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# EXISTENCE RESULTS FOR SYSTEMS OF SECOND-ORDER IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract. In this paper the authors study the existence of solutions to systems of nonlinear second order impulsive differential equations. Their results are established by using vector versions of Perov's fixed point theorem and the nonlinear alternative of Leray-Schauder type. Both approaches are combined with a technique based on vector-valued metrics and matrices that converge to zero. Examples illustrating the results are included.

## 1. Introduction

The theory of impulsive differential equations describes processes that experience a sudden change in their state at certain moments. Processes with such a character arise naturally and often, especially in phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. There has been a significant development in the theory of IDE; see for example the books [**1, 6, 7, 9, 20, 26**] and the papers [**4, 5, 14, 15, 21, 25, 27, 28, 29**].

We are concerned with the existence and uniqueness of solutions of the system of nonlinear second-order singular and impulsive differential equations with two boundary conditions

$$\begin{cases} -u_1''(t) = f_1(t, u_1(t), u_2(t)), & t \in J', \\ -u_2''(t) = f_2(t, u_1(t), u_2(t)), & t \in J', \\ -\Delta u_1' \mid_{t=t_k} = I_{1,k} u_1(t_k), & k = 1, 2, \ldots, m, \\ -\Delta u_2' \mid_{t=t_k} = I_{2,k} u_2(t_k), & k = 1, 2, \ldots, m, \\ \alpha u_1(0) - \beta u_1'(0) = 0, \quad \alpha u_2(0) - \beta u_2'(0) = 0, & \\ \gamma u_1(1) + \delta u_1'(1) = 0, \quad \gamma u_2(1) + \delta u_2'(1) = 0, & \end{cases} \tag{1.1}$$

where $\alpha$, $\beta$, $\gamma$, $\delta \geq 0$, $\rho = \beta\gamma + \alpha\gamma + \alpha\delta > 0$, $J = [0, 1]$, $0 < t_1 < t_2 < \cdots < t_m < 1$, $J' = J \setminus \{t_1, t_2, \ldots, t_m\}$, $f_i \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $I_{i,k} \in C(\mathbb{R}, \mathbb{R})$, $i = 1, 2$, $k \in \{1, 2, \cdots, m\}$, $\Delta u' \mid_{t=t_k} = u_1(t_k^+) - u_1(t_k^-)$, and $\Delta u_2' \mid_{t=t_k} = u_2(t_k^+) - u_2(t_k^-)$ in which $u_1'(t_k^+)$, $u_2'(t_k^+)$, $u_1'(t_k^-)$, and $u_2'(t_k^-)$) denote the right and left hand limits of $u_1'(t)$ and $u_2'(t)$ at $t = t_k$, respectively.

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In recent years, many authors have studied existence of solution for system of differential equations by using vector versions of fixed point theorems; see, for example, [**3, 10, 11, 12, 13, 23, 24**].

In [**3**], Bolojan and Precup studied implicit first order differential systems with nonlocal conditions by using a vector version of Krasnosel'skii's theorem, vector-valued norms, and matrices having spectral radius less than one.

In [**23**], the authors studied existence results for systems with nonlinear coupled nonlocal initial condition by using the Perov, Schauder, and Leray Schauder fixed point principles combined with a technique based on vector valued matrices that converge to zero.

Recently Zhang and Wang [**30**] studied nonlocal Cauchy problems for a class of implicit impulsive fractional relaxation differential systems by using vector versions of fixed point theorems, splitting the Lipschitz or linear growth conditions on the nonlinear terms into two parts, and then applying techniques that use convergent matrices and vector-valued norms.

We begin with some preliminary results and introduce the notion of matrices converging to zero. In Section 3, we give sufficient conditions for the existence and uniqueness of solutions to system (1.1) via an application of the Perov fixed point theorem. In Section 4 we use a non linear alternative of Leray-Schauder type to obtain additional existence results. Both of these approaches make use of convergent matrices and vector norms. Some examples are given in Section 5.

## 2. Preliminaries

In this section, we recall some concepts and notation to be used in what follows. We set $J_0 = [0, t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 1, \dots, m$, $t_{m+1} = 1$, and let $y_k$ be the restriction of the function $y$ to $J_k$. We consider the space

$$\begin{aligned} PC^2(J,\mathbb{R}) = \{y \in C([0,1],\mathbb{R}) : y_k \in C^2(J_k,\mathbb{R}),\ k = 0,\dots,m,\ \text{such that} \\ y'(t_k^-)\ \text{and}\ y'(t_k^+)\ \text{exist and satisfy}\ y'(t_k) = y'(t_k^-)\ \text{ for } k = 1,\dots,n\}. \end{aligned} \tag{2.1}$$

Let $PC^2(J,\mathbb{R}) \times PC^2(J,\mathbb{R})$ be endowed with the *vector norm* $\|\cdot\|$ defined by $\|v\| = (\|u_1\|_{PC^2}, \|u_2\|_{PC^2})$ for $v = (u_1, u_2)$, where for $x \in PC^2(J,\mathbb{R})$, we set $\|x\|_{PC^2} = \sup_{t\in J}|x(t)| + \sup_{t\in J}|x'(t)|$. It is clear that $(PC^2(J,\mathbb{R}) \times PC^2(J,\mathbb{R}), \|\cdot\|_{PC^2})$ is a generalized Banach space.

We also need the space

$$\begin{aligned} PCA(J,\mathbb{R}) = \{y \in C([0,1],\mathbb{R}) : y'_k \in AC^1(J_k,\mathbb{R}),\ k = 0,\dots,m,\ \text{such that} \\ y'(t_k^-)\ \text{and}\ y'(t_k^+)\ \text{exist and satisfy}\ y'(t_k) = y'(t_k^-)\ \text{ for }\ k = 1,\dots,n\} \end{aligned} \tag{2.2}$$

with the *vector norm* $\|\cdot\|$ defined by $\|v\| = (\|u_1\|_{PCA}, \|u_2\|_{PCA})$ for $v = (u_1, u_2)$, where for $x \in PCA(J,\mathbb{R})$, we set $\|x\|_{PCA} = \sup_{t\in J}|x(t)|$.

**Definition 2.1.** Let $X$ be a nonempty set. By a vector-valued metric on $X$ we mean a map $d\colon X \times X \to \mathbb{R}^n$ with the following properties:

(i) $d(u, v) \geq 0$ for all $u$, $v \in X$, and if $d(u, v) = 0$, then $u = v$;

(ii) $d(u, v) = d(v, u)$ for all $u$, $v \in X$;

(iii) $d(u,v) \leq d(u,w) + d(w,v)$ for all $u$, $v$, $w \in X$.

