the following theorem.
Theorem 4.1. If an Einstein \((WMPS)_n\) \((n > 2)\) is a \((WS)_n\), then either the scalar curvature of the manifold vanishes or the sum of the associated 1-forms is zero.

Again, if in an Einstein \((WMPS)_n\) \((n > 2)\) \(r = 0\), then using (1.8), (4.1) and (1.7) in (4.3) we get

\[
+ B(Z)R(Y, X, U, V) + D(U)R(Y, Z, X, V) \\
+ D(V)R(Y, Z, U, X),
\]

(4.14)

which is the defining condition of \((WS)_n\). Hence we can state the following theorem.

Theorem 4.2. If in an Einstein \((WMPS)_n\) \((n > 2)\) the scalar curvature vanishes, then it is a \((WS)_n\).

5. Decomposable \((WMPS)_n\)

A Riemannian manifold \((M^n, g)\) is said to be decomposable or a product manifold ([26], [31]) if it can be expressed as \(M^p \times M_2^{n-p}\) for \(2 \leq p \leq (n - 2)\), that is, in some coordinate neighbourhood of the Riemannian manifold \((M^n, g)\), the metric can be expressed as

\[
ds^2 = g_{ij}dx^idx^j = \bar{g}_{ab}dx^adx^b + g^{*\alpha\beta}dx^\alpha dx^\beta,
\]

where \(g_{ab}\) are functions of \(x^1, x^2, \ldots, x^p\) denoted by \(\bar{x}\) and \(g^{*\alpha\beta}\) are functions of \(x^{p+1}, x^{p+2}, \ldots, x^n\) denoted by \(x^*\); \(a, b, c, \ldots\) run from 1 to \(p\) and \(\alpha, \beta, \gamma, \ldots\) run from \(p + 1\) to \(n\).

The two parts of (5.1) are the metrics of \(M^p (p \geq 2)\) and \(M_2^{n-p} (n-p \geq 2)\) which are called the decompositions of the decomposable manifold \(M^n = M^p \times M_2^{n-p}\) \((2 \leq p \leq n - 2)\).

Let \((M^n, g)\) be a Riemannian manifold such that \(M^p (p \geq 2)\) and \(M_2^{n-p} (n-p \geq 2)\). Here throughout this section each object denoted by a ‘bar’ is assumed to be from \(M_1\) and each object denoted by ‘star’ is assumed to be from \(M_2\).

Let \(X, Y, \bar{Z}, \bar{U}, V \in \chi(M_1)\) and \(X^*, Y^*, Z^*, U^*, V^* \in \chi(M_2)\). Then in a decomposable Riemannian manifold \(M^n = M^p \times M_2^{n-p}\) \((2 \leq p \leq n - 2)\), the following relations hold [33]:

\[
R(X^*, Y^*, \bar{Z}, \bar{U}) = R(\bar{X}, Y^*, Z^*, U^*) = R(\bar{X}, Y^*, \bar{Z}, U^*),
\]

\[
(\nabla_X R)(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) = 0 = (\nabla_X R)(\bar{Y}, \bar{Z}, \bar{U}, V^*) = (\nabla_X R)(\bar{Y}, \bar{Z}, \bar{U}, V^*),
\]

\[
R(X^*, Y^*, Z^*, U^*) = R^*(X^*, Y^*, Z^*, U^*),
\]

\[
S(\bar{X}, \bar{Y}) = S(\bar{X}, \bar{Y}); S(X^*, Y^*) = S^*(X^*, Y^*),
\]

\[
(\nabla_X S)(\bar{Y}, \bar{Z}) = (\nabla_X S)(\bar{Y}, \bar{Z});
\]

\[
(\nabla_X S)(Y^*, Z^*) = (\nabla_X S)(Y^*, Z^*),
\]
and $r = \bar{r} + r^*$, where $r, \bar{r}$ and $r^*$ are scalar curvatures of $M, M_1$ and $M_2$, respectively.

Let us consider a Riemannian manifold $(M^n, g)$, which is a decomposable $(WMPS)_n$. Then $M^n = M_1^p \times M_2^{n-p}(2 \leq p \leq n - 2)$.

Now from (1.7), we get

$$M(Y^*, \bar{Z}, \bar{U}, \bar{V}) = 0 = M(\bar{Y}, Z^*, U^*, V^*)$$

(5.2)...

