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PERIODIC SOLUTIONS FOR A SECOND ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATION WITH ITERATIVE TERMS BY SCHAUDER'S FIXED POINT THEOREM

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Abstract. In this work, the technique of the fixed point of Schauder is applied on the second order nonlinear functional differential equation with an iterative terms

$$\begin{aligned}&\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t)+p(t)\frac{\mathrm{d}}{\mathrm{d}t}x(t)+q(t)x(t)\\&=\frac{\mathrm{d}}{\mathrm{d}t}g\left(t,x(t),x^{[2]}(t),\dots,x^{[n]}(t)\right)+f\left(t,x(t),x^{[2]}(t),\dots,x^{[n]}(t)\right)\end{aligned}$$

for the purpose of proving the existence of periodic solutions.

1. Introduction

In spite of the great importance accorded by mathematicians to the study of the existence of solutions for differential equations and despite their long history and the impressive results of these researches, the necessity required by a wide variety of mathematical models to be more realistic has drawn the attention of vast numbers of researchers to treat differential equations with deviating arguments where a large literature has been reserved for the iterative functional differential equations which is a particular type of the so-called state-dependent delay equations.

This type of equations appears in many fields such as biologic, physics, the engineering technique fields ... They have been extensively studied by many scientists. For example, they arise in the study of infection disease transmission models and describe the motion of charged particles with retarded interaction.

Recently, there has been a growing number of papers devoted to the existence of solutions for the first-order equations. Many of them are dedicated to finding some results about the exact solutions [**2**], other about the analytic solutions [**12, 14**] or the periodic solutions [**18**], etc.

There are several possible approaches to deal with these equations such as fixed point theory, Picard's successive approximation and the nonexpansive operators technique. Among them, see the works [**3, 4, 5, 6, 14, 16, 18**] where the fixed point theory approach was gained a great prominence. For instance, in 1984,

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Eder [**3**] applied the Contraction Principle to establish the existence of the unique monotone solution for the iterative differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = x(x(t)),$$

with $x(t_0) = t_0$ ($t_0 \in [-1, 1]$). Later, Fečkan [**5**] used the same principle to discuss the local solution for the generally iterative differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = f(x(x(t))),$$

with the initial value $x(0) = 0$. In 1990, by virtue of Schauder's fixed point theorem, Wang [**14**] investigated the above equation associated with $x(a) = a$, where a is an endpoint of the well-defined interval. After seven years, Ge and Mo [**6**] studied it associated with $x(t_0) = x_0$ on a given compact interval, where the endpoints of the interval are two adjacent null points of f. Next, we find the works of Zhang [**16**] on the equation

$$\frac{\mathrm{d}}{\mathrm{d}z}x(z) = x(az + bx(z) + c\frac{\mathrm{d}}{\mathrm{d}z}(z)),$$

and Egri and Rus [**4**] on the functional differential equation with parameter

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = f(t, x(t), x(x(t))) + \lambda,$$

which are based on using of Schauder's fixed point theorem in order to prove analytic solutions and some existence theorems. Also, we mention the paper of Zhao and Liu [**18**] on the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = c_1(t)x^{[1]}(t) + c_2(t)x^{[2]}(t) + \cdots + c_n(t)x^{[n]}(t),$$

where the authors used Krasnoselskii's fixed point theorem to establish the existence of periodic solutions.

There exist only few results for higher order equations in the literature. Generally, the main difficulty is that the study of this type of problem needs very difficult techniques and leads to a vast mathematical arsenal.

In this work inspired by [**18**] and the references [**2, 3, 4, 5, 6, 7, 9, 12, 14, 16, 17**], we use Schauder's and Banach fixed point theorems and some existence results of solutions of first and second order delay differential equations as [**1, 8, 10, 11, 15**] to prove the existence and uniqueness of periodic solutions of the second order nonlinear functional differential equation with iterative terms

$$\begin{aligned} &\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) + p(t)\frac{\mathrm{d}}{\mathrm{d}t}x(t) + q(t)x(t) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}g\left(t, x(t), x^{[2]}(t), \ldots, x^{[n]}(t)\right) + f\left(t, x(t), x^{[2]}(t), \ldots, x^{[n]}(t)\right), \end{aligned} \tag{1.1}$$

where $x(t) = t$, $x^{[1]}(t) = x(t)$, $x^{[2]}(t) = x(x(t)), \ldots, x^{[n]}(t) = x^{[n-1]}(x(t))$, p and q are positive continuous real-valued functions and the functions $f, g \colon \mathbb{R}^{n+1} \to \mathbb{R}$ are continuous with respect to their arguments.

This paper is organized as follows.

In order to convert the considered equation (1.1) into an integral equation and to give the proofs of Lemmas 6–8 and Theorems 2 and 3, we first start by providing some basic results about Green's function of a second order linear differential equation and the well-known Schauder*'s* fixed point theorem. We present the main results of the work in Section 3. The main idea is to use the powerful and reliable technique of the fixed point method where some conditions are established for ensuring the existence and uniqueness of periodic solutions.