Here, if $x$, $y \in \mathbb{R}^n$, $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n)$, by $x \leq y$ we mean $x_i \leq y_i$ for $i = 1, 2, \cdots, n$.

We call the pair $(X, d)$ a *generalized metric space* with

$$d(x,y) := \begin{pmatrix} d_1(x,y) \\ \vdots \\ d_n(x,y) \end{pmatrix}.$$

Notice that $d$ is a generalized metric space on $X$ if and only if $d_i$, $i = 1, 2, \cdots, n$, are metrics on $X$. Similarly, we speak about a vector-valued norm on a linear space $X$ as being a mapping $\|\cdot\| \colon X \to \mathbb{R}^n_+$ with: $\|x\| = 0$ only for $x = 0$; $\|\lambda x\| = |\lambda|\|x\|$ for $x \in X$, $\lambda \in \mathbb{R}$; and $\|x + y\| \leq \|x\| + \|y\|$ for every $x$, $y \in X$. To any vector-valued norm $\|\cdot\|$ we can associate the vector valued metric $d(x,y) := \|x - y\|$, and we say that $(X, \|\cdot\|)$ is a *generalized Banach space* if $X$ is complete with respect to $d$.

**Definition 2.2.** A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1.

In other words, all the eigenvalues of $M$ are in the open unit disc, i.e., $|\lambda| < 1$ for every $\lambda \in \mathbb{C}$ with $\det(M - \lambda I) = 0$, where $I$ denote the identity matrix in $\mathcal{M}_{n\times n}(\mathbb{R})$.

**Theorem 2.1** ([**22**])**.** *Let $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$, the following assertions are equivalent:*

(a) *$M$ is convergent to zero;*

(b) *$M^k \to 0$ as $k \to \infty$;*

(c) *The matrix $(I - M)$ is nonsingular and*

$$(I - M)^{-1} = I + M + M^2 + \cdots + M^k + \cdots;$$

(d) *The matrix $(I - M)$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements.*

**Definition 2.3.** Let $(X, d)$ be a generalized metric space. An operator $T \colon X \to X$ is called contractive associated with the above $d$ on $X$, if there exists a convergent to zero matrix $M$ such that $d(T(x), T(y)) \leq M d(x,y)$ for all $x$, $y \in X$.

**Definition 2.4.** We say $f_i \colon [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, is an $L^1$-Carathéodory function if

1. $f_i(\cdot, x, y)$ is measurable for any $(x,y) \in \mathbb{R} \times \mathbb{R}$,
2. $f_i(t, \cdot, \cdot)$ is continuous for almost every $t \in [0,1]$,
3. for each $r_1$, $r_2 > 0$, there exists $\phi_{r_1,r_2} \in \mathrm{L}^1([0,+\infty))$ such that

$$|f(t,x,y)| \leq \Phi_{r_1,r_2}(t)$$

for all $x \in \mathbb{R}$ with $|x| \leq r_1$, $y \in \mathbb{R}$ with $|y| \leq r_2$, and almost all $t \in [0,1]$.

Next, we recall the vector version of Perov's fixed point theorem. For the proof and additional details we refer to [**17**] and [**19**].

**Theorem 2.2** (Perov's fixed point theorem)**.** *Suppose that $(X, d)$ is a complete generalized metric space and $T\colon X \to X$ is a contractive operator with Lipschitz matrix $M$. Then $T$ has a unique fixed point $u$, and for each $u_0 \in X$,*

$$d(T^k(u_0), u) \le M^k (I - M)^{-1} d(u_0, T(u_0)) \qquad \textit{where } k \in \mathbb{N}.$$

In Section 4 we make use of the following form of the nonlinear alternative of Leray-Schauder type.

**Theorem 2.3** ([**8**])**.** *Let $E$ be a Banach space, $C$ a closed, convex subset of $E$, and $U$ be an open subset of $C$ with $0 \in U$. Suppose that $N: U \to C$ is a continuous, compact (that is, $N(U)$ is a relatively compact subset of $C$) map. Then:*

(i) *Either $N$ has a fixed point in $U$, or*

(ii) *There exists $x \in \partial U$ (the boundary of $U$ in $C$) and $\lambda \in (0, 1)$ with $x = \lambda N(x)$.*

## 3. Main Results I

In this section of our paper we give sufficient conditions for the existence and uniqueness of solutions to problem (1.1) using Perov's fixed point theorem. We begin with a lemma that will aid in transforming problem (1.1) into a fixed point problem that will be used in this section as well as later in the paper.

**Lemma 3.1.** *The vector $(u_1, u_2) \in PC^2(J, \mathbb{R}) \times PC^2(J, \mathbb{R})$ is a solution of the differential system* (1.1) *if and only if*

$$\begin{cases} u_1(t) = \displaystyle\int_0^1 G(t,s) f_1(s, u_1(s), u_2(s))\mathrm{d}s + \sum_{k=1}^{m} G(t, t_k) I_{1,k}(u_1(t_k)), \\ u_2(t) = \displaystyle\int_0^1 G(t,s) f_2(s, u_1(s), u_2(s))\mathrm{d}s + \sum_{k=1}^{m} G(t, t_k) I_{2,k}(u_2(t_k)), \end{cases} \tag{3.1}$$

*where*

$$G(t,s) = \frac{1}{\rho} \begin{cases} (\gamma + \delta - \gamma t)(\beta + \alpha s), & 0 \le s \le t \le 1, \\ (\beta + \alpha t)(\gamma + \delta - \gamma s), & 0 \le t \le s \le 1. \end{cases} \tag{3.2}$$

*Proof.* Let $(u_1, u_2) \in PC^2(J, \mathbb{R}) \times PC^2(J, \mathbb{R})$ be a solution of system (1.1). It is easy to see by an integration of (1.1) that

$$u_i'(t) = u_i'(0) - \int_0^t f_i(s, u_1(s), u_2(s))\mathrm{d}s - \sum_{0<t_k<t} I_{i,k}(u_i(t_k)) \qquad \text{for } i = 1, 2. \tag{3.3}$$

Integrating again, we obtain

$$\begin{aligned} u_i(t) = u_i(0) + u_i'(0)t - \int_0^t (t-s) f_1(s, u_1(s), u_2(s))\mathrm{d}s \\ - \sum_{0<t_k<t} I_{i,k}(u_i(t_k))(t - t_k) \qquad \text{for } i = 1, 2. \end{aligned} \tag{3.4}$$

