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ANALYSIS OF SEMILOCAL CONVERGENCE UNDER W-CONTINUITY CONDITION ON SECOND ORDER FRÉCHET DERIVATIVE IN BANACH SPACES

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Abstract. This article includes semilocal convergence of Homier type method in Banach spaces by considering recurrence relation technique under w-continuity condition which is more generalized than Lipschitz continuity condition. The existence and uniqueness theorem has been formed along with an error bound. A numerical example has been presented to vindicate the theoretical results.

1. Introduction

The techniques to find the solution of nonlinear equations are commonly used in analytical and numerical branches of mathematics. Newton's method is a prominent method to solve this kind of problems. Newton-Kantorovich theorem [**15**] is commonly used for the convergence of Newton's method. Furthermore, in the last few years, many research papers have been devoted to analyze the semilocal convergence of various iterative methods for solving nonlinear problems. A few years ago, in the reference [**17**], the fifth order iterative method for solving systems of nonlinear equations was given by

$$\begin{aligned} b_j &= a_j - \frac{1}{2}\Gamma_j T(a_j), \\ c_j &= a_j - [T'(b_j)]^{-1}T(a_j), \\ a_{j+1} &= c_j - [2[T'(b_j)]^{-1} - \Gamma_j]T(c_j), \end{aligned} \tag{1.1}$$

where $\Gamma_j = [T'(a_j)]^{-1}$. The aim of this paper is to discuss the approximate solution a^* of a nonlinear equation

$$T(a) = 0, \tag{1.2}$$

where nonlinear operator T is defined from the set U to set V; U and V are two Banach spaces defined in the open subset Ω. In the article [**14**], the above scheme was extended in Banach spaces by assuming the below mentioned assumptions:

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(A1) $\|\Gamma_0\| \leq \tau$,
(A2) $\|\Gamma_0 T(a_0)\| \leq \rho$,
(A3) $\|T''(a)\| \leq K$,
(A4) there exists a positive real number L such that

$$\|T''(a) - T''(b)\| \leq L\|a - b\|, \qquad a, b \in \Omega,\ L > 0,$$

and studied their semilocal convergence. It was observed that some of the nonlinear equations may not satisfy the Lipschitz continuity condition (A4). Many articles such as references ([**10**], [**4**], [**2**], [**3**], [**9**]) are devoted to find the solution a^* of the nonlinear integral equation of mixed Hammerstein types, which is given by

$$a(s) + \sum_{i=1}^{m} \int_e^f Q_i(s,t) N_i(a(t)) \mathrm{d}t = u(s), \qquad s \in [e, f], \tag{1.3}$$

where u, Q_i and N_i are well-known functions $-\infty < e < f < +\infty$, $(i = 1, 2, \ldots, m.)$. Out of them, some particular types of Hammerstein equation do not fulfill the condition (A4) (one example has been considered in the numerical testing section). To settle this circumstance, we may consider the more general continuity condition, i.e., w-continuity ([**6**], [**11**]) given by

$$\|T''(a) - T''(b)\| \leq w(\|a - b\|), a, b \in \Omega, \tag{1.4}$$

where $w(a)$ is defined as continuous real function for $a > 0$ and $w(0) \geq 0$. In the literature ([**7**], [**12**], [**13**], [**8**], [**1**], [**16**], [**18**]), various papers were published to relax the condition (A4) for higher convergence schemes.

In this paper, we discuss the semilocal convergence of a multi-point fifth order iterative method (1.1) for finding solution of nonlinear equations (1.2). The semilocal convergence of the scheme is derived by using recurrence equations and the hypothesis that second order Fréchet derivative fulfill weaker continuity condition which is known as w- continuity condition. The existence and uniqueness theorem is included along with an error estimate. A Numerical example is worked out to justify the significance of present discussions.

2. Recurrence relations

In this article, we denote $D(a, \kappa) = \{b \in U : \|b - a\| < \kappa\}$ and $\overline{D(a, \kappa)} = \{b \in U : \|b - a\| \leq \kappa\}$. Let $T : \Omega \subseteq U \to V$ be twice Fréchet differentiable nonlinear operator from Banach space U to V, where Ω is an open set, $a_0 \in \Omega$. Postulate that

(C1) $\|\Gamma_0\| \leq \tau$,
(C2) $\|\Gamma_0 T(a_0)\| \leq \rho$,
(C3) $\|T''(a)\| \leq K, a \in \Omega$,
(C4) $\|T''(a) - T''(b)\| \leq w(\|a - b\|)$ for all $a, b \in \Omega$, where $w(\mu)$ is a non-decreasing continuous real function for $\mu > 0$ and satisfies that $w(0) \geq 0$,

(C5) there exists a non-negative real function $v \in C[0,1]$ with $v(u) \leq 1$, such that $w(u\mu) \leq v(u)w(\mu)$ for $u \in [0,1]$, $\mu \in (0,+\infty)$.

