Acta Math. Univ. Comenianae
Vol. LXXXIV, 1 (2015), pp. 103–121

COMPLETE FRACTIONAL MONOTONE APPROXIMATION

G. A. ANASTASSIOU

Abstract. The theory of complete fractional simultaneous monotone uniform polynomial approximation with rates using mixed fractional linear differential operators is developed and presented in the paper.

To achieve that, we establish first ordinary simultaneous polynomial approximation with respect to the highest order right and left fractional derivatives of the function under approximation using their moduli of continuity. Then we derive the complete right and left fractional simultaneous polynomial approximation with rates, as well we treat their affine combination. Based on the last and elegant analytical techniques, we derive preservation of monotonicity by mixed fractional linear differential operators. We study some special cases.

1. Introduction

The topic of monotone approximation introduced in [**5**] has become a major trend in approximation theory. A typical problem in this subject is: for a positive integer k, approximate a given function whose k-th derivative is grater or equal to zero by polynomials that have this property.

In [**2**], the authors replaced the kth derivative with a linear differential operator of order k. We mention this motivating result.

Theorem 1. *Let h, k, p be integers, $0 \leq h \leq k \leq p$, and let f be a real function, $f^{(p)}$ continuous in $[-1,1]$ with first modulus of continuity $\omega_1\left(f^{(p)}, x\right)$ there. Let $a_j(x)$, $j = h, h+1, \ldots, k$ be real functions, defined and bounded on $[-1,1]$, and assume that $a_h(x)$ is either $\geq$ some number $\alpha > 0$ or $\leq$ some number $\beta < 0$ throughout $[-1,1]$. Consider the operator*

$$L = \sum_{j=h}^{k} a_j(x) \left[\frac{d^j}{dx^j}\right] \tag{1}$$

and throughout $[-1,1]$, suppose that

$$L(f) \geq 0. \tag{2}$$

Received April 23, 2014.
2010 *Mathematics Subject Classification.* Primary 26A33, 41A10, 41A17, 41A25, 41A29.
Key words and phrases. Monotone Approximation; Caputo fractional derivative; fractional linear differential operator; modulus of continuity.

Then, for every integer $n \geq 1$, there is a real polynomial $Q_n(x)$ of degree $\leq n$ such that

$$L(Q_n) \geq 0 \text{ throughout } [-1,1] \tag{3}$$

and

$$\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq C n^{k-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right), \tag{4}$$

where C is independent of n or f.

The purpose of this article is to extend completely Theorem 1 to the fractional level. All involved ordinary derivatives will become now fractional derivatives and even more we will have fractional simultaneous approximation.

We need the following definitions

Definition 2. ([**3**, p. 50]) Let $\alpha > 0$ and $\lceil \alpha \rceil = m$, ($\lceil \cdot \rceil$ ceiling of the number). Consider $f \in AC^m([0,1])$ (space of functions f with $f^{(m-1)} \in AC([0,1])$, absolutely continuous functions), $z \in [0,1]$. We define the left Caputo fractional derivative of f of order α as follows:

$$\left(D_{*z}^{\alpha} f\right)(x) = \frac{1}{\Gamma(m-\alpha)} \int_z^x (x-t)^{m-\alpha-1} f^{(m)}(t)\, \mathrm{d}t \tag{5}$$

for any $x \in [z,1]$, where Γ is the gamma function.

We set

$$\begin{aligned} &D_{*z}^{0} f(x) = f(x), \\ &D_{*z}^{m} f(x) = f^{(m)}(x) \qquad \text{for all } x \in [z,1]. \end{aligned} \tag{6}$$

Definition 3 ([**4**])**.** Let $\alpha > 0$ and $\lceil \alpha \rceil = m$. Consider $f \in AC^m([0,1])$, $z \in [0,1]$. We define the right Caputo fractional derivative of f of order α as follows:

$$\left(D_{z-}^{\alpha} f\right)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^z (t-x)^{m-\alpha-1} f^{(m)}(t)\, \mathrm{d}t \tag{7}$$

for any $x \in [0,z]$.

We set

$$\begin{aligned} &D_{z-}^{0} f(x) = f(x), \\ &D_{z-}^{m} f(x) = (-1)^m f^{(m)}(x) \qquad \text{for all } x \in [0,z]. \end{aligned} \tag{8}$$

Remark 4 (to Definitions 2, 3). Let $n \in \mathbb{N}$ with $f^{(n)} \in AC^m([0,1])$, where $\alpha > 0$, $\lceil \alpha \rceil = m$ with $\alpha \notin \mathbb{N}$, and $\lceil n + \alpha \rceil = n + \lceil \alpha \rceil = n + m$, then

$$\begin{aligned} \left(D_{*z}^{\alpha} f^{(n)}\right)(x) &= \frac{1}{\Gamma(m-\alpha)} \int_z^x (x-t)^{m-\alpha-1} \left(f^{(n)}(t)\right)^{(m)} \mathrm{d}t \\ &= \frac{1}{\Gamma((n+m)-(n+\alpha))} \int_z^x (x-t)^{(n+m)-(n+\alpha)-1} f^{(n+m)}(t)\, \mathrm{d}t \\ &= D_{*z}^{n+\alpha} f(x). \end{aligned} \tag{9}$$

That is,

$$\left(D_{*z}^{\alpha} f^{(n)}\right)(x) = D_{*z}^{n+\alpha} f(x), \qquad \text{for all } x \in [z,1]. \tag{10}$$

Similarly, we get

$$\begin{aligned}
\left(D_{z-}^{\alpha} f^{(n)}\right)(x) &= \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^z (t-x)^{m-\alpha-1} \left(f^{(n)}(t)\right)^{(m)} \mathrm{d}t \\
&= \frac{(-1)^{n+m}(-1)^n}{\Gamma((n+m)-(n+\alpha))} \int_x^z (t-x)^{(n+m)-(n+\alpha)-1} f^{(n+m)}(t)\mathrm{d}t \\
&= (-1)^n D_{z-}^{n+\alpha} f(z).
\end{aligned} \tag{11}$$

That is,

$$\left(D_{z-}^{\alpha} f^{(n)}\right)(x) = (-1)^n D_{z-}^{n+\alpha} f(z) \qquad \text{for all } x \in [0,z]. \tag{12}$$

Next we consider $f \in C([0,1])$ and the Bernstein polynomials

$$(B_N f)(t) = \sum_{k=0}^{N} f\left(\frac{k}{N}\right) \binom{N}{k} t^k (1-t)^{N-k}$$

for all $t \in [0,1]$, $N \in \mathbb{N}$, of degree N.

We have $B_N 1 = 1$, and B_N are positive linear operators.

Theorem 5 ([**1**])**.** *Let $0 < \alpha < 1$, $r > 0$ and $f \in AC([0,1])$ such that $f' \in L_\infty([0,1])$.*

Then we have

$$\begin{aligned}
\|B_N f - f\|_\infty \le{}& \frac{1}{\Gamma(\alpha+1)} \left(1 + \frac{1}{(\alpha+1)r}\right) \\
&\Big[\sup_{x\in[0,1]} \omega_1 \left(D_{x-}^{\alpha} f, r \left\|B_n\left(|\cdot - x|^{\alpha+1} \chi_{[0,x]}(\cdot), x\right)\right\|_\infty^{\frac{1}{(\alpha+1)}}\right)_{[0,x]} \\
&\cdot \left\|B_n\left(|\cdot - x|^{\alpha+1} \chi_{[0,x]}(\cdot), x\right)\right\|_\infty^{\frac{\alpha}{(\alpha+1)}} \\
&+ \sup_{x\in[0,1]} \omega_1 \left(D_{*x}^{\alpha} f, r \left\|B_n\left(|\cdot - x|^{\alpha+1} \chi_{[x,1]}(\cdot), x\right)\right\|_\infty^{\frac{1}{(\alpha+1)}}\right)_{[x,1]} \\
&\cdot \left\|B_n\left(|\cdot - x|^{\alpha+1} \chi_{[x,1]}(\cdot), x\right)\right\|_\infty^{\frac{\alpha}{(\alpha+1)}} \Big] \qquad \textit{for all } N \in \mathbb{N}.
\end{aligned} \tag{13}$$

χ above stands for the characteristic function, also the two first moduli of continuity are over the intervals $[0,x]$ and $[x,1]$, respectively, as indicated.