Letting $t = 1$ in (3.3) and (3.4), we have

$$u_i'(1) = u_i'(0) - \int_0^1 f_i(s, u_1(s), u_2(s))\mathrm{d}s - \sum_{k=1}^m I_{i,k}(u_i(t_k)) \qquad \text{for } i = 1,2. \tag{3.5}$$

$$\begin{aligned} u_i(1) = u_i(0) + u_i'(0) - \int_0^1 (1-s) f_i(s, u_1(s), u_2(s))\mathrm{d}s \\ - \sum_{k=1}^m I_{i,k}(u_i(t_k))(1-t_k) \qquad \text{for } i = 1,2. \end{aligned} \tag{3.6}$$

Therefore,

$$\begin{aligned} \gamma u_i(1) + \delta u_i'(1) = \gamma u_i(0) + (\gamma+\delta)u_i'(0) - \int_0^1 (\gamma+\delta-\gamma s) f_i(s, u_1(s), u_2(s))\mathrm{d}s \\ - \sum_{k=1}^m I_{i,k}(u_i(t_k))(\gamma+\delta-\gamma t_k) \qquad \text{for } i = 1,2. \end{aligned}$$

We then have

$$\alpha u_i(0) - \beta u_i'(0) = 0 \qquad \text{for } i = 1,2,$$

and

$$\gamma u_i(0) + (\gamma+\delta)u_i'(0) = \int_0^1 (\gamma+\delta-\gamma s) f_i(s, u_1(s), u_2(s))\mathrm{d}s + \sum_{k=1}^m I_{i,k}(u_i(t_k))(\gamma+\delta-\gamma t_k)$$

for $i = 1,2$. An application of Cramer's method yields

$$u_i(0) = \frac{\beta}{\rho}\Big[\int_0^1 (\gamma+\delta-\gamma s) f_i(s, u_1(s), u_2(s))\mathrm{d}s + \sum_{k=1}^m (\gamma+\delta-\gamma t_k) I_{i,k}(u_i(t_k))\Big]$$

and

$$u_i'(0) = \frac{\alpha}{\rho}\Big[\int_0^1 (\gamma+\delta-\gamma s) f_i(s, u_1(s), u_2(s))\mathrm{d}s + \sum_{k=1}^m (\gamma+\delta-\gamma t_k) I_{i,k}(u_i(t_k))\Big].$$

Thus,

$$\begin{aligned} u_i(t) = {} & \frac{\beta}{\rho}\Big[\int_0^1 (\gamma+\delta-\gamma s) f_i(s, u_1(s), u_2(s))\mathrm{d}s + \sum_{k=1}^m (\gamma+\delta-\gamma t_k) I_{i,k}(u_i(t_k))\Big] \\ & + \frac{\alpha t}{\rho}\Big[\int_0^1 (\gamma+\delta-\gamma s) f_i(s, u_2(s), u_2(s))\mathrm{d}s + \sum_{k=1}^m (\gamma+\delta-\gamma t_k) I_{i,k}(u_i(t_k))\Big] \\ & - \int_0^t (t-s) f_i(s, u_1(s), u_2(s))\mathrm{d}s - \sum_{0<t_k<t} (t-t_k) I_{i,k}(u_i(t_k)) \;\; \text{for } i = 1,2. \end{aligned}$$

We then have

$$\begin{aligned}u_i(t) = &\frac{1}{\rho}\Big(\int_0^1 (\alpha t+\beta)(\gamma+\delta-\gamma s)f_i(s,u_i(s),u_i(s))\mathrm{d}s \\ &- \int_0^t (t-s)(\rho)f_i(s,u_1(s),u_2(s))\mathrm{d}s\Big) \\ &+ \frac{1}{\rho}\Big(\sum_{k=1}^{m}(\alpha t+\beta)(\gamma+\delta-\gamma t_k)I_{i,k}(u_i(t_k)) \\ &- \sum_{0<t_k<t}(t-t_k)(\rho)I_{i,k}(u_i(t_k))\Big)\end{aligned}$$

for $i = 1, 2$. Hence

$$\begin{cases} u_1(t) = \displaystyle\int_0^1 G(t,s)f_1(s,u_1(s),u_2(s))\mathrm{d}s + \sum_{k=1}^{m} G(t,t_k)I_{1,k}(u_1(t_k)), \\ u_2(t) = \displaystyle\int_0^1 G(t,s)f_2(s,u_1(s),u_2(s))\mathrm{d}s + \sum_{k=1}^{m} G(t,t_k)I_{2,k}(u_2(t_k)), \end{cases}$$

where $G(t,s)$ is given in (3.2).

Conversely, if the vector $(u_1,u_2)$ is a solution of (3.1), then

$$u_i(t) = \int_0^1 G(t,s)f_i(s,u_i(s),u_i(s))\mathrm{d}s + \sum_{k=1}^{m} G(t,t_k)I_{i,k}(u_i(t_k)) \quad \text{for } i = 1,2.$$

i.e.,

$$\begin{aligned}u_i(t) = &\int_0^t \frac{1}{\rho}(\gamma+\delta-\gamma t)(\beta+\alpha s)f_i(s,u_1(s),u_2(s))\mathrm{d}s \\ &+ \int_t^1 \frac{1}{\rho}(\beta+\alpha t)(\gamma+\delta-\gamma s)f_i(s,u_1(s),u_2(s))\mathrm{d}s \\ &+ \sum_{t_k<t}\frac{1}{\rho}(\gamma+\delta-\gamma t)(\beta+\alpha t_k)I_{i,k}(u_i(t_k)) \\ &+ \sum_{t_k>t}\frac{1}{\rho}(\beta+\alpha t)(\gamma+\delta-\gamma t_k)I_{i,k}(u_i(t_k)) \qquad \text{for } i=1,2,\ t\neq t_k,\end{aligned}$$

and

$$\begin{aligned}u_i'(t) = &\frac{-\gamma}{\rho}\int_0^t (\beta+\alpha s)f_i(s,u_1(s),u_2(s))\mathrm{d}s \\ &+ \frac{\alpha}{\rho}\int_t^1 (\gamma+\delta-\gamma s)f_i(s,u_1(s),u_2(s))\mathrm{d}s \\ &+ \frac{-\gamma}{\rho}\sum_{t_k<t}(\beta+\alpha t_k)I_{i,k}(u_i(t_k)) \\ &+ \frac{\alpha}{\rho}\sum_{t_k>t}(\gamma+\delta-\gamma t_k)I_{i,k}(u_i(t_k)) \qquad \text{for } i=1,2,\ t\neq t_k.\end{aligned}$$

