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THE GENERAL CASE ON THE ORDER OF APPEARANCE OF PRODUCT OF CONSECUTIVE LUCAS NUMBERS

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ABSTRACT. Let F_n and L_n be the nth Fibonacci number and Lucas number, respectively. The order of appearance of m in the Fibonacci sequence, denoted by $z(m)$, is the smallest positive integer k such that m divides F_k. The formula for $z(L_nL_{n+1}L_{n+2}\cdots L_{n+k})$ has been recently obtained by Marques for $1 \leq k \leq 3$, and by Marques and Trojovský for $k = 4$. In this article, we extend the results to the cases $k = 5$ and $k = 6$. Our method gives a general idea on how to obtain the formulas of $z(L_nL_{n+1}\cdots L_{n+k})$ for every $k \geq 1$.

1. INTRODUCTION

Throughout this article, we write $(a_1, a_2, \ldots, a_k)$ and $[a_1, a_2, \ldots, a_k]$ for the greatest common divisor and the least common multiple of $a_1, a_2, \ldots, a_k$, respectively.

The Fibonacci sequence $(F_n)_{n\geq1}$ is defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$, and the Lucas sequence $(L_n)_{n\geq1}$ is defined by the same recursive pattern with initial values $L_1 = 1$ and $L_2 = 3$. For each positive integer m, the order (or the rank) of appearance of m in the Fibonacci sequence, denoted by $z(m)$, is the smallest positive integer k such that m divides F_k. The divisibility property of Fibonacci and Lucas numbers, and the behavior of the order of appearance have been a popular area of research, see [**1, 4, 6, 7, 11, 13, 14, 15, 21, 22**] and references therein for additional details and history. We also refer the reader to [**12, 16, 17, 18, 19**] for some recent results concerning with Fibonacci and Lucas numbers.

Recently, Marques [**9**] obtained the formulas for

$$z(L_nL_{n+1}), z(L_nL_{n+1}L_{n+2}), \quad \text{and} \quad z(L_nL_{n+1}L_{n+2}L_{n+3}).$$

Then Marques and Trojovský [**10**] extended the above to $z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4})$. In this article, we modify the method used in [**5**], and obtain $z(L_nL_{n+1}\cdots L_{n+k})$ for $k = 5$ and $k = 6$. In fact, our method gives an algorithm to compute $z(L_nL_{n+1}\cdots L_{n+k})$ for any given $k \geq 1$.

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2. Auxiliary Results

In this section, we give some lemmas that will be used in the proof of the main theorems. Recall that for a prime p and a positive integer n, the p-adic order of n, denoted by $v_p(n)$, is the exponent of p in the prime factorization of n. The next lemma will be used to calculate 2-adic and 3-adic orders of Fibonacci numbers.

Lemma 2.1. ([**8**, Lengyel]) *For each $n \geq 1$, we have*

$$v_2(F_n) = \begin{cases} 0 & \text{if } n \equiv 1, 2 \pmod 3, \\ 1 & \text{if } n \equiv 3 \pmod 6, \\ v_2(n) + 2 & \text{if } n \equiv 0 \pmod 6, \end{cases}$$

$v_5(F_n) = v_5(n)$, and if p is a prime, $p \neq 2$, and $p \neq 5$, then

$$v_p(F_n) = \begin{cases} v_p(n) + v_p(F_{z(p)}) & \text{if } n \equiv 0 \pmod{z(p)}, \\ 0 & \text{if } n \not\equiv 0 \pmod{z(p)}. \end{cases}$$

In particular,

$$v_3(F_n) = \begin{cases} v_3(n) + 1 & \text{if } n \equiv 0 \pmod 4, \\ 0 & \text{if } n \not\equiv 0 \pmod 4. \end{cases}$$

In addition, we have

$$v_2(L_n) = \begin{cases} 0 & \text{if } n \equiv 1, 2 \pmod 3, \\ 2 & \text{if } n \equiv 3 \pmod 6, \\ 1 & \text{if } n \equiv 0 \pmod 6, \text{ and} \end{cases}$$

for all primes $p \notin \{2, 5\}$,

$$v_p(L_n) = \begin{cases} v_p(n) + v_p(F_{z(p)}) & \text{if } z(p) \text{ is even and } n \equiv \frac{z(p)}{2} \pmod{z(p)}, \\ 0 & \text{otherwise.} \end{cases}$$

In particular,

$$v_3(L_n) = \begin{cases} v_3(n) + 1 & \text{if } n \equiv 2 \pmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

The next lemma are well-known and will be applied throughout this article. The proof can be found, for example, in [**3, 7, 20**].