Remark 6 (to Theorem 5). Next, we choose $r = \frac{1}{\alpha+1}$, $p = \frac{2}{\alpha+1} > 1$, $q = \frac{2}{1-\alpha} > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

We observe that

$$\begin{aligned} & B_n\left(|\cdot - x|^{\alpha+1}\chi_{[0,x]}(\cdot), x\right), \ B_n\left(|\cdot - x|^{\alpha+1}\chi_{[x,1]}(\cdot), x\right) \\ & \le B_n\left(|\cdot - x|^{\alpha+1}, x\right) = \sum_{k=0}^{N}\left|x - \frac{k}{N}\right|^{\alpha+1}\binom{N}{k}x^k(1-x)^{N-k} \end{aligned} \tag{14}$$

(by discrete Hölder's inequality)

$$\begin{aligned} & \le \left(\sum_{k=0}^{N}\left(x - \frac{k}{N}\right)^2\binom{N}{k}x^k(1-x)^{N-k}\right)^{\frac{\alpha+1}{2}} = \left(\frac{x(1-x)}{N}\right)^{\frac{\alpha+1}{2}} \\ & \le \frac{1}{(4N)^{\frac{\alpha+1}{2}}} = \frac{1}{\left(2\sqrt{N}\right)^{\alpha+1}} \qquad \text{for all } x \in [0,1]. \end{aligned} \tag{15}$$

We have proved the following important auxilliary result.

Theorem 7. *Let $0 < \alpha < 1$, $f \in AC([0,1])$ with $f' \in L_\infty([0,1])$, $N \in \mathbb{N}$. Then*

$$\begin{aligned} \|B_N f - f\|_\infty \le & \frac{2^{1-\alpha}}{\Gamma(\alpha+1)N^{\frac{\alpha}{2}}}\Big[\sup_{x\in[0,1]}\omega_1\Big(D_{x-}^{\alpha}f, \frac{1}{2(\alpha+1)N^{\frac{1}{2}}}\Big)_{[0,x]} \\ & + \sup_{x\in[0,1]}\omega_1\Big(D_{*x}^{\alpha}f, \frac{1}{2(\alpha+1)N^{\frac{1}{2}}}\Big)_{[x,1]}\Big] =: T_N^{\alpha}(f) < \infty. \end{aligned} \tag{16}$$

Proof. By (13) and (15).
By [**1**], we get that the quantity within the bracket of (16) is finite. □

So as $N \to \infty$, we derive $B_N f \overset{u}{\to} f$ (uniformly) with rates.

2. Main Result

We give the following simultaneous approximation fractional result.

Theorem 8. *Let $\beta > 0$, $\beta \notin \mathbb{N}$ with integral part $[\beta] = n \in \mathbb{Z}_+$ such that $\beta = n + \alpha$, where $0 < \alpha < 1$. Let $f \in AC^{n+1}([0,1])$ and $f^{(n+1)} \in L_\infty([0,1])$, $N \in \mathbb{N}$. Set*

$$\begin{aligned} P_{N+n}(f)(x) := & \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}x^k \\ & + \int_0^x\left(\int_0^{x_{n-1}}\cdots\left(\int_0^{x_1}B_N\left(f^{(n)}\right)(t_1)\,\mathrm{d}t_1\right)\mathrm{d}x_1\ldots\right)\mathrm{d}x_{n-1} \\ = & \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}x^k + \frac{1}{(n-1)!}\int_0^x(x-t)^{n-1}B_N\left(f^{(n)}\right)(t)\mathrm{d}t, \end{aligned} \tag{17}$$

for all $0 \le x \le 1$, a polynomial of degree $(N+n)$. $B_N\left(f^{(n)}\right)$ is the Bernstein polynomial of degree N. If $n = 0$, the sum in (17) collapses.

Set also

$$T_N^{\beta,\alpha}(f) := \frac{2^{1-\alpha}}{\Gamma(\alpha+1)N^{\frac{\alpha}{2}}}\Big[\sup_{x\in[0,1]}\omega_1\left(D_{x-}^{\beta}f, \frac{1}{2(\alpha+1)N^{\frac{1}{2}}}\right)_{[0,x]} + \sup_{x\in[0,1]}\omega_1\left(D_{*x}^{\beta}f, \frac{1}{2\,(\alpha+1)\,N^{\frac{1}{2}}}\right)_{[x,1]}\Big] < \infty \tag{18}$$

for every $N \in \mathbb{N}$.

Then $P_{N+n}^{(i)}(f) = P_{N+n-i}(f)$ *and*

$$\left\| P_{N+n}^{(i)}(f) - f^{(i)} \right\|_{\infty,[0,1]} \le \frac{T_N^{\beta,\alpha}(f)}{(n-i)}, \qquad i = 0, 1, \ldots, n. \tag{19}$$

As $N \to \infty$, *we derive with rates* $P_{N+n}^{(i)}(f) \xrightarrow{u} f^{(i)}$.

Proof. Notice that $\lceil\beta\rceil = (n+1) \in \mathbb{N}$. By (16) we have

$$\left\| B_N\left(f^{(n)}\right) - f^{(n)} \right\|_{\infty,[0,1]} \le T_N^{\alpha}\left(f^{(n)}\right) < \infty \tag{20}$$

that is,

$$-T_N^{\alpha}\left(f^{(n)}\right) \le B_N\left(f^{(n)}\right)(t_1) - f^{(n)}(t_1) \le T_N^{\alpha}\left(f^{(n)}\right) \tag{21}$$

for every $0 \le t_1 \le x_1 \le 1$.

Set

$$P_N(f)(x) := B_N\left(f^{(n)}\right)(x). \tag{22}$$

Hence it holds

$$-T_N^{\alpha}\left(f^{(n)}\right)x_1 \le \int_0^{x_1} B_N\left(f^{(n)}\right)(t_1)\,dt_1 - \int_0^{x_1} f^{(n)}(t_1)\,dt_1 \le T_N^{\alpha}\left(f^{(n)}\right)x_1, \tag{23}$$

that is,

$$-T_N^{\alpha}\left(f^{(n)}\right)x_1 \le f^{(n-1)}(0) + \int_0^{x_1} B_N\left(f^{(n)}\right)(t_1)\,dt_1 - f^{(n-1)}(x_1) \le T_N^{\alpha}\left(f^{(n)}\right)x_1. \tag{24}$$

Set

$$P_{N+1}(f)(x) := f^{(n-1)}(0) + \int_0^{x} B_N\left(f^{(n)}\right)(t_1)\,dt_1, \tag{25}$$

that is,

$$P'_{N+1}(f)(x) = P_N(f)(x) \qquad \text{all } x \in [0,1]. \tag{26}$$

Hence

$$-T_N^{\alpha}\left(f^{(n)}\right)x_1 \le P_{N+1}(f)(x_1) - f^{(n-1)}(x_1) \le T_N^{\alpha}\left(f^{(n)}\right)x_1 \tag{27}$$

for all $x_1 \in [0,1]$.

Continuing the procedure, we get

$$-T_N^{\alpha}\left(f^{(n)}\right)\frac{x_2^2}{2} \le f^{(n-2)}(0) + \int_0^{x_2} P_{N+1}(f)(x_1)\,\mathrm{d}x_1 - f^{(n-2)}(x_2) \le T_N^{\alpha}\left(f^{(n)}\right)\frac{x_2^2}{2} \tag{28}$$

for all $0 \le x_1 \le x_2 \le 1$.