Differentiating again, we see that

$$\begin{aligned} u_i''(t) &= \frac{1}{\rho}\Big( -\gamma\int_0^t (\beta+\alpha s) f_i(s,u_1(s),u_2(s))\mathrm{d}s \\ &\quad + \alpha\int_t^1 (\gamma+\delta-\gamma s) f_i(s,u_1(s),u_2(s))\mathrm{d}s\Big)' \\ &= -f_i(s,u_1(s),u_2(s)) \qquad \text{for } i=1,2,\ t\neq t_k. \end{aligned}$$

Since

$$u_i(0) = \frac{\beta}{\rho}\int_0^1 (\gamma+\delta-\gamma s) f_i(s,u_1(s),u_2(s))\mathrm{d}s + \frac{\beta}{\rho}\sum_{k=1}^m (\gamma+\delta-\gamma t_k) I_{i,k}(u_i(t_k)),$$

$$u_i'(0) = \frac{\alpha}{\rho}\int_0^1 f_i(s,u_1(s),u_2(s))\mathrm{d}s + \frac{\alpha}{\rho}\sum_{k=1}^m (\gamma+\delta-\gamma t_k) I_{i,k}(u_i(t_k))$$

for $i=1,2$, we have that $\alpha u_i'(0) = \beta u_i'(0)$ for $i=1,2$. Also, since

$$u_i(1) = \frac{\delta}{\rho}\int_0^1 (\beta+\alpha s) f_i(s,u_1(s),u_2(s))\mathrm{d}s + \frac{\delta}{\rho}\sum_{k=1}^m (\beta+\alpha t_k) I_{i,k}(u_2(t_k)),$$

$$u_i'(1) = -\frac{\gamma}{\rho}\int_0^1 (\beta+\alpha s) f_i(s,u_1(s),u_2(s))\mathrm{d}s + \frac{-\gamma}{\rho}\sum_{t_k<t} (\beta+\alpha t_k) I_{i,k}(u_i(t_k))$$

for $i=1,2$, we have that $\gamma u_i(1) + \delta u_i'(1) = 0$ for $i=1,2$. Hence,

$$u_i(t_k^+) - u_i(t_k^-) = \frac{1}{\rho}(-\gamma(\beta+\alpha t_k) - \alpha(\gamma+\delta-\gamma t_k) I_{i,k}(u_i(t_k)) = -I_{i,k}(u_i(t_k))$$

for i=1,2, and this completes the proof of the lemma. □

We are now ready to present our main result in this section.

**Theorem 3.1.** *Assume that the following conditions are satisfied:*

$(H_1)$ *There exist four positive real constants $P_1$, $P_2$, $P_3$, and $P_4$ such that*

$$\begin{cases} |f_1(t,u_1,u_2) - f_1(t,\bar{u}_1,\bar{u}_2)| \le P_1|u_1-\bar{u}_1| + P_2|u_2-\bar{u}_2|, \\ |f_2(t,u_1,u_2) - f_2(t,\bar{u}_1,\bar{u}_2)| \le P_3|u_1-\bar{u}_1| + P_4|u_2-\bar{u}_2| \end{cases}$$

*for each $u_1$, $u_2$, $\bar{u}_1$, $\bar{u}_2 \in \mathbb{R}$ and each $t \in J$;*

$(H_2)$ *There exist $K_{1,k}$ and $K_{2,k}$ such that*

$$|I_{1,k}(u_1) - I_{1,k}(\bar{u}_1)| \le K_{1,k}|u_1-\bar{u}_1|, \quad k=1,2,\dots,m,$$

*and*

$$|I_{2,k}(u_2) - I_{2,k}(\bar{u}_2)| \le K_{2,k}|u_2-\bar{u}_2|, \quad k=1,2,\dots,m,$$

*for all $u_1$, $u_2$, $\bar{u}_1$, $\bar{u}_2 \in \mathbb{R}$.*

*If the matrix*

$$M := G^* \begin{pmatrix} P_1 + mK_1 & P_2 \\ P_3 & P_4 + mK_2 \end{pmatrix} \tag{3.7}$$

*converges to* 0*, where* $G^* = \sup\{|G(t,s)| : (t,s) \in J \times J\}$*,* $K_1 = \max\{K_{1,k}\}$*, and* $K_2 = \max\{K_{2,k}\}$ *for* $k = 1, 2, \dots, m$*, then the problem* (1.1) *has a unique solution.*

*Proof.* Consider the operator

$$N\colon C(J,\mathbb{R}) \times C(J,\mathbb{R}) \longrightarrow C(J,\mathbb{R}) \times C(J,\mathbb{R})$$

defined by

$$N(u_1,u_2) = (A_1(u_1,u_2), A_2(u_1,u_2)),$$

where

$$A_1(u_1,u_2)(t) = \int_0^1 G(t,s) f_1(s,u_1(s),u_2(s))\mathrm{d}s + \sum_{k=1}^m G(t,t_k) I_{1,k}(u_1(t_k)),$$

and

$$A_2(u_1,u_2)(t) = \int_0^1 G(t,s) f_2(s,u_1(s),u_2(s))\mathrm{d}s + \sum_{k=1}^m G(t,t_k) I_{2,k}(u_2(t_k)).$$

Let $(u_1,u_2), (\bar{u}_1,\bar{u}_2) \in C(J,\mathbb{R}) \times C(J,\mathbb{R})$, then

$$\begin{aligned}
&|A_1(u_1,u_2)(t) - A_1(\bar{u}_1,\bar{u}_2)(t))| \\
&\le \int_0^1 |G(t,s)||f_1(s,u_1(s),u_2(s)) - f_1(s,\bar{u}_1(s),\bar{u}_2(s))|\mathrm{d}s \\
&\quad + \sum_{k=1}^m |G(t,t_k)||I_{1,k}(u_1(t_k)) - I_{1,k}(\bar{u}_1(t_k))| \\
&\le G^* \int_0^1 \Big[P_1|u_1(s) - \bar{u}_1(s)| + P_2|u_2(s) - \bar{u}_2(s)|\Big]\mathrm{d}s + G^* \sum_{k=1}^m K_{1,k}|u_1(t_k) - \bar{u}_1(t_k)| \\
&\le G^* \Big(P_1 + \sum_{k=1}^m K_{1,k}\Big)\|u_1 - \bar{u}_1\|_C + G^* P_2 \|u_2 - \bar{u}_2\|_C \\
&\le G^* \left[(P_1 + mK_1)\|u_1 - \bar{u}_1\|_C + P_2\|u_2 - \bar{u}_2\|_C\right],
\end{aligned}$$

so

$$\|A_1(u_1,u_2) - A_1(\bar{u}_1,\bar{u}_2)\|_C \le G^* \left[(P_1 + mK_1)\|u_1 - \bar{u}_1\|_C + P_2\|u_2 - \bar{u}_2\|_C\right]. \tag{3.8}$$