Lemma 2.2. *Let m, n be positive integers and $d = (m, n)$. Then the following statements hold.*

(i) *For $n \geq 2$, $L_n \mid F_m$ if and only if $2n \mid m$.*
(ii) $(F_m, F_n) = F_d$.
(iii) $$(L_m, L_n) = \begin{cases} L_d & \text{if } \frac{m}{d} \text{ and } \frac{n}{d} \text{ are odd}, \\ 2 & \text{if } (\frac{m}{d} \text{ or } \frac{n}{d} \text{ is even}) \text{ and } 3 \mid d, \\ 1 & \text{if } (\frac{m}{d} \text{ or } \frac{n}{d} \text{ is even}) \text{ and } 3 \nmid d. \end{cases}$$

(iv) $(F_m, L_n) = \begin{cases} L_d & \text{if } \frac{m}{d} \text{ is even and } \frac{n}{d} \text{ is odd,} \\ 2 & \text{if } (\frac{m}{d} \text{ is odd or } \frac{n}{d} \text{ is even) and } 3 \mid d, \\ 1 & \text{if } (\frac{m}{d} \text{ is odd or } \frac{n}{d} \text{ is even) and } 3 \nmid d. \end{cases}$

In particular, any three consecutive Lucas numbers are pairwise relatively prime. In addition, $(L_n, L_{n+3}) = F_{(n,3)}$, $(L_n, L_{n+4}) = F_{(n-2,4)}$, $(L_n, L_{n+5}) = 1$, *and* $(L_n, L_{n+6}) = 2^{v_2(L_n)}$.

Proof. The statements (i), (ii), (iii), and (iv) are well-known. The rest follows from an application of (iii) and the definition of Fibonacci numbers. We only give the proof to the last equality

$$(L_n, L_{n+6}) = 2^{v_2(L_n)}. \tag{1}$$

Let $d = (n, n+6) = (n, 6)$. We divide the calculation into four cases.
Case 1: $2 \mid n$ and $3 \mid n$. Then $d = 6$, $\frac{n}{6}$ or $\frac{n+6}{6}$ is even, and we obtain by (iii) that $(L_n, L_{n+6}) = 2$. We also know from Lemma 2.1 that $v_2(L_n) = 1$. Therefore, (1) holds.
Case 2: $2 \nmid n$ and $3 \mid n$. Then $d = 3$, $\frac{n}{3}$ and $\frac{n+6}{3}$ are odd, so we obtain by (iii) that $(L_n, L_{n+6}) = L_d = L_3 = 4$. By Lemma 2.1, $v_2(L_n) = 2$. So (1) is verified.
Case 3: $2 \mid n$ and $3 \nmid n$. Similar to Case 1, we have $d = 2$, $\frac{n}{2}$ or $\frac{n+6}{2}$ is even, so $(L_n, L_{n+6}) = 1 = 2^0 = 2^{v_2(L_n)}$.
Case 4: $2 \nmid n$ and $3 \nmid n$. Similar to Case 2, we have $d = 1$, n and $n+6$ are odd, and $(L_n, L_{n+6}) = L_d = L_1 = 1 = 2^{v_2(L_n)}$. □

We will also need to know the least common multiple of consecutive integers and make a calculation on some expressions involving the greatest common divisor. Therefore we recall basic results in elementary number theory as follows. For positive integers a, b, c, if $(a, b) = 1$, then $(c, ab) = (c, a)(c, b)$ and $(a, bc) = (a, c)$. In addition, $((a, b), c) = (a, b, c)$, $(a, b) = (b, a)$, $(ca, cb) = c(a, b)$, and if $a \equiv b \pmod{c}$, then $(a, c) = (b, c)$. Recall also that $[a_1, a_2, \ldots, a_k] = [[a_1, a_2, \ldots, a_{k-1}], a_k]$ and $[a, b] = \frac{ab}{(a,b)}$. We use these without further reference.

Lemma 2.3. ([**5**, Lemma 2.3]) *For each $n \in \mathbb{N}$, the following holds.*

$$[n, n+1] = n(n+1),$$

$$[n, n+1, n+2] = \frac{n(n+1)(n+2)}{(2, n)},$$

$$[n, n+1, n+2, n+3] = \frac{n(n+1)(n+2)(n+3)}{2(3, n)},$$

$$[n, n+1, n+2, n+3, n+4] = \frac{n(n+1)(n+2)(n+3)(n+4)}{2(4, n)(3, n(n+1))},$$

$$[n, n+1, n+2, n+3, n+4, n+5] = \frac{n(n+1)(n+2)(n+3)(n+4)(n+5)}{6(5, n)(4, n(n+1))},$$

$$[n, n+1, n+2, n+3, n+4, n+5, n+6] = \frac{n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)}{12(3,n)(5,n(n+1))\left(4,(n+2)\left(2,\frac{n(n+1)}{2}\right)\right)}.$$

Proof. We give this result in [**5**, Lemma 2.3] but it has not been published formally, so we give the proof here for completeness. For each $k \geq 0$, define $g_k : \mathbb{N} \to \mathbb{N}$ by

$$g_k(n) = \frac{n(n+1)\cdots(n+k)}{[n, n+1, \ldots, n+k]}.$$

Farhi [**2**] show that $g_0(n) = g_1(n) = 1$ for every $n \in \mathbb{N}$, and g_k satisfies the recursive relation

$$g_k(n) = (k!, (n+k)g_{k-1}(n)) \text{ for all } k, n \in \mathbb{N}.$$

By the definition of the function $g_k(n)$, we obtain that $[n, n+1, \ldots, n+k] = \frac{n(n+1)\cdots(n+k)}{g_k(n)}$. So we only need to find $g_k(n)$ for $k = 1, 2, 3, 4, 5, 6$. Since each case is similar, we only show the proof in the cases $k = 6$. We have

$$\begin{aligned}
g_6(n) &= (6!, (n+6)g_5(n)) \\
&= (6!, 6(n+6)(5,n)(4,n(n+1))) \\
&= 6(8 \cdot 5 \cdot 3, (n+6)(5,n)(4,n(n+1))) \\
&= 6(8, (n+6)(4,n(n+1)))(5,(n+6)(5,n))(3,n+6) \\
&= 6(8, (n+6)(4,n(n+1)))(5,(n+1)(5,n))(3,n) \\
&= 6(8, (n+6)(4,n(n+1)))(5,5(n+1),n(n+1))(3,n) \\
&= 12\left(4,(n+6)\left(2,\frac{n(n+1)}{2}\right)\right)(5,n(n+1))(3,n) \\
&= 12\left(4,(n+2)\left(2,\frac{n(n+1)}{2}\right)\right)(5,n(n+1))(3,n).
\end{aligned}$$