Set

$$P_{N+2}(f)(x) := f^{(n-2)}(0) + \int_0^{x} P_{N+1}(f)(x_1)\,\mathrm{d}x_1, \tag{29}$$

that is,

$$P'_{N+2}(f)(x) = P_{N+1}(f)(x), \tag{30}$$

and

$$P''_{N+2}(f)(x) = P_N(f)(x) \qquad \text{all } x \in [0,1]. \tag{31}$$

So far we have

$$-T_N^{\alpha}\left(f^{(n)}\right)\frac{x_2^2}{2} \le P_{N+2}(f)(x_2) - f^{(n-2)}(x_2) \le T_N^{\alpha}\left(f^{(n)}\right)\frac{x_2^2}{2} \tag{32}$$

for all $x_2 \in [0,1]$.

Similarly, we derive

$$-T_N^{\alpha}\left(f^{(n)}\right)\frac{x_3^3}{3!} \le f^{(n-3)}(0) + \int_0^{x_3} P_{N+2}(f)(x_2)\mathrm{d}x_2 - f^{(n-3)}(x_3) \le T_N^{\alpha}(f^{(n)})\frac{x_3^3}{3!} \tag{33}$$

for all $0 \le x_2 \le x_3 \le 1$.

Set

$$P_{N+3}(f)(x) := f^{(n-3)}(0) + \int_0^{x} P_{N+2}(f)(x_2)\mathrm{d}x_2, \tag{34}$$

that is,

$$P'_{N+3}(f)(x) = P_{N+2}(f)(x), \tag{35}$$

and

$$P'''_{N+3}(f)(x) = P_N(f)(x) \qquad \text{for all } x \in [0,1]. \tag{36}$$

Hence

$$-T_N^{\alpha}\left(f^{(n)}\right)\frac{x_3^3}{3!} \le P_{N+3}(f)(x_3) - f^{(n-3)}(x_3) \le T_N^{\alpha}\left(f^{(n)}\right)\frac{x_3^3}{3!} \tag{37}$$

for all $x_3 \in [0,1]$.

Continuing as above, after n steps we get

$$-T_N^{\alpha}\left(f^{(n)}\right)\frac{x_n^n}{n!} \le P_{N+n}(f)(x_n) - f(x_n) \le T_N^{\alpha}(f^{(n)})\frac{x_n^n}{n!} \tag{38}$$

with $0 \le x_n \le 1$.

As above

$$P_{N+n}(f)(x) := f(0) + \int_0^x P_{N+n-1}(f)(x_{n-1})\mathrm{d}x_{n-1}, \tag{39}$$

that is,

$$P'_{N+n}f)(x) = P_{N+n-1}(f)(x), \tag{40}$$

and

$$P^{(n)}_{N+n}(f)(x) = P_N(f)(x) = B_N(f^{(n)})(x) \qquad \text{all } x \in [0,1]. \tag{41}$$

So clearly, $P_{N+n}(f)$ has the representations (17), the second one comes from Taylor's theorem.

By (21), we get

$$\left\| P_N(f) - f^{(n)} \right\|_\infty \le T_N^\alpha(f^{(n)}), \tag{42}$$

by (27), we find

$$\left\| P_{N+1}(f) - f^{(n-1)} \right\|_\infty \le T_N^\alpha(f^{(n)}), \tag{43}$$

by (32), we derive

$$\left\| P_{N+2}(f) - f^{(n-2)} \right\|_\infty \le \frac{T_N^\alpha(f^{(n)})}{2}, \tag{44}$$

by (37), we obtain

$$\left\| P_{N+3}(f) - f^{(n-3)} \right\|_\infty \le \frac{T_N^\alpha(f^{(n)})}{3!}, \tag{45}$$

and by (38), we have

$$\| P_{N+n}(f) - f \|_\infty \le \frac{T_N^\alpha(f^{(n)})}{n!}. \tag{46}$$

So, we have proved that

$$\left\| P^{(i)}_{N+n}(f) - f^{(i)} \right\|_\infty \le \frac{T_N^\alpha\left(f^{(n)}\right)}{(n-i)!}, \qquad i = 0, 1, \ldots, n. \tag{47}$$

Based on (10) and (12) we derive that

$$T_n^\alpha(f^{(n)}) = T_N^{\beta,\alpha}(f). \tag{48}$$

By [**1**], the quantity within the bracket of (18) is finite.

The proof of the theorem now is complete. □

We completely left fractionalize Theorem 8, to have

Theorem 9. *Suppose that all terms and assumptions are as in Theorem 8. Consider* $\alpha_j > 0$, $j = 1, \ldots, n \in \mathbb{N}$, *such that* $\alpha_0 = 0 < \alpha_1 \le 1 < \alpha_2 \le 2 < \alpha_3 \le 3 < \cdots < \cdots < \alpha_n \le n$. *Then*

$$\left\| D^{\alpha_j}_{*0}(f) - D^{\alpha_j}_{*0}\left(P_{N+n}(f)\right) \right\|_{\infty,[0,1]} \le \frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)(n-j)!}, \tag{49}$$

$j = 0, 1, \ldots, n$. *Notice that* (49) *generalizes* (19).

Proof. Let $\alpha_j > 0$, $j = 1, \ldots, n$, such that $0 < \alpha_1 \leq 1 < \alpha_2 \leq 2 < \alpha_3 \leq 3 < \cdots < \cdots < \alpha_n \leq n$. That is, $\lceil \alpha_j \rceil = j$, $j = 1, \ldots, n$.

We consider the left Caputo fractional derivatives

$$\begin{aligned} \left(D_{*}^{\alpha_j} f\right)(x) &= \frac{1}{\Gamma(j-\alpha_j)} \int_0^x (x-t)^{j-\alpha_j-1} f^{(j)}(t)\, \mathrm{d}t, \\ \left(D_{*0}^{j} f\right)(x) &= f^{(j)}(x), \end{aligned} \tag{50}$$

and

$$\begin{aligned} \left(D_{*0}^{\alpha_j} (P_{N+n}(f))\right)(x) &= \frac{1}{\Gamma(j-\alpha_j)} \int_0^x (x-t)^{j-\alpha_j-1} \left(P_{N+n}(f)\right)^{(j)}(t)\, \mathrm{d}t, \\ \left(D_{*0}^{j} (P_{N+n}(f))\right)(x) &= \left(P_{N+n}(f)\right)^{(j)}. \end{aligned} \tag{51}$$

We notice that

$$\begin{aligned} &\left| \left(D_{*0}^{\alpha_j} f\right)(x) - \left(D_{*0}^{\alpha_j} (P_{N+n}(f))\right)(x) \right| \\ &\quad = \frac{1}{\Gamma(j-\alpha_j)} \left| \int_0^x (x-t)^{j-\alpha_j-1} f^{(j)}(t) \mathrm{d}t \right. \\ &\qquad\qquad \left. - \int_0^x (x-t)^{j-\alpha_j-1} \left(P_{N+n}(f)\right)^{(j)}(t) \mathrm{d}t \right| \\ &\quad = \frac{1}{\Gamma(j-\alpha_j)} \left| \int_0^x (x-t)^{j-\alpha_j-1} \left(f^{(j)}(t) - (P_{N+n}(f))^{(j)}(t) \right) \mathrm{d}t \right| \\ &\quad \leq \frac{1}{\Gamma(j-\alpha_j)} \int_0^x (x-t)^{j-\alpha_j-1} \left| f^{(j)}(t) - (P_{N+n}(f))^{(j)}(t) \right| \mathrm{d}t \qquad (52) \\ &\quad \overset{(19)}{\leq} \frac{1}{\Gamma(j-\alpha_j)} \left(\int_0^x (x-t)^{j-\alpha_j-1} \mathrm{d}t \right) \frac{T_N^{\beta,\alpha}(f)}{(n-j)!} \\ &\quad = \frac{x^{j-\alpha_j}}{\Gamma(j-\alpha_j)(j-\alpha_j)} \frac{T_N^{\beta,\alpha}(f)}{(n-j)!} = \frac{x^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{T_N^{\beta,\alpha}(f)}{(n-j)!}. \qquad (53) \end{aligned}$$

