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SOME REMARKS ON IDEAL EQUAL BAIRE CLASSES

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Abstract. In this paper, we consider the notion of $\mathcal{I}$-equal convergence introduced by Das, Dutta, and Pal [**9**], and two related notions of convergence, namely, $\mathcal{I}$-discrete, and $\mathcal{I}$-strong uniform equal convergence and primarily investigate some lattice properties of the corresponding Baire classes obtained from a class of functions Φ.

1. Introduction

The concept of convergence of a sequence of real numbers was extended to statistical convergence independently by Fast [**11**], Steinhaus [**22**], and Schoenberg [**21**]. A lot of developments was made on this interesting notion of convergence after the pioneering works of Šalát [**20**] and Fridy [**13**]. Kostyrko et al.[**14**] extended the concept of statistical convergence to $\mathcal{I}$-convergence using the notion of ideals. For the last ten years, a lot of works have been done on $\mathcal{I}$-convergence (see, for example, [**6**], [**7**], [**16**], [**17**] where many more references can be seen). As a natural consequence over the years, researchers applied this new notion of convergence to sequences of functions and some significant investigations were done in [**1**], [**2**], [**8**], [**9**], [**12**], [**15**],[**18**].

In [**4**], Császár and Laczkovich introduced two new types of convergence of sequences of real-valued functions under the name of equal convergence and discrete convergence (see also [**3**], [**5**]), and studied the lattice properties of the corresponding Baire classes. In [**19**], Papanastassiou defined and studied the notions of uniform equal convergence, uniform discrete convergence, and strong uniform equal convergence for sequences of real-valued functions (see also [**10**]).

In the present paper, we consider the notion of $\mathcal{I}$-equal convergence from [**9**], and two related notions of convergence, namely, $\mathcal{I}$-discrete convergence and $\mathcal{I}$-strong uniform equal convergence which is stronger than $\mathcal{I}$-equal convergence for sequences of real-valued functions. We then investigate some lattice properties

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of the Baire classes arising from a given class of functions mainly following the line of [**8**]. However, the methods of proofs are different and the results of this paper truly extend the classical results.

2. Preliminaries

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers. A family $\mathcal{I} \subset 2^Y$ of subsets of a non-empty set Y is said to be an ideal on Y if,

(i) $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$.
(ii) $A \in \mathcal{I}$, $B \subset A$ imply $B \in \mathcal{I}$, while an admissible ideal $\mathcal{I}$ of Y further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$.

If $\mathcal{I}$ is a non-trivial proper ideal in Y (i.e., $Y \notin \mathcal{I}$, $\mathcal{I} \neq \{\emptyset\}$), then the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset Y : \text{there exists } A \in \mathcal{I} \text{ such that } M = Y \setminus A\}$ is a filter in Y. It is called the filter associated with the ideal $\mathcal{I}$.

Recall that a sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be $\mathcal{I}$-convergent to $x \in \mathbb{R}$ if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - x| \geq \varepsilon\} \in \mathcal{I}$ [**14**]. We write $\mathcal{I}\text{-lim } x_n = x$ in this case. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be $\mathcal{I}^*$-convergent to $x \in \mathbb{R}$ if there is a set $M \in \mathcal{F}(\mathcal{I})$, $M = \{m_1 < m_2 < \cdots < m_k < \dots\}$ such that $\lim_{k\to\infty} x_{m_k} = x$ [**14**]. A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be $\mathcal{I}$-divergent to ∞ or $-\infty$ if for any positive real number G, $\{n \in \mathbb{N} : x_n \leq G\} \in \mathcal{I}$ or $\{n \in \mathbb{N} : x_n \geq -G\} \in \mathcal{I}$, respectively, [**17**] (though in [**17**] the terms $\mathcal{I}$-convergent to $+\infty$ and $\mathcal{I}$-convergent to $-\infty$ were used).

We now recall the following types of convergence introduced in [**4**], which were generalized using the notion of ideals in [**9**]. Let X be a non-empty set and let f, f_n, $n = 1, 2, 3, \dots$ be real-valued functions defined on X. f is called the discrete limit of the sequence $\{f_n\}_{n\in\mathbb{N}}$ if for every $x \in X$, there exists $n_0 = n_0(x)$ such that $f(x) = f_n(x)$ for $n \geq n_0$. The terminology is motivated by the fact that this condition means precisely the convergence of the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ to $f(x)$ with respect to the discrete topology of the real line. A function f is said to be the equal limit of the sequence $\{f_n\}_{n\in\mathbb{N}}$ if there exists a sequence of positive reals $\{\varepsilon_n\}_{n\in\mathbb{N}}$ tending to zero such that for every $x \in X$, there exists $n_0 = n_0(x)$ with $|f_n(x) - f(x)| < \varepsilon_n$ for $n \geq n_0$.