□

Next we calculate the least common multiple of consecutive Lucas numbers.

Lemma 2.4. *For each $k \geq 1$, let $P_k = L_n L_{n+1} L_{n+2} \cdots L_{n+k}$. Then the following statements hold for every $n \geq 1$*

(i) $[L_n, L_{n+1}] = L_n L_{n+1}$.

(ii) $[L_n, L_{n+1}, L_{n+2}] = L_n L_{n+1} L_{n+2}$.

(iii) $[L_n, L_{n+1}, L_{n+2}, L_{n+3}] = \frac{P_3}{F_{(n,3)}}$.

(iv) $$[L_n, L_{n+1}, L_{n+2}, L_{n+3}, L_{n+4}] = \begin{cases} \dfrac{P_4}{F_{(n-2,4)}} & \text{if } n \equiv 1 \pmod 3, \\ \dfrac{P_4}{2F_{(n-2,4)}} & \text{if } n \equiv 0, 2 \pmod 3. \end{cases}$$

(v) $$[L_n, L_{n+1}, L_{n+2}, \ldots, L_{n+5}] = \begin{cases} \dfrac{P_5}{6} & \text{if } n \equiv 1, 2 \pmod 4, \\ \dfrac{P_5}{2} & \text{if } n \equiv 0, 3 \pmod 4. \end{cases}$$

$$\text{(vi)}\ \ [L_n, L_{n+1}, L_{n+2}, \ldots, L_{n+6}] = \begin{cases} \dfrac{P_6}{3 \cdot 2^{v_2(L_n)+1}} & \textit{if } n \equiv 0, 1, 2 \pmod 4, \\ \dfrac{P_6}{2^{v_2(L_n)+1}} & \textit{if } n \equiv 3 \pmod 4. \end{cases}$$

Proof. Recall from Lemma 2.2 that three consecutive Lucas numbers are pairwise relatively prime. So (i) and (ii) follow immediately. Then we obtain from (ii) and Lemma 2.2 that

$$\begin{aligned}[L_n, L_{n+1}, L_{n+2}, L_{n+3}] &= [[L_n, L_{n+1}, L_{n+2}], L_{n+3}] = [L_n L_{n+1} L_{n+2}, L_{n+3}] \\ &= \frac{L_n L_{n+1} L_{n+2} L_{n+3}}{(L_n L_{n+1} L_{n+2}, L_{n+3})} = \frac{P_3}{(L_n, L_{n+3})} \\ &= \frac{P_3}{F_{(n,3)}},\end{aligned}$$

which proves (iii).

Similar to the proof of (iii), we obtain

$$[L_n, L_{n+1}, \ldots, L_{n+4}] = \left[\frac{P_3}{F_{(n,3)}}, L_{n+4}\right] = \frac{P_4}{(P_3, F_{(n,3)} L_{n+4})}. \tag{2}$$

<u>*Case 1:*</u> $n \not\equiv 0 \pmod 3$. By Lemma 2.2, the right hand side of (2) is equal to

$$\frac{P_4}{(P_3, L_{n+4})} = \frac{P_4}{(L_n, L_{n+4})(L_{n+1}, L_{n+4})} = \frac{P_4}{F_{(n-2,4)} F_{(n+1,3)}},$$

which is equal to $\frac{P_4}{F_{(n-2,4)}}$ if $n \equiv 1 \pmod 3$ and is equal to $\frac{P_4}{2F_{(n-2,4)}}$ if $n \equiv 2 \pmod 3$.

<u>*Case 2:*</u> $n \equiv 0 \pmod 3$. By Lemma 2.2, the right hand side of (2) is equal to

$$\frac{P_4}{2\left(L_n L_{n+1} L_{n+2} \frac{L_{n+3}}{2}, L_{n+4}\right)} = \frac{P_4}{2(L_n, L_{n+4})(L_{n+1}, L_{n+4})} = \frac{P_4}{2F_{(n-2,4)}}.$$

Next we prove (v). Since L_{n+3}, L_{n+4}, L_{n+5} are pairwise relatively prime, we see that

$$(P_4, L_{n+5}) = (L_n, L_{n+5})(L_{n+1}, L_{n+5})(L_{n+2}, L_{n+5}) = F_{(n-1,4)} F_{(n+2,3)}. \tag{3}$$

<u>*Case 1:*</u> $n \equiv 1 \pmod 3$. Then by (iv) and a similar calculation in the proof of (iv), we obtain

$$[L_n, L_{n+1}, L_{n+2}, \ldots, L_{n+5}] = \left[\frac{P_4}{F_{(n-2,4)}}, L_{n+5}\right] = \frac{P_5}{(P_4, F_{(n-2,4)} L_{n+5})}.$$

By Lemma 2.2, $\left(F_{(n-2,4)}, L_{n+5}\right) = 1$. So we obtain by (3) that

$$\begin{aligned}(P_4, F_{(n-2,4)} L_{n+5}) &= (P_4, F_{(n-2,4)}) F_{(n-1,4)} F_{(n+2,3)} \\ &= 2(P_4, F_{(n-2,4)}) F_{(n-1,4)}.\end{aligned} \tag{4}$$

It is easy to check that if $n \equiv 1, 2 \pmod 4$, then the right hand side of (4) is equal to 6, and if $n \equiv 0, 3 \pmod 4$, then it is equal to 2.