We have proved

$$\begin{aligned} &\left| \left(D_{*0}^{\alpha_j} f\right)(x) - \left(D_{*0}^{\alpha_j} (P_{N+n}(f))\right)(x) \right| \\ &\qquad \leq \frac{x^{j-\alpha_j} T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)(n-j)!} \leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)(n-j)!}, \end{aligned} \tag{54}$$

for every $x \in [0,1]$, which proves the claim. □

We completely right fractionalize Theorem 8, to have

Theorem 10. *Suppose that all terms and assumptions are the same as in Theorem 9. It holds*

$$\left\| D_{1-}^{\alpha_j}(f) - D_{1-}^{\alpha_j}\left(P_{N+n}(f)\right)\right\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma\left(j-\alpha_j+1\right)(n-j)!}, \tag{55}$$

$j = 0, 1, \ldots, n.$
Observe that (55) *generalizes* (19).

Proof. We notice that

$$\begin{aligned}
&\left|\left(D_{1-}^{\alpha_j} f\right)(x) - \left(D_{1-}^{\alpha_j}(P_{N+n}(f))\right)(x)\right| \\
&\quad = \frac{1}{\Gamma(j-\alpha_j)}\Big|\int_x^1 (t-x)^{j-\alpha_j-1} f^{(j)}(t)\mathrm{d}t \\
&\qquad\qquad - \int_x^1 (t-x)^{j-\alpha_j-1}\left(P_{N+n}(f)\right)^{(j)}(t)\mathrm{d}t\Big| \\
&\quad = \frac{1}{\Gamma(j-\alpha_j)}\left|\int_x^1 (t-x)^{j-\alpha_j-1}\left(f^{(j)}(t) - (P_{N+n}(f))^{(j)}(t)\right)\mathrm{d}t\right| \qquad (56)\\
&\quad \leq \frac{1}{\Gamma(j-\alpha_j)}\int_x^1 (t-x)^{j-\alpha_j-1}\left|f^{(j)}(t) - (P_{N+n}(f))^{(j)}(t)\right|\mathrm{d}t \\
&\quad \overset{(19)}{\leq} \frac{1}{\Gamma(j-\alpha_j)}\left(\int_x^1 (t-x)^{j-\alpha_j-1}\mathrm{d}t\right)\frac{T_N^{\beta,\alpha}(f)}{(n-j)!} \\
&\quad = \frac{(1-x)^{j-\alpha_j}}{\Gamma\left(j-\alpha_j+1\right)}\frac{T_N^{\beta,\alpha}(f)}{(n-j)!}. \qquad (57)
\end{aligned}$$

We have proved

$$\begin{aligned}
&\left|\left(D_{1-}^{\alpha_j} f\right)(x) - \left(D_{1-}^{\alpha_j}(P_{N+n}(f))\right)(x)\right| \\
&\qquad \leq \frac{(1-x)^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{T_N^{\beta,\alpha}(f)}{(n-j)!} \leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma\left(j-\alpha_j+1\right)(n-j)!},
\end{aligned} \tag{58}$$

for all $x \in [0,1]$ which proves the claim. □

Next the important corollary follows.

Corollary 11. *Suppose that all terms and assumptions are the same as in Theorem 9. Let $\lambda \in [0,1]$. Then*

$$\begin{aligned}
&\Big\|\left(\lambda D_{*0}^{\alpha_j}(f) + (1-\lambda) D_{1-}^{\alpha_j}(f)\right) \\
&\qquad - \left(\lambda D_{*0}^{\alpha_j}\left(P_{N+n}(f)\right) + (1-\lambda)\, D_{1-}^{\alpha_j}\left(P_{N+n}\left(f\right)\right)\right)\Big\|_{\infty,[0,1]} \\
&\leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma\left(j-\alpha_j+1\right)(n-j)!}, \qquad j = 0, 1, \ldots, n.
\end{aligned} \tag{59}$$

Proof. We see that

$$\begin{aligned}
&\left\|\left(\lambda D_{*0}^{\alpha_j}(f)+(1-\lambda)\,D_{1-}^{\alpha_j}(f)\right)\right.\\
&\qquad\left.-\left(\lambda D_{*0}^{\alpha_j}\left(P_{N+n}(f)\right)+(1-\lambda)\,D_{1-}^{\alpha_j}\left(P_{N+n}f\right)\right)\right\|_{\infty,[0,1]}\\
&=\left\|\lambda\left(D_{*0}^{\alpha_j}(f)-D_{*0}^{\alpha_j}(P_{N+n}(f))\right)\right.\\
&\qquad\left.+(1-\lambda)\left(D_{1-}^{\alpha_j}(f)-D_{1-}^{\alpha_j}(P_{N+n}(f))\right)\right\|_{\infty} \qquad (60)\\
&\le\lambda\left\|D_{*0}^{\alpha_j}(f)-D_{*0}^{\alpha_j}(P_{N+n}(f))\right\|_{\infty}\\
&\qquad+(1-\lambda)\left\|D_{1-}^{\alpha_j}(f)-D_{1-}^{\alpha_j}(P_{N+n}(f))\right\|_{\infty}\\
&\overset{((49),(55))}{\le}\lambda\frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)\,(n-j)!}+(1-\lambda)\,\frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)\,(n-j)!} \qquad (61)\\
&=\frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)\,(n-j)!},
\end{aligned}$$

which proves the claim. □

Next we come to our main result that is the complete fractional simultaneous monotone uniform approximation, using mixed fractional differential operators.

Theorem 12. *Let $\beta>0$, $\beta\notin\mathbb{N}$, $n=[\beta]\in\mathbb{N}$: $\beta=n+\alpha$, $0<\alpha<1$; $f\in AC^{n+1}([0,1])$, and $f^{(n+1)}\in L_\infty([0,1])$. Let $\lambda\in[0,1]$, and $h,k\in\mathbb{Z}_+$ with $0\le h\le k\le n$. When $\lambda\ne1$, we take h to be even. Consider the numbers $\alpha_0=0<\alpha_1\le1<\alpha_2\le2<\alpha_3\le3<\cdots<\cdots<\alpha_n\le n$. Let $\alpha_j(x)$, $j=h,h+1,\dots,k$ be real functions, defined and bounded on $[0,1]$, and suppose that $\alpha_h(x)$ is either $\ge\overline{\alpha}>0$ or $\le\overline{\beta}<0$ on $[0,1]$. We set*

$$l_\tau:\equiv\sup_{x\in[0,1]}\left|\alpha_h^{-1}(x)\alpha_\tau(x)\right|,\qquad\tau=h,\dots,k. \tag{62}$$

$T_N^{\beta,\alpha}(f)$, $N\in\mathbb{N}$, as in (18), $(N\to\infty)$. Consider the mixed fractional linear differential operator

$$L^*:=\sum_{j=h}^{k}\alpha_j(x)\left[\lambda D_{*0}^{\alpha_j}+(1-\lambda)\,D_{1-}^{\alpha_j}\right]. \tag{63}$$

Then for any $N\in\mathbb{N}$, there exists a real polynomial Q_{N+n} of degree $(N+n)$ such that