We also recall the following ideas of convergence of a sequence of functions using the notion of ideals from [**1**]. A sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions is said to be $\mathcal{I}$-pointwise convergent to f if for all $x \in X$ the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent to $f(x)$ and in this case we write $f_n \xrightarrow{\mathcal{I}} f$. The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be $\mathcal{I}$-uniformly convergent to f if for any $\varepsilon > 0$ there exists $A \in \mathcal{I}$ such that for all $n \in A^c$ and for all $x \in X$, $|f_n(x) - f(x)| < \varepsilon$. A function f is said to be the $\mathcal{I}^*$-uniform limit of $\{f_n\}_{n\in\mathbb{N}}$ if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \dots\} \in \mathcal{F}(\mathcal{I})$ such that f is the uniform limit of the subsequence $\{f_{m_k}\}_{k\in\mathbb{N}}$.

3. Main results

Throuhout the paper, we consider $\mathcal{I}$ an admissible ideal.

We start by recalling the following definition from the recent work of Das, Dutta, and Pal [**9**].

Definition 3.1. A sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be $\mathcal{I}$-equally convergent to f if there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive reals with $\mathcal{I}-\lim_n \varepsilon_n = 0$ such that for any $x \in X$, $\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$. In this case, we write $f_n \xrightarrow{\mathcal{I}-e} f$.

Example 3.1. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ and $\mathcal{I} \neq$ Fin, the ideal of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}$ must contain an infinite set A. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions on X defined by

$$f_n = \begin{cases} 0 & \text{for all } n \in \mathbb{N} \setminus A, \\ 1 & \text{for all } n \in A. \end{cases}$$

Thus for any $x \in X$, $\{n \in \mathbb{N} : |f_n(x)| > 1/n\} \subseteq A$. If we let $f \equiv 0$, we have $f_n \xrightarrow{\mathcal{I}-e} f$ but clearly $\{f_n\}_{n\in\mathbb{N}}$ does not equally converge to f.

We first present the following equivalent condition for $\mathcal{I}$-equal convergence.

Theorem 3.2. *Let f_n, $f\colon X \to \mathbb{R}$, $n \in \mathbb{N}$. Then $f_n \xrightarrow{\mathcal{I}-e} f$ if and only if there exists a sequence $\{\rho_n\}_{n\in\mathbb{N}}$ of positive integers $\mathcal{I}$-divergent to ∞ such that*

$$\rho_n|f_n - f| \xrightarrow{\mathcal{I}-e} 0.$$

Proof. First suppose that $f_n \xrightarrow{\mathcal{I}-e} f$. Then there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive reals with $\mathcal{I} - \lim_n\ \varepsilon_n = 0$ such that for any $x \in X$,

$$\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}. \tag{1}$$

Now define a sequence $\{\rho_n\}_{n\in\mathbb{N}}$ as

$$\rho_n = \frac{1}{\sqrt{\varepsilon_n}}, \qquad n \in \mathbb{N}.$$

Obviously $\{\rho_n\}_{n\in\mathbb{N}}$ is a sequence of reals which is $\mathcal{I}$-divergent to ∞. Hence, from (1) for any $x \in X$,

$$\{n \in \mathbb{N} : \rho_n|f_n(x) - f(x)| \geq \sqrt{\varepsilon}_n\} \in \mathcal{I}$$

which implies $\rho_n|f_n - f| \xrightarrow{\mathcal{I}-e} 0$.