2. Preliminaries

For $T > 0$ and $L, M \geq 0$, let

$$P_T = \{x \in \mathcal{C}(\mathbb{R}, \mathbb{R}) : x(t+T) = x(t+T)\},$$

equipped with the norm

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,T]} |x(t)|,$$

and

$$P_T(L, M) = \{x \in P_T : \|x\| \leq L,\ |x(t_2) - x(t_1)| \leq M|t_2 - t_1|,\ \forall t_1, t_2 \in \mathbb{R}\},$$

then $(P_T, \|\cdot\|)$ is a Banach space and $P_T(L, M)$ is a closed convex and bounded subset of P_T.

We assume that p and q are two continuous real-valued functions such that

$$p(t+T) = p(t), \qquad q(t+T) = q(t), \tag{2.1}$$

and

$$\int_0^T p(s)\mathrm{d}s > 0, \qquad \int_0^T q(s)\mathrm{d}s > 0. \tag{2.2}$$

The functions $f(t, x_1, x_2, \ldots, x_n)$ and $g(t, x_1, x_2, \ldots, x_n)$ are supposed to be periodic in t with period T and globally Lipschitz in $x_1, \ldots, x_n$, i.e,

$$\begin{aligned} f(t+T, x_1, \ldots, x_n) &= f(t, x_1, \ldots, x_n) \\ g(t+T, x_1, \ldots, x_n) &= g(t, x_1, \ldots, x_n), \end{aligned} \tag{2.3}$$

and there exist n positive constants k_1, k_2, ..., k_n and n positive constants c_1, c_2, ..., c_n such that

$$|f(t, x_1, \ldots, x_n) - f(t, y_1, \ldots, y_n)| \leq \sum_{i=1}^{n} k_i \|x_i - y_i\|, \tag{2.4}$$

and

$$|g(t, x_1, \ldots, x_n) - g(t, y_1, \ldots, y_n)| \leq \sum_{i=1}^{n} c_i \|x_i - y_i\|. \tag{2.5}$$

Lemma 1 ([**8**]). *Suppose that* (2.1) *and* (2.2) *hold and*

$$\frac{R_1\left[\exp\left(\int_0^T p(u)\mathrm{d}u\right) - 1\right]}{Q_1 T} \geq 1, \tag{2.6}$$

where

$$R_1 = \max_{t\in[0,T]} \left| \int_t^{t+T} \frac{\exp\left(\int_t^s p(u)\mathrm{d}u\right)}{\exp\left(\int_0^T p(u)\mathrm{d}u\right) - 1} q(s)\mathrm{d}s \right|$$

and

$$Q_1 = \left(1 + \exp\left(\int_0^T p(u)\mathrm{d}u\right)\right)^2 R_1^2.$$

Then there are continuous and T-periodic functions a and b such that $b(t) > 0$, $\int_0^T a(u)\mathrm{d}u > 0$, and

$$a(t) + b(t) = p(t), \qquad \frac{\mathrm{d}}{\mathrm{d}t} b(t) + a(t)b(t) = q(t) \qquad \text{for all } t \in \mathbb{R}.$$

Lemma 2 ([**15**])**.** *Suppose the conditions of Lemma 1 hold and $\phi \in P_T$. Then the equation*

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} x(t) + p(t)\frac{\mathrm{d}}{\mathrm{d}t} x(t) + q(t)x(t) = \phi(t)$$

has a T-periodic solution. Moreover, the periodic solution can be expressed as

$$x(t) = \int_t^{t+T} G(t,s)\phi(s)\mathrm{d}s,$$

where

$$\begin{aligned} G(t,s) = & \frac{\int_t^s \exp\left[\int_t^u b(v)\mathrm{d}v + \int_u^s a(v)\mathrm{d}v\right]\mathrm{d}u}{\left[\exp\left(\int_0^T a(u)\mathrm{d}u\right) - 1\right]\left[\exp\left(\int_0^T b(u)\mathrm{d}u\right) - 1\right]} \\ & + \frac{\int_s^{t+T} \exp\left[\int_t^u b(v)\mathrm{d}v + \int_u^{s+T} a(v)\mathrm{d}v\right]\mathrm{d}u}{\left[\exp\left(\int_0^T a(u)\mathrm{d}u\right) - 1\right]\left[\exp\left(\int_0^T b(u)\mathrm{d}u\right) - 1\right]}. \end{aligned} \tag{2.7}$$

Corollary 1 ([**15**])**.** *Green's function G satisfies the following properties*

$$\begin{aligned} & G(t,t+T) = G(t,t), \qquad G(t+T,s+T) = G(t,s), \\ & \frac{\partial}{\partial s} G(t,s) = a(s)G(t,s) - \frac{\exp(\int_t^s b(v)\mathrm{d}v)}{\exp(\int_0^T b(v)\mathrm{d}v) - 1}, \\ & \frac{\partial}{\partial t} G(t,s) = -b(t)G(t,s) + \frac{\exp(\int_t^s a(v)\mathrm{d}v)}{\exp(\int_0^T a(v)\mathrm{d}v) - 1}. \end{aligned} \tag{2.8}$$