<u>Case 2</u>: $n \equiv 0, 2 \pmod 3$. Similar to Case 1, we have

$$[L_n, L_{n+1}, L_{n+2}, \ldots, L_{n+5}] = \frac{P_5}{(P_4, 2F_{(n-2,4)}L_{n+5})}.$$

Using Lemma 2.2 or Lemma 2.1 it is easy to check that 2, $F_{(n-2,4)}$, and L_{n+5} are pairwise relatively prime. This and (3) imply that

$$\begin{aligned}(P_4, 2F_{(n-2,4)}L_{n+5}) &= 2(P_4, F_{(n-2,4)})F_{(n-1,4)}F_{(n+2,3)}\\ &= 2(P_4, F_{(n-2,4)})F_{(n-1,4)},\end{aligned}$$

which is the same as (4). This proves (v). Next we prove (vi).
<u>Case 1:</u> $n \equiv 1, 2 \pmod 4$ and $n \equiv 1, 2 \pmod 3$. Similar to the proof of (v), we have

$$[L_n, L_{n+1}, L_{n+2}, \ldots, L_{n+6}] = \frac{P_6}{(P_5, 6L_{n+6})}.$$

We obtain by Lemma 2.2 that L_{n+6} is relatively prime to $L_{n+3}L_{n+4}L_{n+5}$ and obtain by Lemma 2.1 that $6 \mid L_{n+4}L_{n+5}$. Therefore,

$$\begin{aligned}(P_5, 6L_{n+6}) &= 6\left(L_nL_{n+1}L_{n+2}L_{n+3}\frac{L_{n+4}L_{n+5}}{6}, L_{n+6}\right)\\ &= 6(L_n, L_{n+6})(L_{n+1}, L_{n+6})(L_{n+2}, L_{n+6})\\ &= 6 \cdot 2^{v_2(L_n)}F_{(n,4)} = 3 \cdot 2^{v_2(L_n)+1}.\end{aligned} \tag{5}$$

<u>Case 2:</u> $n \equiv 1, 2 \pmod 4$ and $n \equiv 0 \pmod 3$. Similar to Case 1, we obtain that $2 \mid L_{n+3}$, $3 \mid L_{n+4}L_{n+5}$, and $\frac{L_{n+6}}{2}$ is relatively prime to $\frac{L_{n+3}}{2}$, L_{n+4}, and L_{n+5}. Thus

$$(P_5, 6L_{n+6}) = 12\left(\frac{L_n}{2}L_{n+1}L_{n+2}\frac{L_{n+3}}{2}\frac{L_{n+4}L_{n+5}}{3}, \frac{L_{n+6}}{2}\right) = 3 \cdot 2^{v_2(L_n)+1},$$

which is the same as (5).

So Cases 1 and 2 lead to the same formula for $[L_n, L_{n+1}, \ldots, L_{n+6}]$.
<u>Case 3:</u> $n \equiv 0, 3 \pmod 4$ and $n \equiv 1, 2 \pmod 3$. Similar to Case 1,

$$[L_n, L_{n+1}, \ldots, L_{n+6}] = \frac{P_6}{(P_5, 2L_{n+6})}$$

and

$$(P_5, 2L_{n+6}) = 2(L_n, L_{n+6})(L_{n+1}, L_{n+6})(L_{n+2}, L_{n+6}) = 2^{v_2(L_n)+1}F_{(n,4)}. \tag{6}$$

<u>Case 4:</u> $n \equiv 0, 3 \pmod 4$ and $n \equiv 0 \pmod 3$. Similar to Case 3,

$$[L_n, L_{n+1}, \ldots, L_{n+6}] = \frac{P_6}{(P_5, 2L_{n+6})}, \text{ and } (P_5, 2L_{n+6}) = 2^{v_2(L_n)+1}F_{(n,4)},$$

which is the same as (6).

In Cases 3 and 4, if $n \equiv 0 \pmod 4$, then $F_{(n,4)} = 3$, and if $n \equiv 3 \pmod 4$, then $F_{(n,4)} = 1$, which imply the desired result. This completes the proof. □

3. Main Results

We modify the result of Khaochim and Pongsriiam [**5**, Theorem 3.1] so that it is applicable to $z(L_nL_{n+1}\dots L_{n+k})$. In fact, the next theorem describes the general strategy in obtaining the formula for $z(L_nL_{n+1}\dots L_{n+k})$ for every $k \geq 1$.