Conversely, assume that $\rho_n|f_n - f| \xrightarrow{\mathcal{I}-e} 0$, where $\{\rho_n\}_{n\in\mathbb{N}}$ is a sequence of positive integers $\mathcal{I}$-divergent to ∞. Then we can find a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of positive reals satisfying $\mathcal{I} - \lim_n\ \lambda_n = 0$ such that for any $x \in X$, $\{n \in \mathbb{N} : \rho_n|f_n(x) - f(x)| \geq \lambda_n\} \in \mathcal{I}$. Let us define a sequence $\{\theta_n\}_{n\in\mathbb{N}}$ by

$$\theta_n = \frac{\lambda_n}{\rho_n}, \qquad n \in \mathbb{N}.$$

Then $\mathcal{I}-\lim_n \theta_n = 0$ and for any $x \in X$, $\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \theta_n\} \in \mathcal{I}$. This completes the proof of the theorem. □

Theorem 3.3. *Let $f_n \colon X \to \mathbb{R}$, $n \in \mathbb{N}$. If $f_n \xrightarrow{\mathcal{I}-e} 0$, then $f_n^2 \xrightarrow{\mathcal{I}-e} 0$.*

Proof. There exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive reals with $\mathcal{I}-\lim_n \varepsilon_n = 0$ such that for any $x \in X$,

$$\{n \in \mathbb{N} : |f_n(x)| \geq \varepsilon_n\} \in \mathcal{I}.$$

Then we have

$$\{n \in \mathbb{N} : |f_n(x)|^2 \geq \varepsilon_n^2\} \in \mathcal{I} \qquad \text{for any } x \in X,$$

and so

$$\{n \in \mathbb{N} : |f_n^2(x)| \geq \varepsilon_n^2\} \in \mathcal{I} \qquad \text{for any } x \in X.$$

Therefore, $f_n^2 \xrightarrow{\mathcal{I}-e} 0$. □

Let Φ be an arbitrary class of functions defined on a non-empty set X. We denote by $\Phi^{\mathcal{I}-e}$, the class of all functions defined on X, which are $\mathcal{I}$-equal limits of sequences of functions belonging to Φ. For any class of functions Φ on X, we first recall the following definitions from [**5**].

Definition 3.4.

(a) Φ is called a lattice if Φ contains all constants and $f, g \in \Phi$ implies $\max(f, g) \in \Phi$ and $\min(f, g) \in \Phi$.
(b) Φ is called a translation lattice if it is a lattice and $f \in \Phi, c \in \mathbb{R}$ implies $f + c \in \Phi$.
(c) Φ is called a congruence lattice if it is a translation lattice and $f \in \Phi$ implies $-f \in \Phi$.
(d) Φ is called a weakly affine lattice if it is a congruence lattice and there is a set $C \subset (0, \infty)$ such that C is not bounded and $f \in \Phi$, $c \in C$ implies $cf \in \Phi$.
(e) Φ is called an affine lattice if it is a congruence lattice and $f \in \Phi$, $c \in \mathbb{R}$ implies $cf \in \Phi$.
(f) Φ is called a subtractive lattice if it is a lattice and $f, g \in \Phi$ implies $f - g \in \Phi$.
(g) Φ is called an ordinary class if it is a subtractive lattice, $f, g \in \Phi$ implies $f.g \in \Phi$ and $f \in \Phi$, $f(x) \neq 0$ for all $x \in X$, implies $1/f \in \Phi$.

Theorem 3.5. *Let Φ be a class of functions on X. If Φ is a lattice, a translation lattice, a congruence lattice, a weakly affine lattice, an affine lattice, or a subtractive lattice, then so is $\Phi^{\mathcal{I}-e}$.*

Proof. Let Φ be a lattice. Since Φ contains the constant functions, $\Phi^{\mathcal{I}-e}$ also contains the constant functions. Let $f_n \xrightarrow{\mathcal{I}-e} f$. Then there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive reals with $\mathcal{I}-\lim_n \varepsilon_n = 0$ such that for any $x \in X$, $\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$. Now from the relation $\big||f_n|(x) - |f|(x)\big| \leq \big|f_n(x) - f(x)\big|$, it immediately follows that $\{n \in \mathbb{N} : \big||f_n|(x) - |f|(x)\big| \geq \varepsilon_n\} \in \mathcal{I}$ for any $x \in X$, i.e., $|f_n| \xrightarrow{\mathcal{I}-e} |f|$.