Now, we describe the following scalars functions p, δ, and φ as

$$p(u) = \frac{u(6-u)(4-u)}{(2-u)^3} + \frac{2}{(2-u)}, \tag{2.1}$$

$$\delta(u) = \frac{1}{1-up(u)}, \tag{2.2}$$

$$\begin{aligned} \varphi(u,v) = &\left[\frac{u^2}{(2-u)} + \frac{uvJ_1}{(2-u)} + \frac{vJ_1}{2}\right]\psi(u,v) \\ &+ \left[vJ_3 + \frac{vJ_1}{2} + \frac{u^2}{(2-u)}\right]\frac{(6-u)}{(2-u)}\psi(u,v) \\ &+ \frac{u}{2}\frac{(6-u)^2}{(2-u)^2}\psi(u,v)^2, \end{aligned} \tag{2.3}$$

where

$$\psi(u,v) = \frac{u^3}{2(2-u)^2} + \frac{u^2}{2(2-u)} + \frac{J_1 v}{2} + J_2 v,$$

$$J_1 = \int_0^1 v\left(\frac{u}{2}\right)\mathrm{d}u, \qquad J_2 = \int_0^1 v(u)(1-u)\mathrm{d}u, \qquad J_3 = \int_0^1 v(u)\mathrm{d}u.$$

Let us suppose $q(u) = up(u) - 1$, and since $q(0) = -1$ and $q(2) = +\infty$, hence $q(u)$ has at least one zero of the function $q(u)$ in $(0,2)$ and let us denote it by ϕ. In the following lemmas, we mention some properties which follows by the above defined functions.

Lemma 1. *Suppose that the scalar functions p, δ and φ are defined as mentioned in the equations (2.1), (2.2), and (2.3), respectively, then*

(1) *$p(u) > 1$, $\delta(u) > 1$ for $u \in (0,\phi)$, $p(u)$ and $\delta(u)$ are increasing functions,*

(2) *$\varphi(u,v)$ is also increasing for $u \in (0,\phi)$, $v > 0$.*

Proof. The proof is straightforward. □

Now, we define initial values $\rho_0 = \rho$, $\tau_0 = \tau$, $\sigma_0 = K\tau_0\rho_0$, $\beta_0 = \tau_0\rho_0 w(\rho_0)$, and $r_0 = \delta(\sigma_0)\varphi(\sigma_0,\beta_0)$. Moreover, we also describe the following sequences for $j \geq 0$,

$$\rho_{j+1} = r_j\rho_j, \tag{2.4}$$

$$\tau_{j+1} = \delta(\sigma_j)\tau_j, \tag{2.5}$$

$$\sigma_{j+1} = K\tau_{j+1}\rho_{j+1}, \tag{2.6}$$

$$\beta_{j+1} = \tau_{j+1}\rho_{j+1}w(\rho_{j+1}), \tag{2.7}$$

$$r_{j+1} = \delta(\sigma_{j+1})\varphi(\sigma_{j+1},\beta_{j+1}). \tag{2.8}$$

From the definition of σ_{j+1}, β_{j+1}, and relations (2.4)–(2.7), we can deduce that

$$\begin{aligned} \sigma_{j+1} &= \delta(\sigma_j)r_j\sigma_j, \\ \beta_{j+1} &\leq \delta(\sigma_j)r_j v(r_j)\beta_j. \end{aligned}$$

Lemma 2. *Presume the real functions p, δ, and φ are defined as in equations* (2.1), (2.2), *and* (2.3), *respectively, and*

$$0<\sigma_0<\phi, \qquad \delta(\sigma_0)r_0<1, \tag{2.9}$$

then

(1) $\delta(\sigma_j)>1$ *and* $\delta_j\varphi(\sigma_j,\beta_j)<1$ *for* $j\geq 0$,

(2) *the sequences* $\{\rho_j\}$, $\{\sigma_j\}$, $\{\beta_j\}$, *and* $\{r_j\}$ *are decreasing,*

(3) $p(\sigma_j)\sigma_j<1$ *and* $\delta(\sigma_j)r_j<1$ *for* $j\geq 0$.