1) for $j=h+1,\dots,n$, it holds

$$\begin{aligned}
&\left\|\left(\lambda D_{*0}^{\alpha_j}\left(f(x)\right)+(1-\lambda)\,D_{1-}^{\alpha_j}\left(f(x)\right)\right)\right.\\
&\qquad\left.-\left(\lambda D_{*0}^{\alpha_j}\left(Q_{N+n}(x)\right)+(1-\lambda)\,D_{1-}^{\alpha_j}\left(Q_{N+n}(x)\right)\right)\right\|_{\infty,[0,1]} \qquad (64)\\
&\le\frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)\,(n-j)!},
\end{aligned}$$

2) *for* $j = 1, \ldots, h$, *it holds*

$$
\begin{aligned}
& \left\| \left(\lambda D_{*0}^{\alpha_j}(f) + (1-\lambda)\, D_{1-}^{\alpha_j}(f)\right) \right. \\
& \qquad \left. - \left(\lambda D_{*0}^{\alpha_j}\left(Q_{N+n}\right) + (1-\lambda)\, D_{1-}^{\alpha_j}\left(Q_{N+n}\right)\right) \right\|_{\infty,[0,1]} \\
& \leq \frac{T_N^{\beta,\alpha}(f)}{(h-j)!} \left[\frac{1}{\Gamma(j-\alpha_j+1)} + \left(\sum_{\tau=h}^{k} \frac{l_\tau}{\Gamma(\tau-\alpha_\tau+1)(n-\tau)!} \right) \right. \\
& \qquad \left. \cdot \left\{ \lambda \frac{\Gamma(h-j+1)}{\Gamma(h-\alpha_j+1)} + (1-\lambda) \left[\sum_{\theta=0}^{h-j} \binom{h-j}{\theta} \frac{\Gamma(h-j-\theta+1)}{\Gamma(h-\alpha_j-\theta+1)} \right] \right\} \right],
\end{aligned} \tag{65}
$$

and

3)

$$
\|f - Q_{N+n}\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{h!} \left(\sum_{\tau=h}^{k} \frac{l_\tau}{\Gamma(\tau-\alpha_\tau+1)(n-\tau)!} + 1 \right). \tag{66}
$$

The set

$$
\Lambda := \left\{ (\lambda, x) \in (0,1)^2 : \lambda x^{h-\alpha_h} + (1-\lambda)(1-x)^{h-\alpha_h} \geq \Gamma(h-\alpha_h+1) \right\} \tag{67}
$$

is not empty.

1. *We assume that* $L^* f(x) \geq 0$ *for every* $x \in \Lambda$. *Then* $L^*(Q_{N+n}) \geq 0$.
2. *If* $L^* f(0) \geq 0$ *and* $0 \leq \lambda \leq 1 - \Gamma(h-\alpha_h+1)$, *then* $L^*(Q_{N+n})(0) \geq 0$.
3. *If* $L^* f(1) \geq 0$ *and* $\Gamma(h-\alpha_h+1) \leq \lambda \leq 1$, *then* $L^*(Q_{N+n})(1) \geq 0$.
4. *Given* $L^* f(0) \geq 0$, $\lambda = 0$, *we get* $L^*(Q_{N+n})(0) \geq 0$.
5. *Given* $L^* f(1) \geq 0$, $\lambda = 1$, *we derive* $L^*(Q_{N+n})(1) \geq 0$.
6. *Let* $\lambda = 0$, h *even*, $h > \alpha_h$, $L^* f(x) \geq 0$ *for* $x \in (0,1)$ *such that* $x \leq 1 - \Gamma(h-\alpha_h+1)^{\frac{1}{h-\alpha_h}}$, *then* $L^*(Q_{N+n})(x) \geq 0$.

Finally:

7. *Let* $\lambda = 1$, $h > \alpha_h$, *and* $x \in (0,1)$ *with* $x \geq (\Gamma(h-\alpha_h+1))^{\frac{1}{h-\alpha_h}}$. *Assume that* $L^* f(x) \geq 0$, *then* $L^*(Q_{N+n})(x) \geq 0$.

Proof. Let h, k be integers $0 \leq h \leq k \leq n$, and $\alpha_0 = 0 < \alpha_1 \leq 1 < \alpha_2 \leq 2 < \alpha_3 \leq 3 < \cdots < \cdots < \alpha_n \leq n$, that is, $\lceil \alpha_j \rceil = j$, $j = 1, \ldots, n$. We set

$$
l_{j_*} :\equiv \sup_{x \in [0,1]} \left| \alpha_h^{-1}(x) \alpha_{j_*}(x) \right| < \infty, \qquad h \leq j_* \leq k, \tag{68}
$$

and

$$
\rho_N := T_N^{\beta,\alpha}(f) \left(\sum_{j_*=h}^{k} \frac{l_{j_*}}{\Gamma(j_*-\alpha_{j_*}+1)(n-j_*)!} \right). \tag{69}
$$

I. Throughout $[0,1]$, suppose that $\alpha_h(x) \geq \overline{\alpha} > 0$. Call

$$
Q_{N+n}(x) := P_{N+n}(f)(x) + \rho_N \frac{x^h}{h!}, \tag{70}
$$

where $P_{N+n}(f)(x)$ the same as in (17).

Then by (59), we obtain

$$\begin{aligned}
&\left\| \left(\lambda D_{*0}^{\alpha_j} + (1-\lambda) D_{1-}^{\alpha_j}\right) \left(f(x) + \rho_N \frac{x^h}{h!}\right) \right. \\
&\qquad \left. - \left(\lambda D_{*0}^{\alpha_j} + (1-\lambda) D_{1-}^{\alpha_j}\right) (Q_{N+n}(x)) \right\|_{\infty,[0,1]} \\
&\leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)(n-j)!},
\end{aligned} \tag{71}$$

for all $0 \leq j \leq n$.

When $h+1 \leq j \leq n$, immediately by (71), we obtain

$$\begin{aligned}
&\left\| \left(\lambda D_{*0}^{\alpha_j} + (1-\lambda) D_{1-}^{\alpha_j}\right) (f(x)) \right. \\
&\qquad \left. - \left(\lambda D_{*0}^{\alpha_j} + (1-\lambda) D_{1-}^{\alpha_j}\right) (Q_{N+n}(x)) \right\|_{\infty,[0,1]} \\
&\leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)(n-j)!},
\end{aligned} \tag{72}$$

that proves (64).

Next we treat the case of $1 \leq j \leq h$. After we get calculations

$$D_{*0}^{\alpha_j}\left(\frac{x^h}{h!}\right) = \frac{\Gamma(h-j+1)\, x^{h-\alpha_j}}{\Gamma(h-\alpha_j+1)(h-j)!}, \tag{73}$$

and

$$\begin{aligned}
D_{1-}^{\alpha_j}\left(\frac{x^h}{h!}\right) &= \frac{1}{(h-j)!} \\
&\cdot \left[\sum_{\theta=0}^{h-j} \left(\binom{h-j}{\theta}\right) (-1)^{h+\theta} \left\{\frac{\Gamma(h-j-\theta+1)}{\Gamma(h-\alpha_j-\theta+1)} (1-x)^{h-\alpha_j-\theta}\right\}\right].
\end{aligned} \tag{74}$$

Hence by (71), we have

$$\begin{aligned}
&\left\| \left(\lambda D_{*0}^{\alpha_j} + (1-\lambda) D_{1-}^{\alpha_j}\right) (f(x)) + \rho_N \left\{ \lambda \frac{\Gamma(h-j+1)\, x^{h-\alpha_j}}{\Gamma(h-\alpha_j+1)(h-j)!} \right.\right. \\
&\quad \left. + \frac{(1-\lambda)}{(h-j)!} \left[\sum_{\theta=0}^{h-j} \binom{h-j}{\theta} (-1)^{h+\theta} \left\{\frac{\Gamma(h-j-\theta+1)}{\Gamma(h-\alpha_j-\theta+1)} (1-x)^{h-\alpha_j-\theta}\right\}\right]\right\} \\
&\quad \left. - \left(\lambda D_{*0}^{\alpha_j} + (1-\lambda) D_{1-}^{\alpha_j}\right) (Q_{N+n}(x)) \right\|_{\infty,[0,1]} \\
&\leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)(n-j)!}
\end{aligned} \tag{75}$$

for all $1 \leq j \leq h$.