Theorem 3.1. *Let $n \geq 2$, $k \geq 1$, $a = 2[n, n+1, \dots, n+k]$, $b = L_nL_{n+1}\dots L_{n+k}$, and $f_k(n) = \frac{L_nL_{n+1}L_{n+2}\dots L_{n+k}}{[L_n, L_{n+1}, L_{n+2}, \dots, L_{n+k}]}$. Then the following hold:*

(i) *$b \mid f_k(n)F_{aj}$ for every $j \geq 1$.*

(ii) *$z(b) = aj$, where j is the smallest positive integer such that $b \mid F_{aj}$. In fact, j is the smallest positive integer such that $v_p(b) \leq v_p(F_{aj})$ for every prime p dividing $f_k(n)$.*

Proof. Since $2(n+i) \mid a$ for all $0 \leq i \leq k$, we obtain by Lemma 2.2(i) that $L_{n+i} \mid F_a$ for all $0 \leq i \leq k$. So $[L_n, L_{n+1}, \dots, L_{n+k}] \mid F_a$. Therefore, $b \mid f_k(n)F_a$. Since $F_a \mid F_{aj}$,

$$b \mid f_k(n)F_{aj} \quad \text{for every } j \geq 1.$$

This proves (i). Next let $z(b) = \ell$. Then $b \mid F_\ell$. Therefore $L_{n+i} \mid F_\ell$ for all $0 \leq i \leq k$. Then we obtain by Lemma 2.2(i) that $2(n+i) \mid \ell$ for all $0 \leq i \leq k$, which implies that $a \mid \ell$. Thus $\ell = aj$ for some $j \in \mathbb{N}$. By the definition of $z(b)$, we see that j is the smallest positive integer such that

$$b \mid F_{aj}. \tag{7}$$

Note that (7) is equivalent to $v_p(b) \leq v_p(F_{aj})$ for every prime p. But by (i), if p is a prime and $p \nmid f_k(n)$, then

$$v_p(b) \leq v_p(f_k(n)F_{aj}) = v_p(F_{aj}).$$

Therefore, (7) is equivalent to

$$v_p(b) \leq v_p(F_{aj}) \quad \text{for every prime } p \text{ dividing } f_k(n). \tag{8}$$

Hence $z(b) = \ell = aj$ and j is the smallest positive integer satisfying (8). This proves (ii). □

Next, we give another proof of Marques's result to demonstrate an application of Theorem 3.1.

Example 3.2. Let n, k, a, b, and $f_k(n)$ be the quantities defined in Theorem 3.1. Then by Lemma 2.4, $f_k(n) = 1$ for $k \in \{1, 2\}$. In this case, we can choose $j = 1$ and obtain by Theorem 3.1 and Lemma 2.3 that

$$z(L_nL_{n+1}) = 2[n, n+1] = 2n(n+1) \qquad \text{and}$$

$$z(L_nL_{n+1}L_{n+2}) = 2[n, n+1, n+2] = \frac{2n(n+1)(n+2)}{(n,2)}.$$

Assume that $k = 3$. Then by Lemma 2.4, we have $f_k(n) = F_{(n,3)}$. If $3 \nmid n$, then we can choose $j = 1$ and obtain by Theorem 3.1 and Lemma 2.3 that $z(b) = a = 2[n, n+1, n+3, n+3] = \frac{2n(n+1)(n+2)(n+3)}{2(3,n)} = n(n+1)(n+2)(n+3)$. Assume that $3 \mid n$. Then $f_k(n) = 2$ and we need to find the smallest j such that

$v_2(b) \le v_2(F_{aj})$. Since $12 \mid aj$, we obtain by Lemma 2.1 that $v_2(F_{aj}) = v_2(aj)+2 \ge 4 > v_2(L_n) + v_2(L_{n+3}) = v_2(b)$ for every j. So we can choose $j = 1$ and obtain $z(b) = a = \frac{n(n+1)(n+2)(n+3)}{3}$. In any case,

$$z(L_n L_{n+1} L_{n+2} L_{n+3}) = \frac{n(n+1)(n+2)(n+3)}{(n,3)}.$$

This gives all main results of Marques in [**9**].

Theorem 3.3. *Let $n \ge 1$, $a = 2[n, n+1, n+2, n+3, n+4]$, and $b = L_n L_{n+1} L_{n+2} L_{n+3} L_{n+4}$. Then*

$$z(b) = \begin{cases} 3a & \text{if } n \equiv 2, 14, 18, 30 \pmod{36}, \\ a & \text{otherwise.} \end{cases}$$

Proof. It is easy to check that the result holds for $n = 1, 2$. So assume that $n \ge 3$.

<u>Case 1:</u> $n \equiv 1 \pmod 3$. Then by Lemma 2.4 and Theorem 3.1, we have $b \mid F_{(n-2,4)} F_{aj}$ for every $j \ge 1$ and we would like to find the smallest j such that $b \mid F_{aj}$. If $n \not\equiv 2 \pmod 4$, then $F_{(n-2,4)} = 1$, so we can choose $j = 1$ and obtain $z(b) = a$. So assume that $n \equiv 2 \pmod 4$. Then $F_{(n-2,4)} = 3$ and by Theorem 3.1, we only need to consider $v_3(b)$ and $v_3(F_{aj})$. Since $n \equiv 1 \pmod 3$ and $n \equiv 2 \pmod 4$, we obtain by Lemma 2.1 that $v_3(b) = v_3(L_n) + v_3(L_{n+4}) = v_3(n) + v_3(n+4) + 2 = 2$. Since $4 \mid n+2$ and $n+2 \mid aj$, $4 \mid aj$. Similarly, $3 \mid aj$. So we obtain by Lemma 2.1 that for every $j \ge 1$, $v_3(F_{aj}) = v_3(aj) + 1 \ge 2 = v_3(b)$. Thus we can choose $j = 1$ and obtain $z(b) = a$. This shows $z(b) = a$ whenever $n \equiv 1 \pmod 3$.