Proof. Using the equation (2.9) and Lemma 1, it follows that $\delta(\sigma_0)>1$ and $\delta(\sigma_0)\varphi(\sigma_0,\beta_0)<1$. Relations (2.4)–(2.7) conclude that $\rho_1<\rho_0$, $\sigma_1<\sigma_0$, $\beta_1<\beta_0$ and $r_1<r_0$. Now, we find that $\delta(\sigma_1)\varphi(\sigma_1,\beta_1)<\delta(\sigma_0)\varphi(\sigma_0,\beta_0)$, and thus, we showed the next part for $j=0$. So, we can say that $p(\sigma_1,\beta_1)\sigma_1<p(\sigma_0,\beta_0)\sigma_0<1$ and $\delta^2(\sigma_1)\varphi(\sigma_1,\beta_1)<\delta^2(\sigma_0)\varphi(\sigma_0,\beta_0)<1$, which is true for the third part when $j=0$. By mathematical induction, it can be proved that all these parts are true for all $j\geq 0$. □

Lemma 3 ([**14**])**.** *Let $T\colon\Omega\subseteq U\to V$ be a continuously twice Fréchet differentiable nonlinear function from the Banach spaces U to V, Ω is a open set, then*

$$\begin{aligned}
T(a_{j+1}) &= \frac{1}{2}T''(a_j)(w_j-a_j)[T'(b_j)]^{-1}[T''(a_j)(w_j-b_j)\\
&\quad+\int_0^1[T''(a_j+\frac{u}{2}(w_j-a_j))-T''(a_j)]\mathrm{d}u(w_j-a_j)]\Gamma_jT(c_j)\\
&\quad+\int_0^1[T''(a_j+u(w_j-a_j))-T''(a_j)](w_j-a_j)\mathrm{d}u(a_{j+1}-c_j)\\
&\quad-\frac{1}{2}\int_0^1[T''(a_j+\frac{u}{2}(w_j-a_j))-T''(a_j)](w_j-a_j)\mathrm{d}u(a_{j+1}-c_j)\\
&\quad+\frac{1}{2}\int_0^1[T''(a_j+\frac{u}{2}(w_j-a_j))-T''(a_j)]\mathrm{d}u(w_j-a_j)\Gamma_jT(c_j)\\
&\quad+\int_0^1[T'(c_j+u(a_{j+1}-c_j))-T'(w_j)](a_{j+1}-c_j)\mathrm{d}u,
\end{aligned} \tag{2.10}$$

where $w_j=a_j-\Gamma_jT(a_j)$.

Lemma 4. *Suppose that the scalar operators p, δ and φ are specified in the expressions* (2.1)–(2.3), *respectively, and let $\xi\in(0,1)$, then $p(\xi u)<p(u)$, $\delta(\xi u)<\delta(u)$, and $\varphi(\xi u,\xi v)<\xi^2\varphi(u,v)$, $\varphi(\xi u,\xi^{(1+\theta)}v)<\xi^{(2+2\theta)}\varphi(u,v)$ for $u\in(0,\phi)$.*

Proof. Since,

$$p(u)=\frac{u(4-u)(6-u)}{(2-u)^3}+\frac{2}{(2-u)}, \qquad \xi\in(0,1),$$

hence, we obtain

$$p(\xi u) = \frac{\xi u(4-\xi u)(6-\xi u)}{(2-\xi u)^3} + \frac{2}{(2-\xi u)} \leq p(u).$$

Similarly, we have

$$\delta(\xi u) < \delta(u),$$

and

$$\begin{aligned}\varphi(\xi u, \xi v) = & \left[\frac{(\xi u)^2}{(2-\xi u)} + \frac{(\xi u \xi v)J_1}{(2-\xi u)} + \frac{\xi v J_1}{2}\right]\psi(\xi u, \xi v) \\ & + \left[\xi v J_3 + \frac{\xi v J_1}{2} + \frac{\xi u^2}{(2-\xi u)}\right]\frac{(6-\xi u)}{(2-\xi u)}\psi(\xi u, \xi v) \\ & + \frac{\xi u}{2}\frac{(6-\xi u)^2}{(2-\xi u)^2}\psi(\xi u, \xi v)^2 \\ \leq & \ \xi^2 \varphi(u, v).\end{aligned}$$

In the same manner, we can show that

$$\varphi(\xi u, \xi^{(1+\theta)} v) \leq \xi^{(2+2\theta)}\varphi(u, v).$$

This completes the proof. □

For $j = 0$, the presumption of Γ_0 intends the occurrence of w_0 and b_0. Therefore, we obtain

$$\|w_0 - a_0\| = \|\Gamma_0 T(a_0)\| \leq \rho_0 \tag{2.11}$$

and

$$\|b_0 - a_0\| \leq \frac{\rho_0}{2}. \tag{2.12}$$

Thus w_0, $b_0 \in D(a_0, R\rho)$, at that point $R = \frac{p(\sigma_0)}{1-r_0}$.