By (75) and triangle inequality we get

$$\begin{aligned}
&\left\| \left(\lambda D_{*0}^{\alpha_j} + (1-\lambda) D_{1-}^{\alpha_j}\right)(f) - \left(\lambda D_{*0}^{\alpha_j} + (1-\lambda) D_{1-}^{\alpha_j}\right)(Q_{N+n}) \right\|_{\infty,[0,1]} \\
&\leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)(n-j)!} \qquad (76)\\
&\quad + \frac{\rho_N}{(h-j)!} \left\{ \lambda \frac{\Gamma(h-j+1)}{\Gamma(h-\alpha_j+1)} + (1-\lambda) \left[\sum_{\theta=0}^{h-j} \binom{h-j}{\theta} \frac{\Gamma(h-j-\theta+1)}{\Gamma(h-\alpha_j-\theta+1)} \right] \right\} \\
&= \frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)(n-j)!} + \frac{T_N^{\beta,\alpha}(f)}{(h-j)!} \left(\sum_{j_*=h}^{k} \frac{l_{j_*}}{\Gamma(j_*-\alpha_{j_*}+1)(n-j_*)!} \right) \\
&\quad \cdot \left\{ \lambda \frac{\Gamma(h-j+1)}{\Gamma(h-\alpha_j+1)} + (1-\lambda) \left[\sum_{\theta=0}^{h-j} \binom{h-j}{\theta} \frac{\Gamma(h-j-\theta+1)}{\Gamma(h-\alpha_j-\theta+1)} \right] \right\} \\
&\leq \frac{T_N^{\beta,\alpha}(f)}{(h-j)!} \left[\frac{1}{\Gamma(j-\alpha_j+1)} + \left(\sum_{\tau=h}^{k} \frac{l_\tau}{\Gamma(\tau-\alpha_\tau+1)(n-\tau)!} \right) \right. \qquad (77)\\
&\quad \left. \left\{ \lambda \frac{\Gamma(h-j+1)}{\Gamma(h-\alpha_j+1)} + (1-\lambda) \left[\sum_{\theta=0}^{h-j} \binom{h-j}{\theta} \frac{\Gamma(h-j-\theta+1)}{\Gamma(h-\alpha_j-\theta+1)} \right] \right\} \right] =: K.
\end{aligned}$$

So we have derived

$$\begin{aligned}
&\left\| \left(\lambda D_{*0}^{\alpha_j} + (1-\lambda) D_{1-}^{\alpha_j}\right)(f) \right. \\
&\qquad \left. - \left(\lambda D_{*0}^{\alpha_j} + (1-\lambda) D_{1-}^{\alpha_j}\right)(Q_{N+n}) \right\|_{\infty,[0,1]} \leq K,
\end{aligned} \tag{78}$$

$j = 1, \ldots, h$, proving (65).

When $j = 0$ from (71) we obtain

$$\left\| f(x) + \rho_N \frac{x^h}{h!} - Q_{N+n}(x) \right\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{n!}. \tag{79}$$

Hence

$$\begin{aligned}
\|f - Q_{N+n}\|_{\infty,[0,1]} &\leq \frac{\rho_N}{h!} + \frac{T_N^{\beta,\alpha}(f)}{n!} \\
&= \frac{T_N^{\beta,\alpha}(f)}{h!} \left(\sum_{\tau=h}^{k} \frac{l_\tau}{\Gamma(\tau-\alpha_\tau+1)(n-\tau)!} \right) + \frac{T_N^{\beta,\alpha}(f)}{n!} \qquad (80)\\
&\leq \frac{T_N^{\beta,\alpha}(f)}{h!} \left(\sum_{\tau=h}^{k} \frac{l_\tau}{\Gamma(\tau-\alpha_\tau+1)(n-\tau)!} + 1 \right), \qquad (81)
\end{aligned}$$

proving (66).

Furthermore, if (λ, x) is in the critical set Λ, see (67), and $L^* f(x) \geq 0$, we get

$$\begin{aligned}
\alpha_h^{-1} L^*(Q_{N+n}) &= \alpha_h^{-1}(x) L^*(f(x)) \\
&\quad + \rho_N \left\{ \lambda \frac{x^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} + (1-\lambda) \frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} \right\}
\end{aligned} \tag{82}$$

(when $\lambda \in [0,1)$, we assume that h is even)

$$+\sum_{j=h}^{k} \alpha_h^{-1}(x)\alpha_j(x)\left\{\left(\lambda D_{*0}^{\alpha_j} + (1-\lambda) D_{1-}^{\alpha_j}\right)\left[Q_{N+n}(x) - f(x) - \rho_N \frac{x^h}{h!}\right]\right\}$$

$$\overset{(71)}{\geq} \rho_N \left\{\lambda \frac{x^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} + (1-\lambda)\frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}\right\} - \sum_{j=h}^{k} l_j \frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)(n-j)!}$$

$$= \rho_N \left\{\lambda \frac{x^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} + (1-\lambda)\frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} - 1\right\}$$

$$= \frac{\rho_N}{\Gamma(h-\alpha_h+1)}\left\{\lambda x^{h-\alpha_h} + (1-\lambda)(1-x)^{h-\alpha_h} - \Gamma(h-\alpha_h+1)\right\}$$

$$=: A(x,\lambda). \tag{83}$$

The set

$$\Lambda := \left\{(\lambda, x) \in (0,1)^2 : \lambda x^{h-\alpha_h} + (1-\lambda)(1-x)^{h-\alpha_h} \geq \Gamma(h-\alpha_h+1)\right\} \tag{84}$$

is not empty.

If $h = \alpha_h$, then $\Lambda = (0,1)^2$. Assume that $\alpha_h < h$.
Let us choose $\lambda = x = \delta \in (0,1)$, some want to find specific examples of

$$\delta^{1+h-\alpha_h} + (1-\delta)^{1+h-\alpha_h} \geq \Gamma(h-\alpha_h+1). \tag{85}$$

The minimum value of Γ over $(0,\infty)$ is $\Gamma(1.46163) \simeq 0.885603$, we take $1+h-\alpha_h = 1.46163$ and $\delta = 0.99$.

Hence

$$\begin{aligned}(0.99)^{1.46163} + (0.01)^{1.46163} &= 0.985417497 + 0.001193274 \\ &= 0.986610771 > 0.885603.\end{aligned} \tag{86}$$

Similarly, we have that $\Gamma(1.4) = 0.887264$, and by taking $\delta = 0.95$ and $1+h-\alpha_h = 1.4$, we get

$$\begin{aligned}(0.95)^{1.4} + (0.05)^{1.4} &= 0.930707144 + 0.015085441 \\ &= 0.945792585 > 0.887264.\end{aligned} \tag{87}$$

Hence $\Lambda \neq \emptyset$.

Hence over Λ, we get $A(x,\lambda) \geq 0$, thus $L^*(Q_{N+n}) \geq 0$.

We know that $\Gamma(1) = 1$, $\Gamma(2) = 1$, and Γ is convex and positive on $(0,\infty)$. In general $0 \leq h - \alpha_h < 1$, hence $1 \leq h - \alpha_h + 1 < 2$ and $0 < \Gamma(h-\alpha_h+1) \leq 1$, that is $1 - \Gamma(h-\alpha_h+1) \geq 0$.

Next we argue as in (82)–(83).