The idea used in the following case is still the same as that in Case 1. So our argument will be shorter.

<u>Case 2:</u> $n \equiv 0 \pmod 3$. Then by Lemma 2.4 and Theorem 3.1, we have $b \mid 2F_{(n-2,4)} F_{aj}$ for every $j \ge 1$ and our problem is reduced to finding the smallest positive integer j such that $v_p(b) \le v_p(F_{aj})$ for every prime p dividing $2F_{(n-2,4)}$. Let $j \ge 1$. Since $3 \mid n$ and $n \mid a$, we see that $3 \mid aj$. Similarly, $2 \mid aj$. Therefore, $6 \mid aj$. By Lemma 2.1, $v_2(F_{aj}) = v_2(aj) + 2 \ge 3 = v_2(L_n) + v_2(L_{n+3}) = v_2(b)$. So if $n \not\equiv 2 \pmod 4$, then we can choose $j = 1$ and obtain $z(b) = a$. So assume that $n \equiv 2 \pmod 4$. By Lemmas 2.1 and 2.3, we obtain that

$$\begin{aligned} v_3(b) &= v_3(L_n) + v_3(L_{n+4}) = v_3(n) + v_3(n+4) + 2 = v_3(n) + 2, \text{ and} \\ v_3(F_{aj}) &= v_3(aj) + 1 = v_3\left(\frac{n(n+1)(n+2)(n+3)(n+4)}{(4,n)(3,n(n+1))}\right) + v_3(j) + 1 \\ &= v_3(n) + v_3(n+3) - 1 + v_3(j) + 1 = v_3(n) + v_3(n+3) + v_3(j). \end{aligned}$$

So we need to find the smallest $j \ge 1$ such that $v_3(n+3) + v_3(j) \ge 2$. Note that $n + 3 \equiv 0, 3, 6 \pmod 9$.

(i) If $n + 3 \equiv 0 \pmod 9$, then we can choose $j = 1$ and obtain $z(b) = a$.

(ii) If $n + 3 \equiv 3, 6 \pmod 9$, then $v_3(j) \ge 1$, so we choose $j = 3$ and obtain $z(b) = 3a$.

So in this case, $z(b) = 3a$ when $n \equiv 2 \pmod 4$ and $n \equiv 0, 3 \pmod 9$. Otherwise, $z(b) = a$.

<u>*Case 3:*</u> $n \equiv 2 \pmod 3$. Similar to Case 2, $v_2(b) = v_2(L_{n+1}) + v_2(L_{n+4}) = 3 \le v_2(F_{aj})$, and if $n \not\equiv 2 \pmod 4$, then $z(b) = a$. So assume that $n \equiv 2 \pmod 4$. Then similar to Case 2, we obtain by Lemmas 2.1 and 2.3 that

$$v_3(b) = v_3(L_n) + v_3(L_{n+4}) = v_3(n+4) + 2 \quad \text{and}$$
$$v_3(F_{aj}) = v_3(aj) + 1 = v_3(n+1) + v_3(n+4) + v_3(j).$$

Therefore, $v_3(F_{aj}) \ge v_3(b) \Leftrightarrow v_3(n+1) + v_3(j) \ge 2$.

(i) If $n+1 \equiv 0 \pmod 9$, then we can choose $j = 1$ and obtain $z(b) = a$.

(ii) If $n+1 \equiv 3, 6 \pmod 9$, then $v_3(j) \ge 1$ so we choose $j = 3$ and obtain $z(b) = 3a$.

So in this case, $z(b) = 3a$ when $n \equiv 2 \pmod 4$ and $n \equiv 2, 5 \pmod 9$. Otherwise $z(b) = a$. This completes the proof. □

Now the result of Marques and Trojovský [**10**] follows immediately from Theorem 3.3 and Lemma 2.3.

Corollary 3.4 ([**10**])**.** *Let $n \ge 1$ and $b = L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}$. Then*

$$z(b) = \begin{cases} n(n+1)(n+2)(n+3)(n+4) & \text{if } n \equiv 1 \pmod 6, \\ \dfrac{n(n+1)(n+2)(n+3)(n+4)}{2} & \text{if } n \equiv 2, 10, 14, 18, 22, 30, 34 \,(\mathrm{mod}\, 36), \\ \dfrac{n(n+1)(n+2)(n+3)(n+4)}{3} & \text{if } n \equiv 3, 5 \pmod 6, \\ \dfrac{n(n+1)(n+2)(n+3)(n+4)}{4} & \text{if } n \equiv 4 \pmod{12}, \\ \dfrac{n(n+1)(n+2)(n+3)(n+4)}{6} & \text{if } n \equiv 6, 26 \pmod{36}, \\ \dfrac{n(n+1)(n+2)(n+3)(n+4)}{12} & \text{if } n \equiv 0, 8 \pmod{12}. \end{cases}$$

Next we extend the formula of $z(L_nL_{n+1}L_{n+2}\dots L_{n+k})$ to the case $k = 5, 6$.