Theorem 1 (Schauder [**13**])**.** *Let M be a closed convex and bounded subset of a Banach space $(X, \|\cdot\|)$ and let $A\colon M \to M$ be a compact operator, then A has a fixed point.*

Lemma 3. *Suppose* (2.1)–(2.3) *and* (2.6) *hold. If $x \in P_T(L,M)$, then x is a solution of* (1.1) *if and only if*

$$\begin{aligned} x(t) = & \int_t^{t+T} \left[E(t,s) - a(s)G(t,s)\right] g\left(s, x(s), x^{[2]}(s), \ldots, x^{[n]}(s)\right)\mathrm{d}s \\ & + \int_t^{t+T} G(t,s) f\left(s, x(s), x^{[2]}(s), \ldots, x^{[n]}(s)\right)\mathrm{d}s, \end{aligned}$$

where

$$E(t,s)=\frac{\exp(\int_t^s b(v)\mathrm{d}v)}{\exp(\int_0^T b(v)\mathrm{d}v)-1}. \tag{2.9}$$

Proof. Suppose that $x\in P_T(L,M)$ is a solution of (1.1). By Lemma 2, we have

$$\begin{aligned}x(t)=&\int_t^{t+T}G(t,s)\Big[\frac{\mathrm{d}}{\mathrm{d}s}g\big(s,x(s),x^{[2]}(s),\dots,x^{[n]}(s)\big)\Big]\mathrm{d}s\\&+\int_t^{t+T}G(t,s)\Big[f\big(s,x(s),x^{[2]}(s),\dots,x^{[n]}(s)\big)\Big]\,\mathrm{d}s.\end{aligned}$$

An integration by parts gives

$$\begin{aligned}&\int_t^{t+T}G(t,s)\Big[\frac{\mathrm{d}}{\mathrm{d}s}g\big(s,x(s),x^{[2]}(s),\dots,x^{[n]}(s)\big)\Big]\mathrm{d}s\\&=\Big[G(t,s)g\big(s,x(s),x^{[2]}(s),\dots,x^{[n]}(s)\big)\Big]_t^{t+T}\\&\quad-\int_t^{t+T}\Big(\frac{\mathrm{d}}{\mathrm{d}s}G(t,s)\Big)g\big(s,x(s),x^{[2]}(s),\dots,x^{[n]}(s)\big)\mathrm{d}s.\end{aligned}$$

Since

$$\Big[G(t,s)g\big(s,x(s),x^{[2]}(s),\dots,x^{[n]}(s)\big)\Big]_t^{t+T}=0,$$

from (2.8), we obtain

$$\begin{aligned}&\int_t^{t+T}G(t,s)\Big[\frac{\mathrm{d}}{\mathrm{d}s}g\big(s,x(s),x^{[2]}(s),\dots,x^{[n]}(s)\big)\Big]\mathrm{d}s\\&=\int_t^{t+T}g\big(s,x(s),x^{[2]}(s),\dots,x^{[n]}(s)\big)\,[E(t,s)-a(s)G(t,s)]\,\mathrm{d}s.\end{aligned}$$

Consequently,

$$\begin{aligned}x(t)=&\int_t^{t+T}[E(t,s)-a(s)G(t,s)]\,g\big(s,x(s),x^{[2]}(s),\dots,x^{[n]}(s)\big)\mathrm{d}s\\&+\int_t^{t+T}G(t,s)f\big(s,x(s),x^{[2]}(s),\dots,x^{[n]}(s)\big)\mathrm{d}s.\end{aligned}$$

The proof is completed. □

Lemma 4 ([**15**]). *Let* $A=\int_0^T p(u)\mathrm{d}u$ *and* $B=T^2\exp(\frac{1}{T}\int_0^T\ln(q(u)\mathrm{d}u)$. *If*

$$A^2\geq 4B, \tag{2.10}$$

then

$$\min\Big\{\int_0^T a(u)\mathrm{d}u,\int_0^T b(u)\mathrm{d}u\Big\}\geq\frac{1}{2}\big(A-\sqrt{A^2-4B}\big):=l$$

and

$$\max\Big\{\int_0^T a(u)\mathrm{d}u,\int_0^T b(u)\mathrm{d}u\Big\}\leq\frac{1}{2}\big(A+\sqrt{A^2-4B}\big):=m.$$

Corollary 2 ([**15**]). *Functions G and E satisfy*

$$\frac{T}{(\mathrm{e}^m - 1)^2} \le G(t,s) \le \frac{T \exp(\int_0^T p(u)\mathrm{d}u)}{(\mathrm{e}^l - 1)^2}, \qquad |E(t,s)| \le \frac{\mathrm{e}^m}{\mathrm{e}^l - 1}.$$