Assume $N(a_0) = \Gamma_0[T'(a_0) - T'(b_0)]$, then

$$\|N(a_0)\| \leq K\tau\|b_0 - a_0\| \leq \frac{\sigma_0}{2}. \tag{2.13}$$

Applying Banach lemma, we can say that the operator $[I + N(a_0)]^{-1}$ exists and fulfills

$$\|[I + N(a_0)]^{-1}\| \leq \frac{2}{2-\sigma_0} \qquad \text{if } \sigma_0 < 2. \tag{2.14}$$

Hence, we get

$$\|c_0 - a_0\| \leq \frac{2}{(2-\sigma_0)}\rho_0. \tag{2.15}$$

Similarly,

$$\|c_0 - w_0\| \leq \frac{\sigma_0}{2-\sigma_0}\rho_0. \tag{2.16}$$

Again by virtue of Banach lemma, we observe that $[T'(b_0)]^{-1}$ exists and

$$\|[T'(b_0)]^{-1}\| \leq \frac{\tau_0}{1-\frac{1}{2}\sigma_0}. \tag{2.17}$$

Now, since

$$\begin{aligned}\|T(c_0)\| &\leq \|I+[T'(b_0)]^{-1}[T'(a_0)-T'(b_0)]\|\|T'(a_0)-T'(b_0)\|\|\Gamma_0 T(a_0)\| \\ &\quad + \|\int_0^1 (T'(a_0+u(c_0-a_0)-T'(a_0))\mathrm{d}u(c_0-a_0)\| \\ &\leq \frac{K\rho_0^2}{2-\sigma_0}+\frac{2K\rho_0^2}{(2-\sigma_0)^2},\end{aligned} \tag{2.18}$$

appropriately, we obtain

$$\|a_1-c_0\| \leq \frac{(6-\sigma_0)}{(2-\sigma_0)}\tau_0\|T(c_0)\|, \tag{2.19}$$

and

$$\begin{aligned}\|a_1-a_0\| &\leq \|a_1-c_0\|+\|c_0-a_0\| \\ &\leq \left[\frac{(6-\sigma_0)(4-\sigma_0)\sigma_0}{(2-\sigma_0)}+\frac{2}{(2-\sigma_0)}\right]\rho_0 \\ &\leq p(\sigma_0)\rho_0.\end{aligned} \tag{2.20}$$

Using the hypothesis $r_0 < 1/\delta(\sigma_0) < 1$, we obtain that $a_1 \in D(a_0, R\rho)$. For $\sigma_0 \leq \phi$ and $p(\sigma_0) \leq p(\phi)$, we get

$$\begin{aligned}\|I-\Gamma_0 T'(a_1)\| &\leq \|\Gamma_0\|\|T'(a_0)-T'(a_1)\| \\ &\leq K\tau_0\|a_1-a_0\| \\ &\leq \sigma_0 p(\sigma_0) \\ &< 1.\end{aligned} \tag{2.21}$$

Furthermore,

$$\begin{aligned}\|\Gamma_1\| &\leq \frac{\tau_0}{(1-\sigma_0 p(\sigma_0))} = \delta(\sigma_0)\tau_0 \\ &= \tau_1 < 1.\end{aligned} \tag{2.22}$$

Now, we try to find the norm of $T(c_0)$. From the result obtained in [**5**], we have

$$\begin{aligned}T(c_n) = &\int_0^1 T''(w_n+u(c_n-w_n))(1-u)\mathrm{d}u(c_n-w_n)^2 \\ &-\int_0^1 T''(b_n+u(w_n-b_n))(w_n-b_n)\mathrm{d}u \\ &[T'(b_n)]^{-1}[T'(b_n)-T'(a_n)](w_n-a_n) \\ &+\int_0^1 [T''(a_n+u(w_n-a_n))-T''(a_n)](1-u)\mathrm{d}u(w_n-a_n)^2 \\ &+\frac{1}{2}\int_0^1 [T''(a_n)-T''(a_n+\frac{1}{2}u(w_n-a_n)]\mathrm{d}u(w_n-a_n)^2,\end{aligned} \tag{2.23}$$

and hence, we can obtain

$$\|T(c_0)\| \leq \frac{1}{2}K\|c_0 - w_0\|^2 + K^2\|T'(b_0)^{-1}\|\|w_0 - b_0\|\|b_0 - a_0\|\|w_0 - a_0\| + \frac{w(\rho_0)}{2}J_1\|w_0 - a_0\|^2 + w(\rho_0)J_2\|w_0 - a_0\|^2. \tag{2.24}$$