Similarly,

$$\begin{aligned}
&|A_2(u_1,u_2)(t) - A_2(\bar{u}_1,\bar{u}_2)(t)| \\
&\le \int_0^1 |G(t,s)||f_2(s,u_1(s),u_2(s)) - f_2(s,\bar{u}_1(s),\bar{u}_2(s))|\mathrm{d}s \\
&\quad + \sum_{k=1}^m |G(t,t_k)||I_{2,k}(u_2(t_k)) - I_{2,k}(\bar{u}_2(t_k))|
\end{aligned}$$

$$\leq G^*\int_0^1 [P_3|u_1(s)-\bar{u}_1(s)|+P_4|u_2(s)-\bar{u}_2(s)|]\,\mathrm{d}s+G^*\sum_{k=1}^m K_{2,k}|u_2(t_k)-\bar{u}_2(t_k)|$$

$$\leq G^*P_3\|u_1-\bar{u}_1\|_C+G^*\Big(P_4+\sum_{k=1}^m K_{2,k}\Big)\|u_2-\bar{u}_2\|_C$$

$$\leq G^*\left[P_3\|u_1-\bar{u}_1\|_C+(P_4+mK_2)\,\|u_2-\bar{u}_2\|_C\right],$$

and so

(3.9)

$$\|A_2(u_1,u_2)-A_2(\bar{u}_1,\bar{u}_2)\|_{PC^2}\leq G^*\left[P_3||u_1-\bar{u}_1||_C+(P_4+mK_2)\,||u_2-\bar{u}_2||_C\right].$$

From (3.8) and (3.9), we obtain

$$\begin{bmatrix}\|A_1(u_1,u_2)-A_1(\bar{u}_1,\bar{u}_2)\|_{PC}\\ \|A_2(u_1,u_2)-A_2(\bar{u}_1,\bar{u}_2)\|_{PC}\end{bmatrix}\leq M\begin{bmatrix}\|u_1-\bar{u}_1\|_{PC}\\ \|u_2-\bar{u}_2\|_{PC}\end{bmatrix},$$

where

$$M=G^*\begin{pmatrix}P_1+mK_1 & P_2\\ P_3 & P_4+mK_2\end{pmatrix}.$$

Then by (3.7), $N$ is a contraction, so by Perov's fixed point theorem (Theorem 2.2 above), $N$ has a unique fixed point that in turn is a solution of system (1.1). □

## 4. Main Results II

In this section we give an existence result based on the non linear alternative of Leray-Schauder type. We need following conditions to obtain our result:

($C_1$) The functions $f_1$ and $f_2$ are $L^1$-Carathéodory functions;

($C_2$) There exist functions $p$, $q$, $h$, $g$, $\tilde{q}$, and $\bar{h}\in L^1([0,1],\mathbb{R}^+)$ and constants $\alpha_1$, $\alpha_2$, $\alpha_3$, and $\alpha_4\in[0,1)$ such that

$$|f_1(t,u_1,u_2)|\leq p(t)|u_1|^{\alpha_1}+q(t)|u_2|^{\alpha_2}+h(t)$$

for each $t\in J$ and $u_1,\ u_2\in\mathbb{R}$ and

$$|f_2(t,u_1,u_2)|\leq\tilde{p}(t)|u_1|^{\alpha_3}+\tilde{q}(t)|u_2|^{\alpha_4}+\bar{h}(t)$$

for each $t\in J$ and $u_1,u_2\in\mathbb{R}$;

($C_3$) There exist constants $c_k$, $b_k$, $c_k^*$, $b_k^*\in\mathbb{R}^+$ and $\beta_k$, $\beta_k^*\in[0,1)$ such that

$$|I_{1,k}(u_1)|\leq c_k+b_k|u_1|^{\beta_k},\quad k=1,2,\ldots,m,\ u_1\in\mathbb{R}$$

and

$$|I_{2,k}(u_2)|\leq c_k^*+b_k^*|u_2|^{\beta_k^*},\quad k=1,2,\ldots,m,\ u_2\in\mathbb{R}.$$

**Theorem 4.1.** *If conditions* ($C_1$)–($C_3$) *hold, then the system* (1.1) *has at least one solution.*

*Proof.* Let $N$ be the operator defined in the proof of Theorem 3.1. To show that $N$ is continuous let $(u_{1,n}, u_{2,n})$ be a sequence such that $(u_{1,n}, u_{2,n}) \to (\tilde{u}_1, \tilde{u}_2) \in C(J, \mathbb{R}) \times C(J, \mathbb{R})$ as $n \to \infty$. Then,

$$\begin{aligned}
&|A_1(u_{1,n}, u_{2,n})(t) - A_1(\tilde{u}_1, \tilde{u}_2)(t)| \\
&\leq \int_0^1 |G(t,s)||f_1(s, u_{1,n}(s), u_{2,n}(s)) - f_1(s, \tilde{u}_1(s), \tilde{u}_2(s))|\mathrm{d}s \\
&\quad + \sum_{k=1}^m |G(t,t_k)||I_{1,k}(u_n(t_k)) - I_{1,k}(\tilde{u}_1(t_k))| \\
&\leq G^* \int_0^1 |f_1(s, u_{1,n}(s), u_{2,n}(s)) - f_1(s, \tilde{u}_1(s), \tilde{u}_2(s))|\mathrm{d}s \\
&\quad + G^* \sum_{k=1}^m |I_{1,k}(u_{1,n}(t_k)) - I_{1,k}(\tilde{u}_1(t_k))|.
\end{aligned}$$

Since $f_1$ is an $L_1$-Carathéodory function and $I_{1,k}$, $k = 1, 2, \ldots, m$, are continuous, by the Lebesgue dominated convergence theorem,

$$\|A_1(u_{1,n}, u_{2,n}) - A_1(\tilde{u}_1, \tilde{u}_2)\|_C \to 0 \qquad \text{as } n \to \infty.$$

Similarly,

$$\|A_2(u_{1,n}, u_{2,n}) - A_2(\tilde{u}_1, \tilde{u}_2)\|_C \to 0 \qquad \text{as } n \to \infty.$$

Thus, $N$ is continuous.