Lemma 5 ([**18**]). *For any $\varphi, \psi \in P_T(L, M)$,*

$$\left\| \varphi^{[m]} - \psi^{[m]} \right\| \le \sum_{j=0}^{m-1} M^j \left\| \varphi - \psi \right\|.$$

3. Main Results

In this section, we use Schauder's fixed point Theorem 1 to prove the existence of periodic solutions of the equation (1.1). For this and by virtue of Lemma 3, we define an operator $H\colon P_T(L, M) \to P_T$ by

$$(3.1) \qquad \begin{aligned} (H\varphi)(t) = & \int_t^{t+T} [E(t,s) - a(s)G(t,s)]\, g\left(s, \varphi(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)\right) \mathrm{d}s \\ & + \int_t^{t+T} G(t,s) f\left(s, \varphi(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)\right) \mathrm{d}s, \end{aligned}$$

and we show that it is continuous, compact on $P_T(L, M)$ and $H\varphi \in P_T(L, M)$ for all $\varphi \in P_T(L, M)$.

We denote

$$\alpha = \frac{T \exp\left(\int_0^T p(u)\mathrm{d}u\right)}{(\mathrm{e}^l - 1)^2}, \qquad \beta = \frac{\mathrm{e}^m}{\mathrm{e}^l - 1}, \qquad \gamma = \exp\Big(\int_0^T b(v)\mathrm{d}v\Big),$$

$$\delta = \frac{1}{\left[\exp\left(\int_0^T a(u)\mathrm{d}u\right) - 1\right]\left[\exp\left(\int_0^T b(u)\mathrm{d}u\right) - 1\right]},$$

$$(3.2) \qquad \lambda_1 = \max_{t \in [0,T]} |a(t)|, \qquad \lambda_2 = \max_{t \in [0,T]} |b(t)|,$$

$$\rho_1 = \max_{t \in [0,T]} |f(t, 0, 0, \dots, 0)|, \qquad \rho_2 = \max_{t \in [0,T]} |g(t, 0, 0, \dots, 0)|,$$

$$\zeta_1 = \rho_1 + L \sum_{i=1}^{n} k_i \sum_{j=0}^{j=i-1} M^j, \qquad \zeta_2 = \rho_2 + L \sum_{i=1}^{n} c_i \sum_{j=0}^{j=i-1} M^j.$$

Lemma 6. *Suppose that conditions* (2.1)–(2.6), *and* (2.10) *hold. Then the operator $H\colon P_T(L, M) \to P_T$ given by* (3.1), *is continuous and compact.*

Proof. Since $P_T(L, M)$ is a uniformly bounded and equicontinuous subset of the space of continuous functions on the compact $[0, T]$, we can apply the Ascoli-Arzela theorem to confirm that $P_T(L, M)$ is a compact subset from this space. Also since any continuous operator maps compact sets into compact sets, then to show that H is a compact operator it suffices to show that it is continuous.

For $\varphi, \theta \in P_T(L, M)$, we have

$$
\begin{aligned}
&\left|\left(H\varphi\right)(t)-\left(H\theta\right)(t)\right| \\
&\leq \int_t^{t+T}|E(t,s)|\left|g\big(s,\varphi(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\big)-g\big(s,\theta(s),\theta^{[2]}(s),\ldots,\theta^{[n]}(s)\big)\right|\mathrm{d}s \\
&\quad+\int_t^{t+T}|a(s)||G(t,s)|\left|g\big(s,\varphi(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\big)-g\big(s,\theta(s),\theta^{[2]}(s),\ldots,\theta^{[n]}(s)\big)\right|\mathrm{d}s \\
&\quad+\int_t^{t+T}|G(t,s)|\left|f\big(s,\varphi(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\big)-f\big(t,\theta(s),\theta^{[2]}(s),\ldots,\theta^{[n]}(s)\big)\right|\mathrm{d}s.
\end{aligned}
$$

By (2.4) and (2.5), Corollary 2, and notations (3.2), we obtain

$$
\left|\left(H\varphi\right)(t)-\left(H\theta\right)(t)\right| \leq \left(\beta+\alpha\lambda_1\right)T\sum_{i=1}^{n}c_i\left\|\varphi^{[i]}-\theta^{[i]}\right\|+\alpha T\sum_{i=1}^{n}k_i\left\|\varphi^{[i]}-\theta^{[i]}\right\|.
$$