Thus, we attain

$$\|a_1 - c_0\| \leq \frac{(6-\sigma_0)}{(2-\sigma_0)}\tau_0\|T(c_0)\| \leq \frac{(6-\sigma_0)}{(2-\sigma_0)}\left[\left(\frac{1}{2}\frac{\sigma_0^3}{(2-\sigma_0)^2} + \frac{\sigma_0^2}{2(2-\sigma_0)} + \frac{J_1\beta_0}{2} + J_2\beta_0\right)\rho_0\right]. \tag{2.25}$$

By Lemma 3, we can write

$$\|T(a_1)\| \leq \left[\frac{\sigma_0}{(2-\sigma_0)}(K\rho_0 + w(\rho_0)\rho_0 J_1) + \frac{\rho_0 w(\rho_0)J_1}{2}\right]\|\Gamma_0 T(z_0)\| + \left[\rho_0 w(\rho_0)J_3 + \frac{\rho_0}{2}w(\rho_0)J_1 + \frac{K\sigma_0\rho_0}{(2-\sigma_0)}\|a_1 - c_0\| + \frac{K}{2}\|a_1 - c_0\|^2\right]. \tag{2.26}$$

Considering the equations (2.22) and (2.26), it follows that

$$\|w_1 - a_1\| \leq \|\Gamma_1 T(a_1)\| \leq \delta(\sigma_0)\varphi(\sigma_0, \beta_0)\rho_0 = r_0\rho_0 = \rho_1. \tag{2.27}$$

Since $p(\sigma_0) > 1$, hence

$$\|w_1 - a_0\| \leq \|w_1 - a_1\| + \|a_1 - a_0\| < p(\sigma_0)(1 + r_0)\rho < R\rho, \tag{2.28}$$

which gives $w_1 \in D(a_0, R\rho)$. So that we can conclude

$$\begin{aligned} K\|\Gamma_1\|\|\Gamma_1 T(a_1)\| &\leq \delta(\sigma_0)r_0\sigma_0 = \sigma_1, \\ \|\Gamma_1\|\|\Gamma_1 T(a_1)\|w(\|\Gamma_1 T(a_1)\|) &\leq \tau_0\rho_0 w(\rho_0) = \beta_1. \end{aligned} \tag{2.29}$$

Continuing this procedure, we can establish the following system of recurrence relations.

Lemma 5. *Using the presumption of* Lemma 3 and Lemma 4*, the hypotheses* $(C1) - (C5)$ *preserve. For all* $n \geq 0$ *the following properties are true*

(1) *There exists* $\Gamma_j = [T'(a_j)]^{-1}$ *and* $\|\Gamma_j\| \leq \tau_j$,
(2) $\|\Gamma_j T(a_j)\| \leq \rho_j$,
(3) $K\|\Gamma_j\|\|\Gamma_j T(a_j)\| \leq \sigma_j$,
(4) $\|\Gamma_j\|\|\Gamma_j T(a_j)\|w(\|\Gamma_j T(a_j)\|) \leq \beta_j$,
(5) $\|a_{j+1} - a_j\| \leq p(\sigma_j)\rho_j$,
(6) $\|w_j - a_0\| \leq R\rho$,
(7) $\|c_j - a_0\| \leq R\rho$,

(8) $\|a_{j+1} - a_0\| \le R\rho$, *where* $R = \dfrac{p(\sigma_0)}{1 - \delta(\sigma_0)\varphi(\sigma_0, \beta_0)}$.

Proof. The proof of results (1)–(5) is obvious by applying preliminaries results and induction principle. So, we prove only results (6)–(8). Using Lemma 2 and assumptions (C1)–(C5), we obtain

$$
\begin{aligned}
\|w_j - a_0\| &\le \|w_j - a_j\| + \|a_j - a_0\| \le p(\sigma_0)\sum_{i=0}^{j}\rho_i, \\
\|c_j - a_0\| &\le \|c_j - a_j\| + \|a_j - a_0\| \\
&\le \frac{2}{2-\sigma_j}\rho_j + \sum_{i=0}^{j-1}\|a_{i+1} - a_i\| \\
&\le p(\sigma_j)\rho_j + \sum_{i=0}^{j-1} p(\sigma_0) \le p(\sigma_0)\sum_{i=0}^{j-1}\rho_i,
\end{aligned} \tag{2.30}
$$

and

$$
\begin{aligned}
\|a_{j+1} - a_0\| &\le \sum_{i=0}^{j}\|a_{i+1} - a_i\| \\
&\le \sum_{i=0}^{j} p(\sigma_i)\rho_i \le p(\sigma_0)\sum_{i=0}^{j}\left(\prod_{j=0}^{i-1} r_j\right)\rho.
\end{aligned} \tag{2.31}
$$