In order to show that $N$ maps bounded sets into bounded sets in $C(J, \mathbb{R}) \times C(J, \mathbb{R})$, it suffices to show that for any $q > 0$ there exists a positive constant vector $l = (l_1, l_2)$ such that for each $(u_1, u_2) \in B_q = \{(u_1, u_2) \in C(J, \mathbb{R}) \times C(J, \mathbb{R})$ $\|u_1\|_C \leq q, \|u_2\|_C \leq q\}$, we have

$$\|N(u_1, u_2)\|_C \leq \|l\|.$$

For each $t \in J$, we have

$$\begin{aligned}
&|A_1(u_1, u_2)(t)| \\
&\leq \int_0^1 |G(t,s)||f_1(s, u_1(s), u_2(s))| + \sum_{k=1}^m |G(t,t_k)||I_{1,k}(u_1(t_k))| \\
&\leq G^* \int_0^1 (p(s)|u_1(s)|^{\alpha_1} + q(s)|u_2(s)|^{\alpha_2} + h(s))\, \mathrm{d}s + G^* \sum_{k=1}^m \left(c_k + b_k|u_1(t_k)|^{\beta_k}\right) \\
&\leq G^* \|u_1\|_C^{\alpha_1} \int_0^1 p(s)\mathrm{d}s + G^* \|u_2\|_C^{\alpha_2} \int_0^1 q(s)\mathrm{d}s + G^* \int_0^1 h(s)\mathrm{d}s + G^* \sum_{k=1}^m \Big(c_k + b_k \|u_1\|_C^{\beta_k}\Big) \\
&\leq G^* q^{\alpha_1} \|p\|_{L^1} + G^* q^{\alpha_2} \|q\|_{L^1} + G^* \|h\|_{L^1} + G^* \sum_{k=1}^m \left(c_k + b_k q^{\beta_k}\right).
\end{aligned}$$

Hence

$$\|A_1(u_1, u_2)\|_{PC} \leq G^* q^{\tilde{\alpha}} \Big(\|p\|_{L^1} + \|q\|_{L^1} + \sum_{k=1}^m b_k\Big) + G^* \Big(\|h\|_{L^1} + \sum_{k=1}^m c_k\Big) := l_1$$

where

$$\tilde{\alpha} = \max\{\alpha_1, \alpha_2, \beta_k : k = 1, 2, \cdots, m\}.$$

Similarly, we have

$$\|A_2(u_1, u_2)\|_C \le G^* q^{\bar{\alpha}} \Big( \|\tilde{p}\|_{L^1} + \|\tilde{q}\|_{L^1} + \sum_{k=1}^{m} b_k^* \Big) + G^* \Big( \|\bar{h}\|_{L^1} + \sum_{k=1}^{m} c_k^* \Big) := l_2,$$

where

$$\bar{\alpha} = \max\{\alpha_3, \alpha_4, \beta_k^* : k = 1, 2, \cdots, m\},$$

which is what we needed to show.

Next we show that $N$ maps bounded sets into equicontinuous sets of $C([0,1],\mathbb{R})\times C(J,\mathbb{R})$. Let $B_q$ be the bounded set obtained above. Let $r_1$, $r_2 \in J$ with $r_1 < r_2$ and $u \in B_q$, then we have

$$\begin{aligned}
&|A_1(u_1, u_2)(r_2) - A_1(u_1, u_2)(r_1)| \\
&\le \int_0^1 |G(r_2, s) - G(r_1, s)||f_1(s, u_1(s), u_2(s))|\mathrm{d}s \\
&\quad + \sum_{k=1}^{m} |G(r_2, t_k) - (G(r_1, t_k)||I_{1,k}(u_1(t_k))| \\
&\le \int_0^1 |G(r_2, s) - G(r_1, s)|[(p(s)|u_1(s)|^{\alpha_1} + q(s)|u_2(s)|^{\alpha_2} + h(s))]\mathrm{d}s \\
&\quad + \sum_{k=1}^{m} |G(r_2, t_k) - G(r_1, t_k)| \left(c_k + b_k|u_1(s)|^{\beta_k}\right) \\
&\le q^{\alpha_1} \int_0^1 |G(r_2, s) - G(r_1, s)|p(s)\mathrm{d}s + q^{\alpha_2} \int_0^1 |G(r_2, s) - G(r_1, s)|q(s)\mathrm{d}s \\
&\quad + \int_0^1 |G(r_2, s) - G(r_1, s)|h(s)\mathrm{d}s + \sum_{k=1}^{m} |G(r_2, t_k) - G(r_1, t_k)| \left(c_k + b_k q^{\beta_k}\right).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&|A_2(u_1, u_2)(r_2) - A_2(u_1, u_2)(r_1)| \\
&\le q^{\alpha_3} \int_0^1 |G(r_2, s) - G(r_1, s)|\tilde{p}(s)\mathrm{d}s + q^{\alpha_4} \int_0^1 |G(r_2, s) - G(r_1, s)|\tilde{q}(s)\mathrm{d}s \\
&\quad + \int_0^1 |G(r_2, s) - G(r_1, s)|\bar{h}(s)\mathrm{d}s + \sum_{k=1}^{m} |G(r_2, t_k) - G(r_1, t_k)| \left(c_k^* + b_k^* q^{\beta_k}\right).
\end{aligned}$$

Notice that the terms on the right-hand side in the above two expressions tend to zero as $|r_2 - r_1| \to 0$. We can now apply the Arzelà-Ascoli theorem to conclude that $N\colon B_M \to C(J,\mathbb{R}) \times C(\bar{J},\mathbb{R})$ is completely continuous.

Next, let $(u_1, u_2) \in C(J,\mathbb{R}) \times C(J,\mathbb{R})$ with $(u_1, u_2) = \lambda N(u_1, u_2)$ for some $0 < \lambda < 1$. Then $u_1 = \lambda A_1(u_1, u_2)$ and $u_2 = \lambda A_2(u_1, u_2)$. Thus, for $t \in [0, 1]$, we have

$$
\begin{aligned}
|u_1(t)| &\leq \int_0^1 |G(t,s)||f_1(s,u_1(s),u_2(s))| + \sum_{k=1}^{m} |G(t,t_k)||I_{1,k}(u_1(t_k))| \\
&\leq G^*\int_0^1 [(p(s)|u_1(s)|^{\alpha_1} + q(s)|u_2(s)|^{\alpha_2} + h(s)]\, \mathrm{d}s + G^*\sum_{k=1}^{m}\big(c_k + b_k|u_1(t_k)|^{\beta_k}\big) \\
&\leq G^*\|u_1\|_C^{\alpha_1}\int_0^1 p(s)\mathrm{d}s + G^*\|u_2\|_C^{\alpha_2}\int_0^1 q(s)\mathrm{d}s + G^*\int_0^1 h(s)\mathrm{d}s \\
&\quad + G^*\sum_{k=1}^{m}\left(c_k + b_k\|u_1\|_C^{\beta_k}\right).
\end{aligned}
$$