From Lemma 5, it follows that

$$
\left|\left(H\varphi\right)(t)-\left(H\theta\right)(t)\right| \leq T\sum_{i=1}^{n}\left(\left(\beta+\alpha\lambda_1\right)c_i+\alpha k_i\right)\sum_{j=0}^{j=i-1}M^j\left\|\varphi-\theta\right\|,
$$

which shows that the operator H is continuous. Therefore, H is compact. □

Lemma 7. *For any* $t_1, t_2 \in \mathbb{R}$*,*

$$
\int_{t_1}^{t_1+T}\left|G\left(t_2,s\right)-G\left(t_1,s\right)\right|\mathrm{d}s \leq T\,\mathrm{e}^{2m}\,\delta\left[T\lambda_2\gamma\left(2\,\mathrm{e}^{2m}+1\right)+\mathrm{e}^{m}+1\right]\left|t_2-t_1\right|.
$$

Proof. Let $t_1, t_2 \in \mathbb{R}$, we have

$$
\begin{aligned}
&\left|G\left(t_2,s\right)-G\left(t_1,s\right)\right| \\
&\leq \frac{\left|\int_{t_2}^{s}\exp\left[\int_{t_2}^{u}b(v)\mathrm{d}v+\int_u^s a(v)\mathrm{d}v\right]\mathrm{d}u-\int_{t_1}^{s}\exp\left[\int_{t_1}^{u}b(v)\mathrm{d}v+\int_u^s a(v)\mathrm{d}v\right]\mathrm{d}u\right|}{\left[\exp\left(\int_0^T a(u)\mathrm{d}u\right)-1\right]\left[\exp\left(\int_0^T b(u)\mathrm{d}u\right)-1\right]} \\
&\quad+\frac{\left|\int_s^{t_2+T}\exp\left[\int_{t_2}^{u}b(v)\mathrm{d}v+\int_u^{s+T}a(v)\mathrm{d}v\right]\mathrm{d}u\right.}{\left[\exp\left(\int_0^T a(u)\mathrm{d}u\right)-1\right]\left[\exp\left(\int_0^T b(u)\mathrm{d}u\right)-1\right]} \\
&\quad-\frac{\left.\int_s^{t_1+T}\exp\left[\int_{t_1}^{u}b(v)\mathrm{d}v+\int_u^{s+T}a(v)\mathrm{d}v\right]\mathrm{d}u\right|}{\left[\exp\left(\int_0^T a(u)\mathrm{d}u\right)-1\right]\left[\exp\left(\int_0^T b(u)\mathrm{d}u\right)-1\right]}.
\end{aligned}
$$

But

$$\Big|\int_{t_2}^{s}\exp\Big[\int_{t_2}^{u}b(v)\mathrm{d}v+\int_{u}^{s}a(v)\mathrm{d}v\Big]\mathrm{d}u-\int_{t_1}^{s}\exp\Big[\int_{t_1}^{u}b(v)\mathrm{d}v+\int_{u}^{s}a(v)\mathrm{d}v\Big]\mathrm{d}u\Big|$$

$$=\Big|\int_{t_2}^{s}\exp\Big[\int_{u}^{s}a(v)\mathrm{d}v\Big]\exp\Big[\int_{s}^{u}b(v)\mathrm{d}v\Big]\Big(\exp\Big[\int_{t_2}^{s}b(v)\mathrm{d}v\Big]-\exp\Big[\int_{t_1}^{s}b(v)\mathrm{d}v\Big]\Big)\mathrm{d}u$$

$$-\int_{t_1}^{t_2}\Big(\exp\Big[\int_{t_1}^{u}b(v)\mathrm{d}v\Big]\exp\Big[\int_{u}^{s}a(v)\mathrm{d}v\Big]\Big)\mathrm{d}u\Big|$$

$$\leq\Big|\exp\Big[\int_{t_2}^{s}b(v)\mathrm{d}v\Big]-\exp\Big[\int_{t_1}^{s}b(v)\mathrm{d}v\Big]\int_{0}^{T}\Big|\exp\Big[\int_{0}^{T}a(v)\mathrm{d}v\Big]\exp\Big[\int_{0}^{T}b(v)\mathrm{d}v\Big]\Big|\mathrm{d}u$$

$$+\int_{t_1}^{t_2}\Big|\exp\Big[\int_{0}^{T}b(v)\mathrm{d}v\Big]\exp\Big[\int_{0}^{T}a(v)\mathrm{d}v\Big]\Big|\mathrm{d}u\Big|$$

$$\leq T\,\mathrm{e}^{2m}\Big|\exp\Big[\int_{t_2}^{s}b(v)\mathrm{d}v\Big]-\exp\Big[\int_{t_1}^{s}b(v)\mathrm{d}v\Big]\Big|+\mathrm{e}^{2m}\Big|t_2-t_1\Big|.$$