Theorem 3.5. *Let $n \ge 1$, $a = 2[n, n+1, \dots, n+5]$, and $b = L_nL_{n+1}\dots L_{n+5}$. Then*

$$z(b) = \begin{cases} 3a & \text{if } n \equiv 1, 2, 13, 14, 17, 18, 29, 30 \pmod{36}, \\ a & \text{otherwise.} \end{cases}$$

Proof. It is easy to check that the result holds for $n = 1, 2$. So assume that $n \ge 3$. By Lemma 2.4 and Theorem 3.1, we obtain that $b \mid \ell F_{aj}$ for every $j \ge 1$, where $\ell = 2, 6$. So we need to consider only v_2 and v_3 of b and F_{aj}. Remark that $4 \mid aj$ and $3 \mid aj$. So by Lemma 2.1, we obtain $v_2(F_{aj}) = v_2(aj) + 2 \ge 4$. For $n \equiv 0 \pmod 3$, we obtain by Lemma 2.1 that $v_2(b) = v_2(L_n) + v_2(L_{n+3}) = 3$. Similarly, if $n \equiv 1 \pmod 3$, then $v_2(b) = v_2(L_{n+2}) + v_2(L_{n+5}) = 3$, and if $n \equiv 2 \pmod 3$, then $v_2(b) = v_2(L_{n+1}) + v_2(L_{n+4}) = 3$. So in any case,

$$v_2(b) = 3 < v_2(F_{aj}) \qquad \text{for every } j \geq 1. \tag{9}$$

In addition,

(a) if $n \equiv 0 \pmod 4$, then $v_3(b) = v_3(L_{n+2}) = v_3(n+2) + 1$,

(b) if $n \equiv 1 \pmod 4$, then $v_3(b) = v_3(L_{n+1}) + v_3(L_{n+5}) = v_3(n+1) + v_3(n+5) + 2$,

(c) if $n \equiv 2 \pmod 4$, then $v_3(b) = v_3(L_n) + v_3(L_{n+4}) = v_3(n) + v_3(n+4) + 2$,

(d) if $n \equiv 3 \pmod 4$, then $v_3(b) = v_3(L_{n+3}) = v_3(n+3) + 1$.

By Lemmas 2.1 and 2.3, we obtain the following:

(i) If $n \equiv 0 \pmod 3$, then $v_3(F_{aj}) = v_3(aj) + 1 = v_3(a) + v_3(j) + 1 = v_3(n) + v_3(n+3) - 1 + v_3(j) + 1 = v_3(n) + v_3(n+3) + v_3(j)$.

(ii) If $n \equiv 1 \pmod 3$, then $v_3(F_{aj}) = v_3(n+2) + v_3(n+5) + v_3(j)$.

(iii) If $n \equiv 2 \pmod 3$, then $v_3(F_{aj}) = v_3(n+1) + v_3(n+4) + v_3(j)$.

Case 1: $n \equiv 0, 3 \pmod 4$. Then by Theorem 3.1, Lemma 2.4, and (9), we can choose $j = 1$ and obtain $z(b) = a$.

Case 2: $n \equiv 1, 2 \pmod 4$. Then by Theorem 3.1, Lemma 2.4, and (9), we only need to check v_3 of b and F_{aj}.

Case 2.1: $n \equiv 1 \pmod 4$ and $n \equiv 0 \pmod 3$. Then by (b) and (i), we obtain

$$v_3(b) = v_3(n+1) + v_3(n+5) + 2 = 2 \leq v_3(n) + v_3(n+3) + v_3(j) = v_3(F_{aj})$$

for every j. So we choose $j = 1$ and obtain $z(b) = a$.

Case 2.2: $n \equiv 2 \pmod 4$ and $n \equiv 0 \pmod 3$. Then by (c) and (i), $v_3(b) = v_3(n) + 2$ and $v_3(F_{aj}) = v_3(n) + v_3(n+3) + v_3(j)$. So $v_3(F_{aj}) \geq v_3(b)$ if and only if $v_3(n+3) + v_3(j) \geq 2$. Therefore,

(i) if $n + 3 \equiv 0 \pmod 9$, then we choose $j = 1$ and obtain $z(b) = a$,

(ii) if $n + 3 \equiv 3, 6 \pmod 9$, then we choose $j = 3$ and obtain $z(b) = 3a$.

Case 2.3: $n \equiv 2 \pmod 4$ and $n \equiv 1 \pmod 3$. Similar to Case 2.1, we obtain $z(b) = a$.

Case 2.4: $n \equiv 1 \pmod 4$ and $n \equiv 1 \pmod 3$. This case is similar to Case 2.2 and we obtain that $v_3(F_{aj}) \geq v_3(b)$ if and only if $v_3(n+2) + v_3(j) \geq 2$. Therefore

(i) if $n + 2 \equiv 0 \pmod 9$, then $z(b) = a$,

(ii) if $n + 2 \equiv 3, 6 \pmod 9$, then $z(b) = 3a$.