Further we have

$$A(0,\lambda) = \frac{\rho_N}{\Gamma(h-\alpha_h+1)}\{(1-\lambda) - \Gamma(h-\alpha_h+1)\} \geq 0, \tag{88}$$

when $0 \leq \lambda \leq 1 - \Gamma(h-\alpha_h+1)$ which proves the case $L^*(Q_{N+n})(0) \geq 0$, provided $L^* f(0) \geq 0$.

Similarly, we observe that

$$A(1,\lambda) = \frac{\rho_N}{\Gamma(h-\alpha_h+1)}\{\lambda - \Gamma(h-\alpha_h+1)\} \geq 0, \tag{89}$$

where $\Gamma(h-\alpha_h+1) \leq \lambda \leq 1$, that proves the case $L^*(Q_{N+n})(1) \geq 0$, provided $L^* f(1) \geq 0$.

Clearly, we have

$$A(0,0) = \frac{\rho_N}{\Gamma(h-\alpha_h+1)}\{1 - \Gamma(h-\alpha_h+1)\} \geq 0, \tag{90}$$

that proves $L^*(Q_{N+n})(0) \geq 0$ with $\lambda = 0$, provided $L^* f(0) \geq 0$ and

$$A(1,1) = \frac{\rho_N}{\Gamma(h-\alpha_h+1)}\{1 - \Gamma(h-\alpha_h+1)\} \geq 0, \tag{91}$$

that proves $L^*(Q_{N+n})(1) \geq 0$ provided $L^* f(1) \geq 0$ with $\lambda = 1$.

We see also that

$$A(x,0) = \frac{\rho_N}{\Gamma(h-\alpha_h+1)}\left\{(1-x)^{h-\alpha_h} - \Gamma(h-\alpha_h+1)\right\} \geq 0, \tag{92}$$

provided $(1-x) \geq \Gamma(h-\alpha_h+1)^{\frac{1}{h-\alpha_h}}$, equivalently, provided $x \leq 1 - \Gamma(h-\alpha_h+1)^{\frac{1}{h-\alpha_h}}$ with $h > \alpha_h$ and $x \in (0,1)$.

In case $L^*(Q_{N+n})(x) \geq 0$ with $\lambda = 0$ and h even, provided $L^* f(x) \geq 0$. Finally, we observe that

$$A(x,1) = \frac{\rho_N}{\Gamma(h-\alpha_h+1)}\left\{x^{h-\alpha_h} - \Gamma(h-\alpha_h+1)\right\} \geq 0, \tag{93}$$

provided $x \geq \Gamma(h-\alpha_h+1)^{\frac{1}{h-\alpha_h}}$ with $h > \alpha_h$ and $x \in (0,1)$. In case $L^*(Q_{N+n})(x) \geq 0$, with $\lambda = 1$, provided $L^* f(x) \geq 0$.

II. Throughout $[0,1]$, suppose that $\alpha_h(x) \leq \overline{\beta} < 0$. Call now

$$Q_{N+n}(x) := P_{N+n}(f)(x) - \rho_N \frac{x^h}{h!}. \tag{94}$$

Then by (59), we obtain

$$\begin{aligned} &\Big\| \left(\lambda D_{*0}^{\alpha_j} + (1-\lambda) D_{1-}^{\alpha_j}\right)\left(f(x) - \rho_N \frac{x^h}{h!}\right) \\ &\qquad - \left(\lambda D_{*0}^{\alpha_j} + (1-\lambda) D_{1-}^{\alpha_j}\right)(Q_{N+n}(x)) \Big\|_{\infty,[0,1]} \\ &\leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)(n-j)!}, \qquad 0 \leq j \leq n. \end{aligned} \tag{95}$$

As earlier, we obtain the inequalities (64), (65), (66). Furthermore, if $(\lambda, x) \in \Lambda$ and $L^* f(x) \geq 0$, we get

$$\alpha_h^{-1}(x) L^* (Q_{N+n}) = \alpha_h^{-1}(x) L^* (f(x)) - \rho_N \left\{ \lambda \frac{x^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} + (1-\lambda) \frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} \right\}$$

(when $\lambda \in [0,1)$, we assume that h is even)

$$+ \sum_{j=h}^{k} \alpha_h^{-1}(x) \alpha_j(x) \left\{ \left(\lambda D_{*0}^{\alpha_j} + (1-\lambda) D_{1-}^{\alpha_j} \right) \left[Q_{N+n}(x) - f(x) + \rho_N \frac{x^h}{h!} \right] \right\} \tag{96}$$

$$\overset{(95)}{\leq} - \rho_N \left\{ \lambda \frac{x^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} + (1-\lambda) \frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} \right\} + \sum_{j=h}^{k} l_j \frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)(n-j)!}$$

$$= \rho_N \left\{ 1 - \left\{ \lambda \frac{x^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} + (1-\lambda) \frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} \right\} \right\}$$

$$= \frac{\rho_N}{\Gamma(h-\alpha_h+1)} \left\{ \Gamma(h-\alpha_h+1) - \left\{ \lambda x^{h-\alpha_h} + (1-\lambda)(1-x)^{h-\alpha_h} \right\} \right\} \tag{97}$$

$$=: B(x, \lambda).$$

Hence over Λ, we get $B(x, \lambda) \leq 0$, thus $L^* (Q_{N+n}) \geq 0$.

Next we argue as in (96)–(97):

Further we have

$$B(0, \lambda) = \frac{\rho_N}{\Gamma(h-\alpha_h+1)} \{ \Gamma(h-\alpha_h+1) - (1-\lambda) \} \leq 0, \tag{98}$$

when $0 \leq \lambda \leq 1 - \Gamma(h-\alpha_h+1)$, that proves the case $L^* (Q_{N+n})(0) \geq 0$, provided $L^* f(0) \geq 0$.

Similarly, we observe that

$$B(1, \lambda) = \frac{\rho_N}{\Gamma(h-\alpha_h+1)} \{ \Gamma(h-\alpha_h+1) - \lambda \} \leq 0, \tag{99}$$

where $\Gamma(h-\alpha_h+1) \leq \lambda \leq 1$, that proves the case $L^* (Q_{N+n})(1) \geq 0$ provided $L^* f(1) \geq 0$.

Clearly, we have

$$B(0, 0) = \frac{\rho_N}{\Gamma(h-\alpha_h+1)} \{ \Gamma(h-\alpha_h+1) - 1 \} \leq 0, \tag{100}$$

that proves $L^*(Q_{N+n})(0) \geq 0$, provided $L^* f(0) \geq 0$ with $\lambda = 0$.

Also it holds

$$B(1, 1) = \frac{\rho_N}{\Gamma(h-\alpha_h+1)} \{ \Gamma(h-\alpha_h+1) - 1 \} \leq 0, \tag{101}$$

that proves $L^* (Q_{N+n})(1) \geq 0$, provided $L^* f(1) \geq 0$ with $\lambda = 1$.

We see also that

$$B(x,0) = \frac{\rho_N}{\Gamma(h-\alpha_h+1)} \left\{ \Gamma(h-\alpha_h+1) - (1-x)^{h-\alpha_h} \right\} \leq 0, \tag{102}$$

provided $(1-x) \geq \Gamma(h-\alpha_h+1)^{\frac{1}{h-\alpha_h}}$ with $h > \alpha_h$ and $x \in (0,1)$, h is even.

In that case $L^*(Q_{N+n})(x) \geq 0$ with $\lambda = 0$, provided $L^* f(x) \geq 0$.

Finally, we observe that

$$B(x,1) = \frac{\rho_N}{\Gamma(h-\alpha_h+1)} \left\{ \Gamma(h-\alpha_h+1) - x^{h-\alpha_h} \right\} \leq 0, \tag{103}$$

provided that $x \geq \Gamma(h-\alpha_h+1)^{\frac{1}{h-\alpha_h}}$ with $h > \alpha_h$ and $x \in (0,1)$.