Since

$$\int_{t_1}^{t_1+T}\Big|\exp\Big[\int_{t_2}^{s}b(v)\mathrm{d}v\Big]-\exp\Big[\int_{t_1}^{s}b(v)\mathrm{d}v\Big]\Big|\mathrm{d}s$$

$$=\int_{t_1}^{t_1+T}\exp\Big[\int_{t_2}^{s}b(v)\mathrm{d}v\Big]\Big|1-\exp\Big[\int_{t_1}^{t_2}b(v)\mathrm{d}v\Big]\Big|\mathrm{d}s\leq T\|b\|\exp\Big(\int_{0}^{T}b(v)\mathrm{d}v\Big)\Big|t_2-t_1\Big|,$$

we obtain

$$\begin{aligned}&\int_{t_1}^{t_1+T}\Big|\int_{t_2}^{s}\exp\Big[\int_{t_2}^{u}b(v)\mathrm{d}v+\int_{u}^{s}a(v)\mathrm{d}v\Big]\mathrm{d}u\\&-\int_{t_1}^{s}\exp\Big[\int_{t_1}^{u}b(v)\mathrm{d}v+\int_{u}^{s}a(v)\mathrm{d}v\Big]\mathrm{d}u\Big|\mathrm{d}s\\&\leq T\,\mathrm{e}^{2m}\Big(T\|b\|\exp\Big(\int_{0}^{T}b(v)\mathrm{d}v\Big)+1\Big)|t_2-t_1|.\end{aligned}\tag{3.3}$$

Similarly,

$$\Big|\int_{s}^{t_2+T}\exp\Big[\int_{t_2}^{u}b(v)\mathrm{d}v\Big]\exp\Big[\int_{u}^{s+T}a(v)\mathrm{d}v\Big]\mathrm{d}u-\int_{s}^{t_1+T}\exp\Big[\int_{t_1}^{u}b(v)\mathrm{d}v\Big]\exp\Big[\int_{u}^{s+T}a(v)\mathrm{d}v\Big]\mathrm{d}u\Big|$$

$$\leq\Big|\exp\Big[\int_{t_2}^{s}b(v)\mathrm{d}v\Big]-\exp\Big[\int_{t_1}^{s}b(v)\mathrm{d}v\Big]\Big|2T\,\mathrm{e}^{4m}+\mathrm{e}^{3m}\,|t_2-t_1|.$$

Therefore,

$$\begin{aligned}&\int_{t_1}^{t_1+T}\Big|\int_{s}^{t_2+T}\exp\Big[\int_{t_2}^{u}b(v)\mathrm{d}v\Big]\exp\Big[\int_{u}^{s+T}a(v)\mathrm{d}v\Big]\mathrm{d}u\\&-\int_{s}^{t_1+T}\exp\Big[\int_{t_1}^{u}b(v)\mathrm{d}v\Big]\exp\Big[\int_{u}^{s+T}a(v)\mathrm{d}v\Big]\mathrm{d}u\Big|\mathrm{d}s\\&\leq T\Big(2T\,\mathrm{e}^{4m}\,\|b\|\exp\Big(\int_{0}^{T}b(v)\mathrm{d}v\Big)+\mathrm{e}^{3m}\Big)|t_2-t_1|.\end{aligned}\tag{3.4}$$

Thus, it follows from (3.3) and (3.4) that

$$\int_{t_1}^{t_1+T} |G(t_2,s)-G(t_1,s)|\,\mathrm{d}s$$
$$\le T\frac{2T\,\mathrm{e}^{4m}\,\|b\|\exp\big(\int_0^T b(v)\mathrm{d}v\big)+T\,\mathrm{e}^{3m}+\mathrm{e}^{2m}\,T\|b\|\exp\big(\int_0^T b(v)\mathrm{d}v\big)+\mathrm{e}^{2m}}{\big[\exp\big(\int_0^T a(u)\mathrm{d}u\big)-1\big]\big[\exp\big(\int_0^T b(u)\mathrm{d}u\big)-1\big]}|t_2-t_1|$$
$$\le Te^{2m}\delta\left[T\lambda_2\gamma(2\,\mathrm{e}^{2m}+1)+\mathrm{e}^{m}+1\right]|t_2-t_1|.$$

The proof is completed. ∎

Lemma 8. *Suppose that conditions* (2.4) *and* (2.5) *hold. If*

$$T(\beta+\alpha\lambda_1)\zeta_1+T\alpha\zeta_2\le L \tag{3.5}$$

and

$$\left(2\alpha+T\,\mathrm{e}^{2m}\,\delta\left(T\lambda_2\gamma\left(2\,\mathrm{e}^{2m}+1\right)+\mathrm{e}^{m}+1\right)\right)(\lambda_1\zeta_2+\zeta_1)+(2\beta+T\lambda_2\beta)\,\zeta_2\le M, \tag{3.6}$$

then $H\left(P_T(L,M)\right)\subset P_T(L,M)$.