Next we show that if $f_n \xrightarrow{\mathcal{I}-e} f$, $g_n \xrightarrow{\mathcal{I}-e} g$, and $\alpha, \beta \in \mathbb{R}$, then $\alpha f_n + \beta g_n \xrightarrow{\mathcal{I}-e} \alpha f + \beta g$. There exist sequences $\{\varepsilon_n\}_{n\in\mathbb{N}}$ and $\{\lambda_n\}_{n\in\mathbb{N}}$ of positive reals with $\mathcal{I} - \lim_n \varepsilon_n = 0$, $\mathcal{I} - \lim_n \lambda_n = 0$ such that for any $x \in X$,

$$A_x = \{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$$

and

$$B_x = \{n \in \mathbb{N} : |g_n(x) - g(x)| \geq \lambda_n\} \in \mathcal{I}.$$

We can assume that $\alpha \neq 0$ or $\beta \neq 0$. Let $\theta_n = |\alpha|\varepsilon_n + |\beta|\lambda_n$. Hence we have for any $x \in X$,

$$\{n \in \mathbb{N} : |\alpha(f_n - f)(x) + \beta(g_n - g)(x)| \geq \theta_n\} \subseteq A_x \cup B_x \in \mathcal{I}$$

with $\mathcal{I} - \lim_n \theta_n = 0$. Hence $\alpha f_n + \beta g_n \xrightarrow{\mathcal{I}-e} \alpha f + \beta g$.

Next observe that if $f, g \in \Phi^{\mathcal{I}-e}$, $f_n \xrightarrow{\mathcal{I}-e} f$, and $g_n \xrightarrow{\mathcal{I}-e} g$, then in view of the above

$$\max(f_n, g_n) = \frac{f_n + g_n}{2} + \frac{|f_n - g_n|}{2} \xrightarrow{\mathcal{I}-e} \frac{f + g}{2} + \frac{|f - g|}{2} = \max(f, g),$$

which implies that $\max(f, g) \in \Phi^{\mathcal{I}-e}$. Similarly, we can show that $\min(f, g) \in \Phi^{\mathcal{I}-e}$. Thus $\Phi^{\mathcal{I}-e}$ is a lattice. The proofs of the remaining assertions are straightforward. □

Theorem 3.6. *Let Φ be an ordinary class of functions on X and $f, g \in \Phi^{\mathcal{I}-e}$. If f and g are bounded, then $f.g \in \Phi^{\mathcal{I}-e}$.*

Proof. As $f, g \in \Phi^{\mathcal{I}-e}$, there exist sequences $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ in Φ such that $f_n \xrightarrow{\mathcal{I}-e} f$ and $g_n \xrightarrow{\mathcal{I}-e} g$. Consequently, we can find two sequences of positive reals $\{\varepsilon_n\}_{n\in\mathbb{N}}$ and $\{\lambda_n\}_{n\in\mathbb{N}}$ with $\mathcal{I} - \lim_n \varepsilon_n = 0$, $\mathcal{I} - \lim_n \lambda_n = 0$ such that for any $x \in X$,

$$A_x = \{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$$

and

$$B_x = \{n \in \mathbb{N} : |g_n(x) - g(x)| \geq \lambda_n\} \in \mathcal{I}.$$

Since f, g are bounded, there exist $M > 0$ and $K > 0$ such that $|f(x)| \leq M$ and $|g(x)| \leq K$ for all $x \in X$.

Now observe that

$$\begin{aligned}
\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I} &\implies \{n \in \mathbb{N} : \big||f_n(x)| - |f(x)|\big| \geq \varepsilon_n\} \in \mathcal{I} \\
&\implies \{n \in \mathbb{N} : |f_n(x)| - |f(x)| \geq \varepsilon_n\} \in \mathcal{I} \\
&\implies \{n \in \mathbb{N} : |f_n(x)| \geq |f(x)| + \varepsilon_n\} \in \mathcal{I} \\
&\implies \{n \in \mathbb{N} : |f_n(x)| \geq |f(x)| + 1\} \in \mathcal{I} \\
&\qquad (as\ \mathcal{I} - \lim_n \varepsilon_n = 0) \\
&\implies \{n \in \mathbb{N} : |f_n(x)| \geq M + 1\} \in \mathcal{I}.
\end{aligned}$$

Take $P_x = \{n \in \mathbb{N} : |f_n(x)| \geq M+1\}$. To show that $f_n \cdot g_n \xrightarrow{\mathcal{I}-e} f.g$, again observe that for $x \in X$,

$$\begin{aligned}|f_n \cdot g_n(x) - f \cdot g(x)| &= |f_n(x) \cdot g_n(x) - f(x) \cdot g(x)| \\ &= |f_n(x) \cdot g_n(x) - f_n(x) \cdot g(x) + f_n(x) \cdot g(x) - f(x) \cdot g(x)| \\ &\leq |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)|.\end{aligned}$$