Hence,

$$
\|u_1\|_C \leq G^*\|u_1\|_C^{\alpha_1}\|p\|_{L^1} + G^*\|u_2\|_C^{\alpha_2}\|q\|_{L^1} + G^*\|h\|_{L^1} + G^*\sum_{k=1}^{m}\left(c_k + b_k\|u_1\|_C^{\beta_k}\right).
$$

Similarly, we obtain

$$
\|u_2\|_C \leq G^*\|u_1\|_C^{\alpha_3}\|\tilde{p}\|_{L^1} + G^*\|u_2\|_C^{\alpha_4}\|\tilde{q}\|_{L^1} + G^*\|\bar{h}\|_{L^1} + G^*\sum_{k=1}^{m}\left(c_k^* + b_k^*\|u_2\|_C^{\beta_k^*}\right).
$$

Notice that if $\epsilon \leq \delta$ and $\|u\| > 1$, then $\|u\|^{\epsilon} \leq \|u\|^{\delta}$. Thus, $\|u\|^{\epsilon} \leq 1 + \|u\|^{\delta}$ for all $u$.

We then have

$$
\begin{aligned}
&\|u_1\|_C + \|u_2\|_C \\
&\leq G^*\left(\|q\|_{L^1} + \|\tilde{p}\|_{L^1}\right)\left(\|u_1\|_C^{\alpha_3} + \|u_2\|_C^{\alpha_2}\right) + G^*\left(\|p\|_{L^1} + \|\tilde{q}\|_{L^1}\right)\left(\|u_1\|_C^{\alpha_1} + \|u_2\|_C^{\alpha_4}\right) \\
&\quad + G^*\sum_{k=1}^{m}(b_k + b_k^*)\Big(\|u_1\|_C^{\beta_k} + \|u_2\|_C^{\beta_k^*}\Big) + G^*\Big(\sum_{k=1}^{m}(c_k + c_k^*) + \|h\|_{L^1} + \|\bar{h}\|_{L^1}\Big) \\
&\leq G^*\Big(\|q\|_{L^1} + \|\tilde{p}\|_{L^1} + \|p\|_{L^1} + \|\bar{q}\|_{L^1} + \sum_{k=1}^{m}(b_k + b_k^*)\Big)\Big(1 + \|u\|_C^{\alpha^*} + \|v\|_C^{\alpha^*}\Big) \\
&\quad + G^*\Big(\sum_{k=1}^{m}(c_k + c_k^*) + \|h\|_{L^1} + \|\tilde{h}\|_{L^1}\Big) \\
&\leq 2G^*\Big(\|q\|_{L^1} + \|\tilde{p}\|_{L^1} + \|p\|_{L^1} + \|\bar{q}\|_{L^1} + \sum_{k=1}^{m}(b_k + b_k^*)\Big)\Big(\|u\|_C + \|v\|_C\Big)^{\alpha^*} \\
&\quad + G^*\Big(\|q\|_{L^1} + \|\tilde{p}\|_{L^1} + \|p\|_{L^1} + \|\bar{q}\|_{L^1} + \sum_{k=1}^{m}(b_k + b_k^*)\Big) \\
&\quad + G^*\Big(\sum_{k=1}^{m}(c_k + c_k^*) + \|h\|_{L^1} + \|\tilde{h}\|_{L^1}\Big)
\end{aligned}
$$

where

$$
\alpha^* = \max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_k, \beta_k^* : k = 1, 2, \cdots, m\}.
$$

If $\|u_1\|_C + \|u_2\|_C > 1$, then

$$\begin{aligned}
&\frac{\|u_1\|_C + \|u_2\|_C}{\Big(\|u_1\|_C + \|u_2\|_C\Big)^{\alpha^*}} \\
&\leq 2G^*\Big(\|q\|_{L^1} + \|\tilde{p}\|_{L^1} + \|p\|_{L^1} + \|\tilde{q}\|_{L^1} + \sum_{k=1}^{m}(b_k + b_k^*)\Big) \\
&\quad + G^* \frac{\Big(\|q\|_{L^1} + \|\tilde{p}\|_{L^1} + \|p\|_{L^1} + \|\bar{q}\|_{L^1} + \sum_{k=1}^{m}(b_k + b_k^*)\Big)}{\Big(\|u_1\|_C + \|u_2\|_C\Big)^{\alpha^*}} \\
&\quad + G^* \frac{\sum_{k=1}^{m}(c_k + c_k^*) + \|h\|_{L^1} + \|\bar{h}\|_{L^1}}{\Big(\|u\|_C + \|v\|_C\Big)^{\alpha^*}}
\end{aligned}$$

or

$$\begin{aligned}
\Big(\|u_1\|_C + \|u_2\|_C\Big)^{1-\alpha^*} &\leq 2G^*\Big(\|q\|_{L^1} + \|\tilde{p}\|_{L^1} + \|p\|_{L^1} + \|\tilde{q}\|_{L^1} + \sum_{k=1}^{m}(b_k + b_k^*)\Big) \\
&\quad + G^*\Big(\|q\|_{L^1} + \|\tilde{p}\|_{L^1} + \|p\|_{L^1} + \|\bar{q}\|_{L^1} + \sum_{k=1}^{m}(b_k + b_k^*)\Big) \\
&\quad + G^*\Big(\sum_{k=1}^{m}(c_k + c_k^*) + \|h\|_{L^1} + \|\bar{h}\|_{L^1}\Big).
\end{aligned}$$

This implies that

$$\|u_1\|_C + \|u_2\|_C \leq \Big[3G^*\Big(C_1 + \sum_{k=1}^{m}(b_k + b_k^*)\Big) + G^*\Big(\sum_{k=1}^{m}(c_k + c_k^*) + C_2\Big)\Big]^{\frac{1}{1-\alpha^*}} := M_2,$$

where

$$C_1 = \|q\|_{L^1} + \|\tilde{p}\|_{L^1} + \|p\|_{L^1} + \|\tilde{q}\|_{L^1} \quad \text{and} \quad C_2 = \|h\|_{L^1} + \|\bar{h}\|_{L^1}.$$

Consequently

$$\|u_1\|_C \leq M_2 \quad \text{and} \quad \|u_2\|_C \leq M_2.$$

Set

$$U = \{(u_1, u_2) \in C(J, \mathbb{R}) \times C(J, \mathbb{R}) : \ \|u_1\|_C < M_2 + 1 \text{ and } \|u_2\|_C < M_2 + 1\}.$$

From the choice of $U$, there is no $(u_1, u_2) \in \partial U$ such that $(u_1, u_2) = \lambda N(u_1, u_2)$ for some $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative of Leray-Schauder type (Theorem 2.3), the operator $N$ has a fixed point that is a solution of system (1.1). This completes the proof of the theorem. □

## 5. Examples

In this section, we give two examples to illustrate our results above.