Proof. Let $\varphi\in P_T(L,M)$. For having $H\varphi\in P_T(L,M)$, we show that $H\varphi\in P_T$, $\|H\varphi\|\le L$ and $|(H\varphi)(t_2)-(H\varphi)(t_1)|\le M\,|t_2-t_1|$ for all $t_1,t_2\in\mathbb{R}$. It is easy to show that $(H\varphi)(t+T)=(H\varphi)(t)$. By Corollary 2 and notations (3.2), we have

$$|(H\varphi)(t)|\le(\beta+\alpha\lambda_1)\int_t^{t+T}\left|g\big(s,\varphi(s),\varphi^{[2]}(s),\dots,\varphi^{[n]}(s)\big)\right|\mathrm{d}s$$
$$+\alpha\int_t^{t+T}\left|f\big(s,\varphi(s),\varphi^{[2]}(s),\dots,\varphi^{[n]}(s)\big)\right|\mathrm{d}s.$$

From conditions (2.3), (2.4) and Lemma 5, we have

$$\left|f\big(s,\varphi(s),\varphi^{[2]}(s),\dots,\varphi^{[n]}(s)\big)\right|$$
$$\le\left|f\big(s,\varphi(s),\varphi^{[2]}(s),\dots,\varphi^{[n]}(s)\big)-f(s,0,0,\dots,0)\right|+|f(s,0,0,\dots,0)|$$
$$\le\rho_1+\sum_{i=1}^{n}k_i\sum_{j=0}^{j=i-1}M^j\,\|\varphi\|\le\rho_1+L\sum_{i=1}^{n}k_i\sum_{j=0}^{j=i-1}M^j=\zeta_1$$

and

$$\left|g\big(s,\varphi(s),\varphi^{[2]}(s),\dots,\varphi^{[n]}(s)\big)\right|$$
$$\le\left|g\big(s,\varphi(s),\varphi^{[2]}(s),\dots,\varphi^{[n]}(s)\big)-g(s,0,0,\dots,0)\right|+|g(s,0,0,\dots,0)|$$
$$\le\rho_2+\sum_{i=1}^{n}c_i\sum_{j=0}^{j=i-1}M^j\,\|\varphi\|\le\rho_2+L\sum_{i=1}^{n}c_i\sum_{j=0}^{j=i-1}M^j=\zeta_2.$$

So

$$|(H\varphi)(t)|\le T(\beta+\alpha\lambda_1)\zeta_1+T\alpha\zeta_2.$$

Therefore, from (3.5), we obtain

$$\|H\varphi\|\le L.$$

Let $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$, we have

$$\begin{aligned}
&\Big| (H\varphi)(t_2) - (H\varphi)(t_1) \Big| \\
&\leq \Big| \int_{t_2}^{t_2+T} E(t_2, s) g\big(s, \varphi(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\big) \mathrm{d}s \\
&\quad - \int_{t_1}^{t_1+T} E(t_1, s) g\big(s, \varphi(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\big) \mathrm{d}s \Big| \\
&\quad + \Big| \int_{t_2}^{t_2+T} a(s) G(t_2, s) g\big(s, \varphi(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\big) \mathrm{d}s \\
&\quad - \int_{t_1}^{t_1+T} a(s) G(t_1, s) g\big(s, \varphi(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\big) \mathrm{d}s \Big| \\
&\quad + \Big| \int_{t_2}^{t_2+T} G(t_2, s) f\big(s, \varphi(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\big) \mathrm{d}s \\
&\quad - \int_{t_1}^{t_1+T} G(t_1, s) f\big(s, \varphi(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\big) \mathrm{d}s \Big|
\end{aligned}$$

and

$$\begin{aligned}
&\Big| \int_{t_2}^{t_2+T} E(t_2, s) g\big(s, \varphi(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\big) \mathrm{d}s \\
&\qquad - \int_{t_1}^{t_1+T} E(t_1, s) g\big(s, \varphi(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\big) \mathrm{d}s \Big| \\
&\leq \int_{t_2}^{t_1} |E(t_2, s)| \big| g\big(s, \varphi(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\big) \big| \mathrm{d}s \\
&\qquad + \int_{t_1+T}^{t_2+T} |E(t_2, s)| t \big| g\big(s, \varphi(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\big) \big| \mathrm{d}s \\
&\qquad + \int_{t_1}^{t_1+T} |E(t_2, s) - E(t_1, s)| \big| g\big(s, \varphi(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\big) \big| \mathrm{d}s \\
&\leq (2\beta + T\lambda_2\beta)\, \zeta_2 |t_2 - t_1|.
\end{aligned} \tag{3.7}$$

Also

$$\begin{aligned}
&\Big| \int_{t_2}^{t_2+T} a(s) G(t_2, s) g\big(s, \varphi(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\big) \mathrm{d}s \\
&\qquad - \int_{t_1}^{t_1+T} a(s) G(t_1, s) g\big(s, \varphi(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\big) \mathrm{d}s \Big| \\
&\leq \Big| \int_{t_2}^{t_1} a(s) G(t_2, s) g\big(s, \varphi(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\big) \mathrm{d}s \Big| \\
&\qquad + \Big| \int_{t_1+T}^{t_2+T} a(s) G(t_2, s) g\big(s, \varphi(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\big) \mathrm{d}s \Big| \\
&\qquad + \Big| \int_{t_1}^{t_1+T} a(s) \left[G(t_2, s) - G(t_1, s) \right] g\big(s, \varphi(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\big) \mathrm{d}s \Big|.
\end{aligned}$$