Let $\gamma_n = (M+1)\lambda_n + K\varepsilon_n$. It is clear that $\mathcal{I} - \lim_n \gamma_n = 0$ and for $x \in X$,

$$\begin{aligned}&\{n \in \mathbb{N} : |f_n.g_n(x) - f.g(x)| \geq \gamma_n\} \\ &\subseteq \{n \in \mathbb{N} : |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)| \geq \gamma_n\} \\ &\subseteq A_x \cup B_x \cup P_x \in \mathcal{I},\end{aligned}$$

as for any $n \in (A_x \cup B_x \cup P_x)^c = A_x^c \cap B_x^c \cap P_x^c$, we must have

$$\begin{aligned}n \in A_x^c &\implies |f_n(x) - f(x)| < \varepsilon_n, \\ n \in B_x^c &\implies |g_n(x) - g(x)| < \lambda_n, \\ n \in P_x^c &\implies |f_n(x)| < M+1\end{aligned}$$

and so $|f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)| < (M+1)\lambda_n + K\varepsilon_n = \gamma_n$.

This shows that $\{n \in \mathbb{N} : |f_n \cdot g_n(x) - f.g(x)| \geq \gamma_n\} \in \mathcal{I}$, and so $f \cdot g \in \Phi^{\mathcal{I}-e}$. □

Theorem 3.7. *Let Φ be an ordinary class of functions on X. Let $f \in \Phi^{\mathcal{I}-e}$ and $f(x) \neq 0$ for each $x \in X$. If $\frac{1}{f}$ is bounded on X then, $\frac{1}{f} \in \Phi^{\mathcal{I}-e}$.*

Proof. As $f \in \Phi^{\mathcal{I}-e}$, there exists $\{f_n\}_{n\in\mathbb{N}}$ in Φ such that $f_n \xrightarrow{\mathcal{I}-e} f$, i.e., there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive reals with $\mathcal{I} - \lim_n \varepsilon_n = 0$ such that for any $x \in X$,

$$A_x = \{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}.$$

As $1/f$ is bounded, there exists $\lambda > 0$ with $1/|f(x)| \leq \lambda$ for all $x \in X$.

Let us define $g_n \colon X \longrightarrow \mathbb{R}$ by $g_n = \max(f_n, \sqrt{\varepsilon_n})$ for all $n \in \mathbb{N}$. Then, $g_n \in \Phi$ and $g_n(x) \geq \sqrt{\varepsilon_n}$ for all $n \in \mathbb{N}$ and $x \in X$. So for any $x \in X$,

$$\begin{aligned}&\{n \in \mathbb{N} : |g_n(x) - f(x)| \geq \varepsilon_n\} \\ &= \{n \in \mathbb{N} : g_n(x) = f_n(x) \wedge |g_n(x) - f(x)| \geq \varepsilon_n\} \\ &\qquad \cup \{n \in \mathbb{N} : g_n(x) = \sqrt{\varepsilon_n} \wedge |g_n(x) - f(x)| \geq \varepsilon_n\} \\ &\subseteq A_x \cup \{n \in \mathbb{N} : g_n = \sqrt{\varepsilon_n} \wedge g_n(x) - f(x) \geq \varepsilon_n\} \\ &\qquad \cup \{n \in \mathbb{N} : g_n(x) = \sqrt{\varepsilon_n} \wedge -g_n(x) + f(x) \geq \varepsilon_n\} \\ &\subseteq A_x \cup \{n \in \mathbb{N} : f(x) \leq \sqrt{\varepsilon_n} - \varepsilon_n\} \\ &\qquad \cup \{n \in \mathbb{N} : f(x) \geq \varepsilon_n + f_n(x)\} \qquad \text{(as } g_n \geq f_n \text{ for all } n\text{)}.\end{aligned}$$

We have $\{n \in \mathbb{N} : f(x) \geq \varepsilon_n + f_n(x)\} \subseteq A_x$ and as $\varepsilon_n \xrightarrow{\mathcal{I}} 0$, $\sqrt{\varepsilon_n} - \varepsilon_n \xrightarrow{\mathcal{I}} 0$, which implies that

$$D_x = \{n \in \mathbb{N} : f(x) \leq \sqrt{\varepsilon_n} - \varepsilon_n\} \in \mathcal{I}.$$

Therefore, $\{n \in \mathbb{N} : |g_n(x) - f(x)| \geq \varepsilon_n\} \subseteq A_x \cup D_x \cup A_x = A_x \cup D_x \in \mathcal{I}$.