In that case again $L^*(Q_{N+n})(x) \geq 0$, with $\lambda = 1$, provided $L^* f(x) \geq 0$. □

Corollary 13. *Let $\beta > 0$, $\beta \notin \mathbb{N}$, $n = [\beta] \in \mathbb{N} : \beta = n + \alpha$, $0 < \alpha < 1$; $f \in AC^{n+1}([0,1])$, and $f^{(n+1)} \in L_\infty([0,1])$. Let $h, k \in \mathbb{Z}_+$ with $0 \leq h \leq k \leq n$. Consider the numbers $\alpha_0 = 0 < \alpha_1 \leq 1 < \alpha_2 \leq 2 < \alpha_3 \leq 3 < \cdots < \cdots < \alpha_n \leq n$. Let $\alpha_j(x)$, $j = h, h+1, \ldots, k$ be real functions, defined and bounded on $[0,1]$, and suppose that $\alpha_h(x)$ is either $\geq \overline{\alpha} > 0$ or $\leq \overline{\beta} < 0$ on $[0,1]$. We set*

$$l_\tau :\equiv \sup_{x \in [0,1]} \left| \alpha_h^{-1}(x) \alpha_\tau(x) \right|, \qquad \tau = h, \ldots, k. \tag{104}$$

Consider the left fractional linear differential operator

$$L_1 := \sum_{j=h}^{k} \alpha_j(x) D_{*0}^{\alpha_j}. \tag{105}$$

$T_N^{\beta,\alpha}(f)$, $N \in \mathbb{N}$ is the same as in (18).

Then, for any $N \in \mathbb{N}$, there exists a real polynomial Q_{N+n} of degree $(N+n)$ such that

1) *for $j = h+1, \ldots, n$, it holds*

$$\left\| D_{*0}^{\alpha_j} f - D_{*0}^{\alpha_j} Q_{N+n} \right\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)(n-j)!}, \tag{106}$$

2) *for $j = 1, \ldots, h$, it holds*

$$\begin{aligned} &\| D_{*0}^{\alpha_j}(f) - D_{*0}^{\alpha_j}(Q_{N+n}) \|_{\infty,[0,1]} \\ &\leq \frac{T_N^{\beta,\alpha}(f)}{(h-j)!} \left[\frac{1}{\Gamma(j-\alpha_j+1)} \right. \\ &\quad \left. + \left(\sum_{\tau=h}^{k} \frac{l_\tau}{\Gamma(\tau-\alpha_\tau+1)(n-\tau)!} \right) \left(\frac{\Gamma(h-j+1)}{\Gamma(h-\alpha_j+1)} \right) \right], \end{aligned} \tag{107}$$

3)

$$\| f - Q_{N+n} \|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{h!} \left(\sum_{\tau=h}^{k} \frac{l_\tau}{\Gamma(\tau-\alpha_\tau+1)(n-\tau)!} + 1 \right). \tag{108}$$

Further we have

a) *Given* $L_1 f(1) \geq 0$, *then* $L_1 \left(Q_{N+n}\right)(1) \geq 0$.

b) *Let* $h > \alpha_h$ *and* $x \in (0,1)$ *with* $x \geq \left(\Gamma\left(h-\alpha_h+1\right)\right)^{\frac{1}{h-\alpha_h}}$. *Assume that* $L_1 f(x) \geq 0$, *then* $L_1 \left(Q_{N+n}\right)(x) \geq 0$.

Proof. By Theorem 12 and for $\lambda = 1$. □

We finish with the following corollary.

Corollary 14. *Let* $\beta > 0$, $\beta \notin \mathbb{N}$, $n = [\beta] \in \mathbb{N} : \beta = n + \alpha$, $0 < \alpha < 1$; $f \in AC^{n+1}([0,1])$, *and* $f^{(n+1)} \in L_\infty([0,1])$. *Let* $h, k \in \mathbb{Z}_+$ *with* $0 \leq h \leq k \leq n$, h *is even. Consider the numbers* $\alpha_0 = 0 < \alpha_1 \leq 1 < \alpha_2 \leq 2 < \alpha_3 \leq 3 < \cdots < \cdots < \alpha_n \leq n$. *Let* $\alpha_j(x)$, $j = h, h+1, \ldots, k$ *be real functions, defined and bounded on* $[0,1]$, *and suppose* $\alpha_h(x)$ *is either* $\geq \overline{\alpha} > 0$ *or* $\leq \overline{\beta} < 0$ *on* $[0,1]$. *We set*

$$l_\tau :\equiv \sup_{x \in [0,1]} \left|\alpha_h^{-1}(x)\alpha_\tau(x)\right|, \qquad \tau = h, \ldots, k. \tag{109}$$

Consider the right fractional linear differential operator

$$L_2 := \sum_{j=h}^{k} \alpha_j(x) D_{1-}^{\alpha_j}. \tag{110}$$

$T_N^{\beta,\alpha}(f)$, $N \in \mathbb{N}$ *is the same as in* (18).

Then, for any $N \in \mathbb{N}$, *there exists a real polynomial* Q_{N+n} *of degree* $(N+n)$ *such that*

1) *for* $j = h+1, \ldots, n$, *it holds*

$$\left\|D_{1-}^{\alpha_j} f - D_{1-}^{\alpha_j} Q_{N+n}\right\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)\,(n-j)!}, \tag{111}$$

2) *for* $j = 1, \ldots, h$, *it holds*

$$\begin{aligned}
&\|D_{1-}^{\alpha_j}(f) - D_{1-}^{\alpha_j}\left(Q_{N+n}\right)\|_{\infty,[0,1]} \\
&\quad\leq \frac{T_N^{\beta,\alpha}(f)}{(h-j)!}\Bigg[\frac{1}{\Gamma\left(j-\alpha_j+1\right)} \\
&\quad + \left(\sum_{\tau=h}^{k} \frac{l_\tau}{\Gamma\left(\tau-\alpha_\tau+1\right)(n-\tau)!}\right)\left[\sum_{\theta=0}^{h-j}\binom{h-j}{\theta}\frac{\Gamma\left(h-j-\theta+1\right)}{\Gamma\left(h-\alpha_j-\theta+1\right)}\right]\Bigg],
\end{aligned} \tag{112}$$

3)

$$\|f - Q_{N+n}\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{h!}\left(\sum_{\tau=h}^{k} \frac{l_\tau}{\Gamma\left(\tau-\alpha_\tau+1\right)(n-\tau)!} + 1\right). \tag{113}$$

Further we have

a) *If* $L_2 f(0) \geq 0$, *then* $L_2\left(Q_{N+n}\right)(0) \geq 0$.

b) *Let even* $h > \alpha_h$, $x \in (0,1)$ *such that* $x \leq 1 - \left(\Gamma\left(h-\alpha_h+1\right)\right)^{\frac{1}{h-\alpha_h}}$, *and* $L_2 f(x) \geq 0$. *Then* $L_2\left(Q_{N+n}\right)(x) \geq 0$.

Proof. By Theorem 12 and for $\lambda = 0$. □

References

1. Anastassiou G. A., *Fractional Korovkin Theory*, Chaos Solitons Fractals **42** (2009), 2080–2094.
2. Anastassiou G. A. and Shisha O., *Monotone approximation with linear differential operators*, J. Approx. Theory **44** (1985), 391–393.
3. Diethelm K., *The Analysis of Fractional Differential Equations*, Lecture Notes in Math., Vol. 2004, 1st edition, Springer, New York, Heidelberg, 2010.
4. El-Sayed A. M. A. and Gaber M., *On the finite Caputo and finite Riesz derivatives*, Electronic Journal of Theoretical Physics **3(12)** (2006), 81–95.
5. Shisha O., *Monotone approximation*, Pacific J. Math. **15** (1965), 667–671.

G. A. Anastassiou, Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, U.S.A., *e-mail*: `ganastss@memphis.edu`