From Lemma 7, notations (3.2) and conditions (2.4)–(2.5), we obtain

$$\begin{aligned}
&\Big| \int_{t_2}^{t_2+T} a(s)G\,(t_2,s)\, g\big(s,\varphi(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\big)\mathrm{d}s \\
&\quad - \int_{t_1}^{t_1+T} a(s)G\,(t_1,s)\, g\big(s,\varphi(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\big)\mathrm{d}s\Big| \\
&\le \lambda_1\zeta_2\left(2\alpha + Te^{2m}\delta\left(T\lambda_2\gamma\left(2e^{2m}+1\right)+e^m+1\right)\right)|t_2-t_1|\,.
\end{aligned} \tag{3.8}$$

We have also

$$\begin{aligned}
&\Big| \int_{t_2}^{t_2+T} G(t_2,s)f\big(s,\varphi(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\big)\mathrm{d}s \\
&\quad - \int_{t_1}^{t_1+T} G(t_1,s)f\big(s,\varphi(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\big)\mathrm{d}s\Big| \\
&\le \Big| \int_{t_2}^{t_1} G(t_2,s)f\big(s,\varphi(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\big)\mathrm{d}s\Big| \\
&\quad + \Big| \int_{t_1+T}^{t_2+T} G(t_2,s)f\big(s,\varphi(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\big)\mathrm{d}s\Big| \\
&\quad + \Big| \int_{t_1}^{t_1+T} (G(t_2,s)-G(t_1,s))f\big(s,\varphi(s),\varphi^{[2]}(s),\ldots,\varphi^{[n](s)}\big)\mathrm{d}s\Big|.
\end{aligned}$$

From Lemma 7, notations (3.2), and conditions (2.4)–(2.5), we obtain

$$\begin{aligned}
&\Big| \int_{t_2}^{t_2+T} G(t_2,s)f\big(s,\varphi(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\big)\mathrm{d}s \\
&\quad - \int_{t_1}^{t_1+T} G(t_1,s)f\big(s,\varphi(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\big)\mathrm{d}s\Big| \\
&\le \zeta_1\big(2\alpha + T\,\mathrm{e}^{2m}\,\delta\left(T\lambda_2\gamma\left(2\,\mathrm{e}^{2m}+1\right)+\mathrm{e}^m+1\right)\big)|t_2-t_1|.
\end{aligned} \tag{3.9}$$

Thus, it follows from (3.7), (3.8), and (3.9) that

$$\begin{aligned}
&|(H\varphi)\,(t_2)-(H\varphi)\,(t_1)| \\
&\le \big(\big(2\alpha + T\,\mathrm{e}^{2m}\,\delta\left(T\lambda_2\gamma\left(2\,\mathrm{e}^{2m}+1\right)+\mathrm{e}^m+1\right)\big) \\
&\quad \times(\lambda_1\zeta_2+\zeta_1)+(2\beta+T\lambda_2\beta)\zeta_2)|t_2-t_1|.
\end{aligned}$$

From (3.6), we obtain

$$|(H\varphi)\,(t_2)-(H\varphi)\,(t_1)| \le M\,|t_2-t_1|\,. \qquad \square$$

Theorem 2. *Suppose* (2.1)–(2.6), (3.5), *and* (3.6) *hold, then equation* (1.1) *has a solution* $x \in P_T(L,M)$.

Proof. From lemma 3, we see that equation (1.1) has a solution x on $P_T(L,M)$ if and only if the operator H defined by (3.1) has a fixed point.

From Lemma 6 and Lemma 8, all the conditions of Schauder's Theorem 1 are satisfied. Consequently, H has a fixed point on $P_T(L,M)$ and this fixed point is a solution of equation (1.1). □

Theorem 3. *Suppose* (2.1)–(2.5) *and* (2.10) *hold. If*

$$T\sum_{i=1}^{n}\left(\left(\beta+\alpha\lambda_1\right)c_i+\alpha k_i\right)\sum_{j=0}^{j=i-1}M^j<1, \tag{3.10}$$

then equation (1.1) *has a unique solution* $x\in P_T(L,M)$.

Proof. Let $\varphi,\theta\in P_T(L,M)$. Similarly as in the proof of Lemma 6, by using (3.10), we have

$$\begin{aligned}&\left|\left(H\varphi\right)(t)-\left(H\theta\right)(t)\right|\\&\le\left(T\sum_{i=1}^{n}\left(\left(\beta+\alpha\lambda_1\right)c_i+\alpha k_i\right)\sum_{j=0}^{j=i-1}M^j\right)\|\varphi-\theta\|.\end{aligned}$$

By the principle of contractive mapping, H has a unique fixed point on $P_T(L,M)$ and in view of Lemma 1, this fixed point is a solution of equation (1.1). □

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