As $\varepsilon_n \xrightarrow{\mathcal{I}} 0$, $\lambda\sqrt{\varepsilon_n} \xrightarrow{\mathcal{I}} 0$. Then for any $x \in X$,

$$\Big\{n \in \mathbb{N} : |\frac{1}{g_n(x)} - \frac{1}{f(x)}| \geq \lambda\sqrt{\varepsilon_n}\Big\} = \Big\{n \in \mathbb{N} : \frac{|g_n(x) - f(x)|}{|g_n(x)||f(x)|} \geq \lambda\frac{\varepsilon_n}{\sqrt{\varepsilon_n}}\Big\}$$
$$\subseteq A_x \cup D_x \in \mathcal{I}.$$

It now follows from above that $\frac{1}{g_n} \xrightarrow{\mathcal{I}-e} \frac{1}{f}$, and so $\frac{1}{f} \in \Phi^{\mathcal{I}-e}$. □

We now introduce the following generalization of the notion of discrete convergence [**4**].

Definition 3.8. A sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be $\mathcal{I}$-discretely convergent to f if for any $x \in X$, $\{n \in \mathbb{N} : f_n(x) \neq f(x)\} \in \mathcal{I}$. In this case, we write $f_n \xrightarrow{\mathcal{I}-d} f$.

We denote by $\Phi^{\mathcal{I}-d}$, the class of all functions defined on X, which are $\mathcal{I}$-discrete limits of sequences of functions belonging to Φ. Below we study some properties of the class $\Phi^{\mathcal{I}-d}$.

Theorem 3.9. *Let Φ be a class of functions on X. If Φ is a lattice, a translation lattice, a congruence lattice, a weakly affine lattice, an affine lattice, or a subtractive lattice, then so is $\Phi^{\mathcal{I}-d}$.*

Theorem 3.10. *Let Φ be an ordinary class of functions on X. Then $f, g \in \Phi^{\mathcal{I}-d}$ implies $f.g \in \Phi^{\mathcal{I}-d}$. Also if $f \in \Phi^{\mathcal{I}-d}$ is such that $f(x) > 0$ for each $x \in X$, then $\frac{1}{f} \in \Phi^{\mathcal{I}-d}$.*

Proof. Let $f, g \in \Phi^{\mathcal{I}-d}$. Then there exist sequences $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ in Φ such that $f_n \xrightarrow{\mathcal{I}-d} f$ and $g_n \xrightarrow{\mathcal{I}-d} g$.

So, for any $x \in X$,

$$A_x = \{n \in \mathbb{N} : f_n(x) \neq f(x)\} \in \mathcal{I}$$

and

$$B_x = \{n \in \mathbb{N} : g_n(x) \neq g(x)\} \in \mathcal{I}.$$

Then for any $x \in X$,

$$\{n \in \mathbb{N} : f_n(x) \cdot g_n(x) \neq f(x) \cdot g(x)\} \subseteq A_x \cup B_x \in \mathcal{I}$$

which shows that $f_n \cdot g_n \xrightarrow{\mathcal{I}-d} f \cdot g$ and consequently, $f \cdot g \in \Phi^{\mathcal{I}-d}$.

Let $f \in \Phi^{\mathcal{I}-d}$ be such that $f(x) > 0$ for each $x \in X$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in Φ such that $f_n \xrightarrow{\mathcal{I}-d} f$. Let $g_n = \max\{f_n, 1/n\}$ for all $n \in \mathbb{N}$. Then $g_n \in \Phi$ and $g_n \geq 1/n > 0$ for all $n \in \mathbb{N}$. Since $f_n \xrightarrow{\mathcal{I}-d} f$, then for any $x \in X$,

$$A_x = \{n \in \mathbb{N} : f_n(x) \neq f(x)\} \in \mathcal{I}.$$

We show that

$$\{n \in \mathbb{N} : g_n(x) \neq \max\{f(x), 1/n\}\} \subseteq A_x \in \mathcal{I}.$$

If $k \in \{n \in \mathbb{N} : g_n(x) \neq \max\{f(x), 1/n\}\}$, then $g_k(x) \neq \max\{f(x), 1/k\}$, which implies that $g_k(x) \neq f(x)$, otherwise if $g_k(x) = f(x)$, then it follows that $1/k > g_k(x)$, a contradiction to the definition of g_n. Thus $k \in A_x$.