Thus proof is over. □

Now, we consider $\zeta = \delta^2(\sigma_0)\varphi(\sigma_0, \beta_0)$ and $\lambda = 1/\delta(\sigma_0)$. Since $\sigma_1 = \delta^2(\sigma_0)\varphi(\sigma_0, \beta_0)\sigma_0 = \zeta\sigma_0$, $\beta_1 < \zeta\beta_0$, so that by virtue of the equation (2.8) and Lemma 4, it can be written as $r_1 < \delta(\zeta\sigma_0)\varphi(\zeta\sigma_0, \zeta\beta_0) < \zeta^{3^1-1}\delta(\sigma_0)\varphi(\sigma_0, \beta_0) = \lambda\zeta^{3^1}$. Suppose $r_{j+1} \le \lambda\zeta^{3^j}, j \ge 1$. From Lemma 2, we obtain $\sigma_{j+1} < \sigma_j$ and $\delta(\sigma_j)r_j < 1$. Thus

$$
r_{j+1} < \delta(\sigma_j)\varphi(\delta(\sigma_j)r_j\sigma_j, \delta(\sigma_j)r_j\beta_j) < \lambda\zeta^{3^{j+1}}. \tag{2.32}
$$

So, we have $r_j \le \lambda\zeta^{3^j}$, $j \ge 0$, and

$$
\prod_{l=0}^{i-1} r_l \le \prod_{l=0}^{i-1}\lambda\zeta^{3^l} = \lambda^i\zeta^{\frac{3^i-1}{2}}, \qquad i \ge 0. \tag{2.33}
$$

After using the above relation in equation (2.31), we get

$$
\begin{aligned}
\|a_{j+1} - a_0\| &\le p(\sigma_0)\rho\Sigma_{i=0}^{j}\lambda^i\zeta^{\frac{3^i-1}{2}} \\
&\le p(\sigma_0)\rho\frac{1-\lambda^{j+1}\zeta^{\frac{3^j+1}{2}}}{1-d_0} < R\rho.
\end{aligned} \tag{2.34}
$$

Similarly, $\|w_j - a_0\| \le R\rho$, $\|c_j - a_0\| \le R\rho$. This completes the proof.

Lemma 6. *Let* $R = \frac{p(\sigma_0)}{1-r_0}$. *If* $\sigma_0 < s$ *and* $\delta(\sigma_0)r_0 < 1$, *then* $R < 1/\sigma_0$.

Proof. The proof is obvious. □

3. Semilocal convergence

In this section, we focus on deriving the existence and uniqueness theorem, which shows the semilocal convergence of the scheme (1.1).

Theorem 1. *Consider* Ω *is an open subset and* $T\colon \Omega \subseteq U \to V$ *is a twice Fréchet differential operator from Banach spaces* U *to* V. *Suppose that* $a_0 \in \Omega$ *and the hypotheses* (C1)–C5) *are true. Let* $\sigma_0 = K\tau\rho$, $\beta_0 = \tau\rho w(\rho)$, $r_0 = \delta(\sigma_0)\varphi(\sigma_0, \beta_0)$, $\sigma_0 < \phi$, *and* $\delta(\sigma_0)r_0 < 1$, *where* p, δ *and* φ *are given by the equations* (2.1)–(2.3). *Let* $D(a_0, R\rho) \subset \Omega$ *with* $R = p(\sigma_0)/(1 - \delta(\sigma_0)\varphi(\sigma_0, \beta_0)$, *then starting from* a_0, *the sequence* a_j *originated by the scheme* (1.1) *tends to a solution* a^* *of* $T(a) = 0$ *with* a_j, a^* *belonging to* $D(a_0, R\rho)$, *and* a^* *is the unique solution of* $T(a) = 0$ *in* $D(a_0, (2/M\tau) - R\rho) \bigcap \Omega$. *Furthermore, an error bound is given by*

$$\|a_j - a^*\| \le \frac{1}{1 - \lambda\zeta^{3^j}}(p(\sigma_0)\rho\lambda^j\zeta^{(3^j-1)/2}), \tag{3.1}$$

where $\zeta = \delta(\sigma_0)r_0$ *and* $\lambda = 1/\delta(\sigma_0)$.

Proof. Using Lemma 5, first we prove that a_j is a Cauchy sequence. For this, we consider

$$\begin{aligned}\|a_{j+m} - a_j\| &\le \|a_{j+m} - a_{j+m-1}\| + \cdots + \|a_{j+1} - a_j\| \\ &\le \sum_{i=j}^{j+m-1} \|a_{i+1} - a_i\| \le \sum_{i=j}^{j+m-1} p(\sigma_i)\rho_i \\ &\le p(\sigma_0)\rho\lambda^j\zeta^{\frac{3^j-1}{2}}\left(\frac{1 - \lambda^{m+1}\zeta^{\frac{3^j(3^m+1)}{2}}}{1 - \lambda\zeta^{3^j}}\right)\end{aligned} \tag{3.2}$$

which shows that a_j is a Cauchy sequence, and hence there exists a^* such that $\lim_{j\to\infty} a_j = a^*$. Suppose $j = 0, m \to \infty$, we have

$$\|a^* - a_0\| \le R\rho. \tag{3.3}$$

Above result shows that $a^* \in D(a_0, R\rho)$. In the same way, we observe that a^* is a solution $T(a) = 0$. Now

$$\|\Gamma_0 T(a_0)\| \le \|\Gamma_1\|\|T(a_1)\| \le \cdots \le \|\Gamma_j\|\|T(a_j)\| \le \rho_j. \tag{3.4}$$