**Example 5.1.** Consider the impulsive differential system of second order given by

$$(5.1)\quad \begin{cases} -u_1''(t) = \dfrac{1}{6}\dfrac{u_2^2(t)}{1+u_2^2(t)}\sin(2u_1(t)) := f_1(t,u_1(t),u_2(t)), & t \in J \setminus \left\{\frac{1}{4}\right\}, \\ -u_2''(t) = \dfrac{1}{8}\dfrac{u_2^2(t)}{1+u_2^2(t)}\cos(2u_1(t)) := f_2(t,u_1(t),u_2(t)), & t \in J \setminus \left\{\frac{1}{4}\right\}, \\ -\Delta u_1'\left(\frac{1}{4}\right) = \frac{1}{4}\cos\left(u_1\left(\frac{1}{4}\right)\right), & t_1 = \frac{1}{4}, \\ -\Delta u_2'\left(\frac{1}{4}\right) = \frac{1}{3}\sin\left(u_2\left(\frac{1}{4}\right)\right), & \\ u_1(0) = u_1'(0) = 0, \quad u_2(0) = u_2'(0) = 0. & \end{cases}$$

We see that $\alpha = \delta = 1$ and $\beta = \gamma = 0$. Moreover, since

$$\sup_{u_1,u_2\in\mathbb{R}}\left|\frac{\partial f_1(t,u_1,u_2)}{\partial u_1}\right| \le \frac{1}{3}, \qquad \sup_{u_1,u_2\in\mathbb{R}}\left|\frac{\partial f_1(t,u_1,u_2)}{\partial u_2}\right| \le \frac{1}{3},$$

$$\sup_{u_1,u_2\in\mathbb{R}}\left|\frac{\partial f_2(t,u_1,u_2)}{\partial u_1}\right| \le \frac{1}{4}, \qquad \sup_{u_1,u_2\in\mathbb{R}}\left|\frac{\partial f_2(t,u_1,u_2)}{\partial u_2}\right| \le \frac{1}{4},$$

we have

$$|f_1(t,u_1,u_2) - f_1(t,\bar{u}_1,\bar{u}_2)| \le \frac{1}{3}|u_1-\bar{u}_1| + \frac{1}{3}|u_2-\bar{u}_2|$$

and

$$|f_2(t,u_1,u_2) - f_2(t,\bar{u_1},\bar{u}_2)| \le \frac{1}{4}|u_1-\bar{u}_1| + \frac{1}{4}|u_2-\bar{u}_2|.$$

Hence, condition $(H_1)$ holds with $P_1 = \frac{1}{3}$, $P_2 = \frac{1}{3}$, $P_3 = \frac{1}{4}$, and $P_4 = \frac{1}{4}$. Also,

$$|I_{1,1}(u_1) - I_{1,1}(\bar{u_1})| \le \frac{1}{4}|u_1-\bar{u}_1| \quad \text{for each } u,\bar{u} \in \mathbb{R} \text{ and each } t \in [0,1],$$

$$|I_{1,2}(u_2) - I_{1,2}(\bar{u}_2)| \le \frac{1}{3}|u_2-\bar{u}_2| \quad \text{for each } u_2,\bar{u}_2 \in \mathbb{R} \text{ and each } t \in [0,1].$$

Thus, $(H_3)$ holds. From (3.2), the Green's function for the homogeneous problem is given by

$$G(t,s) = \begin{cases} s, & 0 \le s \le t \le 1, \\ t, & 0 \le t \le s \le 1, \end{cases}$$

and we can easily see that

$$G^* = \sup_{(t,s)\in J\times J}|G(t,s)| = 1.$$

For this example $M = \begin{pmatrix} \frac{7}{12} & \frac{1}{3} \\ \frac{1}{4} & \frac{7}{12} \end{pmatrix}$, which has the two eigenvalues $\lambda_1 \simeq 0.872$ and $\lambda_2 \simeq 0.294$. Therefore, $M$ converges to zero. All the conditions in Theorem 3.1 are satisfied, so system (5.1) has a unique solution.

**Example 5.2.** Consider the impulsive differential system

$$(5.2)\quad \begin{cases} -u_1''(t) = t^3 + 2(t-1)^2|u_1(t)|^{0.8} + \mathrm{e}^t|u_2(t)|^{0.3} + 3 & \\ := f_1(t, u_1(t), u_2(t)), & t \in J \setminus \{\frac{1}{2}\}, \\ -u_2''(t) = t^2 + 4t|u_1(t)|^{0.4} + \left(t - \frac{1}{3}\right)^2 |u_2(t)|^{0.6} + 8 & \\ := f_2(t, u_1(t), u_2(t)), & t \in J \setminus \{\frac{1}{2}\}, \\ -\Delta u_1'\left(\frac{1}{2}\right) = \frac{1}{6}\sqrt{u_1\left(\frac{1}{2}\right)}, & t_1 = \frac{1}{2}, \\ -\Delta u_2'\left(\frac{1}{2}\right) = \frac{2}{3}|u_2\left(\frac{1}{2}\right)|^{\frac{2}{5}} + 4, & \\ u_1(0) = u_1'(0) = 0, \quad u_2(0) = u_2'(0) = 0. & \end{cases}$$

We clearly have

$$\begin{cases} |f_1(t, u_1(t), u_2(t))| \leq 2|u_1|^{0.8} + e|u_2|^{0.3} + 4, \\ |f_2(t, u_1(t), u_2(t))| \leq 4|u_1|^{0.4} + \frac{4}{9}|u_2|^{0.6} + 9, \end{cases}$$

and

$$\begin{cases} |I_{1,1}(u_1)| \leq \frac{1}{6}|u_1|^{\frac{1}{2}}, \\ |I_{1,2}(u_2)| \leq \frac{2}{3}|u_2|^{\frac{2}{5}} + 4 \end{cases}$$

for $t \in J$. Now all the hypotheses of Theorem 4.1 are satisfied, so system (5.2) has at least one solution.

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