Now write $B_x = \{n \in \mathbb{N} : \frac{1}{g_n(x)} \neq \frac{1}{\max\{f(x),1/n\}}\}$. Then $B_x \in \mathcal{I}$. The cardinality of $D_x = \{n \in \mathbb{N} : 1/n \geq f(x)\}$ is finite which implies $D_x \in \mathcal{I}$. Therefore, for any $x \in X$,

$$\Big\{n \in \mathbb{N} : \frac{1}{g_n(x)} \neq \frac{1}{f(x)}\Big\}$$
$$\subseteq \{n \in \mathbb{N} : g_n(x) \neq f(x) \wedge f(x) > 1/n\} \cup \{n \in \mathbb{N} : g_n(x) \neq f(x) \wedge f(x) \leq 1/n\}$$
$$\subseteq B_x \cup D_x \in \mathcal{I}.$$

Hence $\frac{1}{f} \in \Phi^{\mathcal{I}-d}$. □

Finally, we introduce the following notion of convergence for a sequence of real-valued functions which is stronger than the notion of $\mathcal{I}$-equal convergence in line of [**19**].

Definition 3.11. A sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be $\mathcal{I}$-strongly equally convergent to f if there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive reals and a set $M \in \mathcal{F}(\mathcal{I})$ with $\sum_{\substack{n=1\\n\in M}}^{\infty} \varepsilon_n < \infty$ such that for any $x \in X$, $\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$. In this case, we write $f_n \xrightarrow{\mathcal{I}-se} f$.

We denote by $\Phi^{\mathcal{I}-se}$, the class of all $\mathcal{I}$-strong equal limits of a class of functions Φ defined on X.

Example 3.2. Let $\{A_i\}_{i\in \mathbb{N}\cup\{0\}}$ be a partition of $\mathbb{N}$ defined by $A_0 = \{1, 3, 5, \dots\}$, $A_1 = \{2\}$, $A_2 = \{4\}$, $A_3 = \{6\}, \dots,$ $A_n = \{2n\}, \dots,$ and let $\mathcal{B} = \{A_i\}_{i\in\mathbb{N}\cup\{0\}}$.

Let $\mathcal{I} = \{D : D$ can be covered by finite number of members from $\mathcal{B}\}$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued functions on a set X defined by

$$f_n = \frac{1}{n+1} \text{ for all } n \in \mathbb{N}.$$

If we take $f \equiv 0$, then obviously f_n $\mathcal{I}$-equally converges to f but as for any $M \in \mathcal{F}(\mathcal{I})$ and $\{\varepsilon_n\}_{n\in\mathbb{N}}$, a sequence of positive reals such that $\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$ for any $x \in X$, we have $\sum_{\substack{n=1\\n\in M}}^{\infty} \varepsilon_n = \infty$, so f_n does not $\mathcal{I}$-strongly equally converge to f.

From Definition 3.11 and the above example, it follows that $\mathcal{I}$-strong equal convergence is stronger than $\mathcal{I}$-equal convergence. As in the case of $\mathcal{I}$-equal convergence we can now easily prove the following results.

Theorem 3.12. *Let $f_n \colon X \to \mathbb{R}$, $n \in \mathbb{N}$. If $f_n \xrightarrow{\mathcal{I}-se} 0$, then $f_n^2 \xrightarrow{\mathcal{I}-se} 0$.*

Theorem 3.13. *Let f and g be bounded on X. If $f_n \xrightarrow{\mathcal{I}-se} f$ and $g_n \xrightarrow{\mathcal{I}-se} g$, then $f_n \cdot g_n \xrightarrow{\mathcal{I}-se} f \cdot g$.*

Proof. As f and g are bounded, let $|f(x)| < N$ and $|g(x)| < K$ for some $N, K \in \mathbb{N}$ for all $x \in X$. Since $f_n \xrightarrow{\mathcal{I}-se} f$ and $g_n \xrightarrow{\mathcal{I}-se} g$, there exist sequences $\{\varepsilon_n\}_{n\in\mathbb{N}}$ and