If, we consider $j \to \infty$ in (3.4), which shows that $\|T(a_j)\| \to 0$ since $p(\sigma_j) < p(\sigma_0)$ and $\rho_j \to 0$. T is a continuous operator in Ω and $T(a^*) = 0$. Let us assume that c^* is another solution of $T(a) = 0$ in $D(a_0, (2/M\tau) - R\rho) \bigcap \Omega$. To prove a^* is a unique solution of the function T, hence

$$\frac{2}{K\tau} - R\rho = \Big(\frac{2}{\sigma_0} - R\Big)\rho > R\rho. \tag{3.5}$$

By Taylor's theorem, we have

$$0 = T(c^*) - T(a^*) = \int_0^1 T'((1-u)a^* + uc^*)\mathrm{d}u(c^* - a^*) \tag{3.6}$$

and

$$\begin{aligned} &\|\Gamma_0\|\| \int_0^1 [T'((1-u)a^* + uc^*) - T'(a_0)]\mathrm{d}u\| \\ &\leq K\tau \int_0^1 [(1-u)\|a^* - a_0\| + u\|c^* - a_0\|]\mathrm{d}u \\ &\leq \frac{K\tau}{2}\Big[R\rho + \frac{2}{K\tau} - R\rho\Big] = 1. \end{aligned} \tag{3.7}$$

Using Banach lemma, we observe that $\int_0^1 T'((1-u)a^* + uc^*)\mathrm{d}u$ is invertible, and hence $c^* = a^*$. Now, by assuming $m \to \infty$ in (3.2), we have the expression (3.1). □

R-order of convergence

Theorem 2. *Suppose the hypothesis of* Lemma 3 *and* Lemma 4 *are true, then R- order of convergence of the scheme* (1.1) *is at least* $3+2\theta$ *and its error estimate is given by*

$$\|a_j - a^*\| \leq \frac{p(\sigma_0)\rho}{\zeta^{\frac{1}{2+2\theta}(1-r_0)}} \left(\zeta^{1/(2+2\theta)}\right)^{(3+2\theta)^j}. \tag{3.8}$$

Proof. Using the hypothesis of Lemma 3 and Lemma 4, the sequence $\{\rho_j\}$ can be converted into the form

$$\begin{aligned} \rho_j &= \delta(\sigma_{j-1})\varphi(\sigma_{j-1}, \beta_{j-1})\rho_{j-1} \\ &= \rho\left(\prod_{i=0}^{j-1} \delta(\sigma_i)\varphi(\sigma_i, \beta_i)\right) \leq \rho\lambda^j \zeta^{\frac{((3+2\theta)^j - 1)}{(2+2\theta)}}. \end{aligned} \tag{3.9}$$

As a result of $\lambda < 1$ and $\zeta < 1$, it concludes that $\rho_j \to 0$ as $j \to \infty$; which shows that the sequence $\{\rho_j\}$ converges to zero. When $j \geq 0$, $m \geq 1$,

$$\begin{aligned} \sum_{i=j}^{j+m} \rho_i &\leq \rho \sum_{i=j}^{j+m} \lambda^i \zeta^{\frac{(3+2\theta)^j - 1}{(2+2\theta)}} \\ &= \rho\zeta^{-1/(2+2\theta)}\left(\lambda^j \zeta^{(3+2\theta)^j/(2+2\theta)} + \sum_{i=j+1}^{j+m} \lambda^i \zeta^{(3+2\theta)^i/(2+2\theta)}\right) \\ &\leq \rho\lambda^j \zeta^{(3+2\theta)^j - 1)/(2+2\theta)}\left(\frac{1 - \lambda^{m+1}\zeta^{\frac{(3+2\theta)^j((3+2\theta)^m + (2+2\theta)}{(2+2\theta)}}}{1 - \lambda\zeta^{(3+2\theta)^j}}\right) \qquad \text{for } j \geq 0. \end{aligned} \tag{3.10}$$

Therefore, $\sum_{j=0}^{\infty} \rho_j$ exists. The theorem immediately follows by using relations obtained above in Theorem 1. □

It is noted that if $\theta = 1$, then the R-order becomes five.