Again from (1.8), we get

$$\nabla_X M(\bar{Y}, Z, U, V) = 0 = (\nabla_X M)(Y^*, Z^*, U^*, V^*).$$

(5.7)
Also from (1.8), we obtain
\[
(\nabla_X \cdot M)(Y^*, Z^*, U^*, V^*)
= A(X^*)M(Y^*, Z^*, U^*, V^*) + B(Y^*)M(X^*, Z^*, U^*, V^*)
+ B(Z^*)M(Y^*, X^*, U^*, V^*) + D(U^*)M(Y^*, Z^*, X^*, V^*)
+ D(V^*)M(Y^*, Z^*, U^*, X^*),
\]
(5.18)

\[
A(\bar{X})M(Y^*, Z^*, U^*, V^*) = 0,
\]
(5.19)

\[
B(\bar{Y})M(X^*, Z^*, U^*, V^*) = 0,
\]
(5.20)

\[
B(\bar{Z})M(Y^*, X^*, U^*, V^*) = 0,
\]
(5.21)

\[
D(\bar{U})M(Y^*, Z^*, X^*, V^*) = 0,
\]
(5.22)

\[
D(\bar{V})M(Y^*, Z^*, U^*, X^*) = 0.
\]
(5.23)

Using the relations from (5.9) to (5.13) we conclude that either
\[
I) \quad A = B = D = 0 \text{ on } M_2,
\]
or
\[
II) \quad M_1 \text{ is } M\text{-projectively-flat.}
\]

Firstly, we consider the case (I). Then from (5.17), it follows that
\[
(\nabla_X \cdot M)(Y^*, Z, \bar{U}, V^*) = 0
\]
which implies by virtue of (5.4)
\[
(\nabla_X \cdot S)(Y^*, V^*) = 0
\]
and hence the decomposition $M_2$ is Ricci symmetric. Also from (5.18), we have
\[
(\nabla_X \cdot M)(Y^*, Z^*, U^*, V^*) = 0
\]
and hence
\[
(\nabla_X \cdot R)(Y^*, Z^*, U^*, V^*) = \frac{1}{2(n-1)}[(\nabla_X \cdot S)(Z^*, U^*) g(Y^*, V^*)
- (\nabla_X \cdot S)(Y^*, U^*) g(Z^*, V^*) + (\nabla_X \cdot S)(Y^*, V^*) g(Z^*, U^*)
- (\nabla_X \cdot S)(Z^*, V^*) g(Y^*, U^*)] = 0,
\]
which yields by virtue of (5.24), that
\[
(\nabla_X \cdot R)(Y^*, Z^*, U^*, V^*) = 0,
\]
that is, the decomposition $M_2$ is locally symmetric.

Secondly, we assume that $M_1$ is $M$-projectively-flat.

Then we have
\[
R(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) = \frac{1}{2(n-1)}[S(\bar{Z}, \bar{U}) g(\bar{Y}, \bar{V}) - S(\bar{Y}, \bar{U}) g(\bar{Z}, \bar{V})
+ S(\bar{Y}, \bar{V}) g(\bar{Z}, \bar{U}) - S(\bar{Z}, \bar{V}) g(\bar{Y}, \bar{U})].
\]
(5.25)

Contracting (5.25) over $\bar{Y}$ and $\bar{V}$, we obtain
\[
S(\bar{Z}, \bar{U}) = \frac{f}{(2n-p)} g(\bar{Z}, \bar{U}).
\]
(5.26)
In view of (5.26), (5.25) yields

\[ R(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) = \frac{\bar{r}}{(2n-p)(n-1)} [g(\bar{Y}, \bar{V})g(\bar{Z}, \bar{U}) - g(\bar{Y}, \bar{U})g(\bar{Z}, \bar{V})], \]

that is, the decomposition \( M_1 \) is a manifold of constant curvature.

Thus we can state the following theorem.

**Theorem 5.1.** Let \((M^n, g)\) be a Riemannian manifold such that \( M = M_1^p \times M_2^{n-p},\) \( 2 \leq p \leq n-2. \) If \( M^n \) is a \((WMPS)_n\) then either (I) or (II) holds.

(I) \( A = 0, \ B = 0, \ D = 0 \) on \( M_2, \) (resp. \( M_1 \)), and hence \( M_2, \) (resp. \( M_1 \)) is Ricci symmetric as well as locally symmetric.

(II) \( M_1 \) (resp. \( M_2 \)) is \( M \)-projectively-flat and hence \( M_1, \) (resp. \( M_2 \)) is a manifold of constant curvature.

**References**


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