$\{\lambda_n\}_{n\in\mathbb{N}}$ of positive reals and sets $P, Q \in \mathcal{F}(\mathcal{I})$ with $\sum\limits_{\substack{n=1\\ n\in P}}^{\infty} \varepsilon_n < \infty$, $\sum\limits_{\substack{n=1\\ n\in Q}}^{\infty} \lambda_n < \infty$ such that for any $x \in X$,

$$A_x = \{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$$

and

$$B_x = \{n \in \mathbb{N} : |g_n(x) - g(x)| \geq \lambda_n\} \in \mathcal{I}.$$

Let $\gamma_n = (N+1)\lambda_n + K\varepsilon_n$. Then we have $\sum\limits_{\substack{n=1\\ n\in P\cap Q}}^{\infty} \gamma_n < \infty$. The rest of the proof is exactly the same as in the proof of Theorem 3.6. □

Theorem 3.14. *Let Φ be a class of functions on X. If Φ is a lattice, a translation lattice, a congruence lattice, a weakly affine lattice, an affine lattice, or a subtractive lattice, then so is $\Phi^{\mathcal{I}-se}$.*

Proof. Let Φ be a lattice. Since Φ contains the constant functions, $\Phi^{\mathcal{I}-se}$ contains the constant functions. Let $f_n \xrightarrow{\mathcal{I}-se} f$. Then there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive reals and a set $M \in \mathcal{F}(\mathcal{I})$ with $\sum\limits_{\substack{n=1\\ n\in M}}^{\infty} \varepsilon_n < \infty$ such that for any $x \in X$, $\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$. Now $\big||f_n|(x) - |f|(x)\big| \leq \big|f_n(x) - f(x)\big|$ for all $x \in X$. Therefore, $\{n \in \mathbb{N} : \big||f_n|(x) - |f|(x)\big| \geq \varepsilon_n\} \in \mathcal{I}$ for any $x \in X$, i.e., $|f_n| \xrightarrow{\mathcal{I}-se} |f|$.

Next we show that if $f_n \xrightarrow{\mathcal{I}-se} f$, $g_n \xrightarrow{\mathcal{I}-se} g$, and $\alpha, \beta \in \mathbb{R}$, then $\alpha f_n + \beta g_n \xrightarrow{\mathcal{I}-se} \alpha f + \beta g$. Indeed, there exist sequences $\{\varepsilon_n\}_{n\in\mathbb{N}}$ and $\{\lambda_n\}_{n\in\mathbb{N}}$ of positive reals, and sets $M_1, M_2 \in \mathcal{F}(\mathcal{I})$ with $\sum\limits_{\substack{n=1\\ n\in M_1}}^{\infty} \varepsilon_n < \infty$, $\sum\limits_{\substack{n=1\\ n\in M_2}}^{\infty} \lambda_n < \infty$ such that for any $x \in X$,

$$A_x = \{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$$

and

$$B_x = \{n \in \mathbb{N} : |g_n(x) - g(x)| \geq \lambda_n\} \in \mathcal{I}.$$

We can assume that $\alpha \neq 0$ or $\beta \neq 0$. Let $\theta_n = |\alpha|\varepsilon_n + |\beta|\lambda_n$ for every $n \in \mathbb{N}$. Hence we have for any $x \in X$,

$$\{n \in \mathbb{N} : |\alpha(f_n - f)(x) + \beta(g_n - g)(x)| \geq \theta_n\} \subseteq A_x \cup B_x \in \mathcal{I}$$

with $\sum\limits_{\substack{n=1\\ n\in M_1\cap M_2}}^{\infty} \theta_n < \infty$. Hence $\alpha f_n + \beta g_n \xrightarrow{\mathcal{I}-se} \alpha f + \beta g$.

Next observe that if $f, g \in \Phi^{\mathcal{I}-se}$, $f_n \xrightarrow{\mathcal{I}-se} f$, and $g_n \xrightarrow{\mathcal{I}-se} g$, then in view of the above,

$$\max(f_n, g_n) = \frac{f_n + g_n}{2} + \frac{|f_n - g_n|}{2} \xrightarrow{\mathcal{I}-se} \frac{f+g}{2} + \frac{|f-g|}{2} = \max(f, g)$$

which implies that $\max(f, g) \in \Phi^{\mathcal{I}-se}$. Similarly we can show that $\min(f, g) \in \Phi^{\mathcal{I}-se}$. Thus $\Phi^{\mathcal{I}-se}$ is a lattice. The proofs of the remaining assertions are straightforward. □

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