4. Numerical result

In this part, we find the solution of mixed Hammerstein type integral equation, which is given as

$$a(s) = 1 + \int_0^1 Q(s,t)\Big(\frac{1}{2}a(t)^{5/2} + \frac{7}{16}a(t)^3\Big)\mathrm{d}t, \qquad s \in [0,1], \tag{4.1}$$

and $T\colon \Omega \subseteq C[0,1] \to C[0,1]$, where $a \in C[0,1]$, $t \in [0,1]$, and Green's function Q is defined as follow

$$Q(s,t) = \begin{cases} (1-s)t & t \le s, \\ s(1-t) & s \le t. \end{cases}$$

To attain the solution of above Hammerstein type integral equation, we take $T(a) = 0$ such that

$$T(a)(s) = a(s) - 1 - \int_0^1 Q(s,t)\Big(\frac{1}{2}a(t)^{5/2} + \frac{7}{16}a(t)^3\Big)\mathrm{d}t, \qquad s \in [0,1], \tag{4.2}$$

where $\Omega = D(0,2)$. First and second Frećhet derivative of T can be obtained as

$$T'(a)b(s) = b(s) - \int_0^1 Q(s,t)\Big(\frac{5}{4}a(t)^{3/2} + \frac{21}{16}a(t)^2\Big)b(t)\mathrm{d}t,$$
$$T''(a)bc(s) = -\int_0^1 Q(s,t)\Big(\frac{15}{8}a(t)^{1/2} + \frac{21}{8}a(t)\Big)b(t)c(t)\mathrm{d}t.$$

It is observe that second derivative of T'' doesn't satisfy the Lipschitz condition because

$$\|T''(a) - T''(b)\| \le \frac{15}{8}\Big\|\int_0^1 Q(s,t)\mathrm{d}t\Big\|\|a-b\|^{1/2} + \frac{21}{8}\Big\|\int_0^1 Q(s,t)\mathrm{d}t\Big\|\|a-b\|,$$

since

$$\begin{aligned} \Big\|\int_0^1 Q(s,t)\mathrm{d}t\Big\| &= \Big\|\int_0^s (1-s)t\mathrm{d}t + \int_s^1 s(1-t)\mathrm{d}t\Big\|, \\ &= \Big\| -\frac{1}{2}\Big(s-\frac{1}{2}\Big)^2 + \frac{1}{8}\Big\| = \frac{1}{8}, \end{aligned} \tag{4.3}$$

hence, we can obtain

$$\|T''(a) - T''(b)\| \le \frac{15}{64}\|a-b\|^{1/2} + \frac{21}{64}\|a-b\|. \tag{4.4}$$

Now for $t \in [0,1]$, we can write

$$w(t\mu) = \frac{15}{64}(t\mu)^{1/2} + \frac{21}{64}(t\mu) \le t^{1/2}w(\mu), \tag{4.5}$$

which implies that

$$v(t) = t^{1/2}.$$

Suppose that the initial approximation is $a_0(t) = 1$. Furthermore, we possess

$$\|T(a_0)\| = \frac{15}{128}$$

and

$$\|I - T'(a_0)\| = \frac{41}{16}\|\int_0^1 Q(s,t)\mathrm{d}t\| = \frac{41}{128} < 1.$$

By Banach lemma, we have

$$\|\Gamma_0\| \leq \frac{1}{1 - (\|I - T'(a_0)\|)} = \frac{128}{87} = \tau.$$

Moreover, we get

$$\|\Gamma_0 T(a_0)\| \leq \|\gamma_0\|\|T(a_0)\| \leq \frac{15}{87} = \rho$$

and

$$\|T''(a)\| \leq \left(\frac{15\sqrt{2}}{8} + \frac{21}{4}\right)\|\int_0^1 Q(s,t)\mathrm{d}t\| = \frac{15\sqrt{2}}{64} + \frac{21}{32} = M.$$

Thus

$$\sigma_0 \backsimeq 0.2505 \tag{4.6}$$

and

$$p(\sigma_0)\sigma_0 \backsimeq 0.539167 \leq 1. \tag{4.7}$$

Above value proves that $\sigma_0 < \phi$ and

$$\delta(\sigma_0)r_0 \backsimeq 0.0477459 \leq 1. \tag{4.8}$$

As a result, it also follows that the solution a^* belongs to $\overline{D(a_0, R\rho)} = \overline{D(1, 0.251\dots)} \subset \Omega = D(0,2)$ following the hypothesis of the Theorem 1. Moreover, it is unique solution in $D(1, 1.118\dots) \cap \Omega$.

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