Case 2.5: $n \equiv 1 \pmod 4$ and $n \equiv 2 \pmod 3$. This case is similar to Cases 2.2 and 2.4, and we obtain that $v_3(F_{aj}) \geq v_3(b)$ if and only if $v_3(n+4) + v_3(j) \geq 2$. So

(i) if $n + 4 \equiv 0 \pmod 9$, then $z(b) = a$,

(ii) if $n + 4 \equiv 3, 6 \pmod 9$, then $z(b) = 3a$.

Case 2.6: $n \equiv 2 \pmod 4$ and $n \equiv 2 \pmod 3$. This case is similar to Cases 2.2, 2.4, and 2.5, and we obtain that $v_3(F_{aj}) \geq v_3(b)$ if and only if $v_3(n+1) + v_3(j) \geq 2$. So

(i) if $n+1 \equiv 0 \pmod 9$, then $z(b) = a$,

(ii) if $n+1 \equiv 3, 6 \pmod 9$, then $z(b) = 3a$.

Combining the result in each case, we obtain the desired formula. □

Theorem 3.6. *Let $n \geq 1$, $a = 2[n, n+1, \dots, n+6]$, and $b = L_n L_{n+1} \dots L_{n+6}$. Then*

$$z(b) = \begin{cases} 3a & \text{if } n \equiv 1, 2, 12, 13, 14, 16, 17, 18, 28, 29 \pmod{36}, \\ a & \text{otherwise.} \end{cases}$$

Proof. The proof of this theorem follows the same idea used previously. So we only give a short proof. Similar to the proof of Theorem 3.5, we only need to evaluate v_2 and v_3 of b and F_{aj}. By Lemma 2.3, we obtain the following:

(i) If $n \equiv 1 \pmod 4$, then $v_2(a) = v_2(n+3) + 1 \geq 3$.

(ii) If $n \equiv 3 \pmod 4$, then $v_2(a) = v_2(n+1) + v_2(n+5) - 1 \geq 3$.

(iii) If $n \equiv 0 \pmod 4$, then $v_2(a) = v_2(n) + v_2(n+4) - 1 \geq 3$.

(iv) If $n \equiv 2 \pmod 4$, then $v_2(a) = v_2(n+2) + v_2(n+6) - 1 \geq 3$.

From (i)-(iv) and Lemma 2.1, we see that

$$v_2(F_{aj}) = v_2(aj) + 2 \geq 5 \qquad \text{for every } j. \tag{10}$$

Next by Lemma 2.1, we obtain the following:

(a) If $n \equiv 1, 2 \pmod 3$, then $v_2(b) = 3$.

(b) If $n \equiv 3 \pmod 6$, then $v_2(b) = 5$.

(c) If $n \equiv 0 \pmod 6$, then $v_2(b) = 4$.

From (a)–(c) and (10), we see that

$$v_2(b) \leq v_2(F_{aj}) \qquad \text{for every } j. \tag{11}$$

<u>*Case 1:*</u> $n \equiv 3 \pmod 4$. Then by Theorem 3.1, Lemma 2.4, and (11), we can choose $j = 1$ and obtain $z(b) = a$.

<u>*Case 2:*</u> $n \equiv 0, 1, 2 \pmod 4$. Then we need to evaluate $v_3(b)$ and $v_3(F_{aj})$. By Lemma 2.1, we obtain the following:

(i) If $n \equiv 0 \pmod 4$, then $v_3(b) = v_3(n+2) + v_3(n+6) + 2$.

(ii) If $n \equiv 1 \pmod 4$, then $v_3(b) = v_3(n+1) + v_3(n+5) + 2$.

(iii) If $n \equiv 2 \pmod 4$, then $v_3(b) = v_3(n) + v_3(n+4) + 2$.

By Lemmas 2.1 and 2.3, we obtain the following:

(a) If $n \equiv 0 \pmod 3$, then $v_3(F_{aj}) = v_3(n) + v_3(n+3) + v_3(n+6) + v_3(j) - 1$.

(b) If $n \equiv 1 \pmod 3$, then $v_3(F_{aj}) = v_3(n+2) + v_3(n+5) + v_3(j)$.

(c) If $n \equiv 2 \pmod 3$, then $v_3(F_{aj}) = v_3(n+1) + v_3(n+4) + v_3(j)$.

Comparing (i)–(iii) and (a)–(c), we see that $v_3(b) \leq v_3(F_{aj})$ for every j in the following cases:

(x) $n \equiv 0 \pmod 4$ and $n \equiv 2 \pmod 3$,

(y) $n \equiv 1 \pmod 4$ and $n \equiv 0 \pmod 3$, and

(z) $n \equiv 2 \pmod 4$ and $n \equiv 1 \pmod 3$.

Hence in the cases (x), (y), and (z), we have $z(b) = a$.

In the other cases, we obtain the following.

Case 2.1: $n \equiv 0 \pmod 4$ and $n \equiv 1 \pmod 3$. Then $v_3(F_{aj}) \geq v_3(b) \Leftrightarrow v_3(n+5)+v_3(j) \geq 2$. So

(i) if $n + 5 \equiv 0 \pmod 9$, then $z(b) = a$,

(ii) if $n + 5 \equiv 3, 6 \pmod 9$, then $z(b) = 3a$.

Case 2.2: $n \equiv 0 \pmod 4$ and $n \equiv 0 \pmod 3$. Then $v_3(F_{aj}) \geq v_3(b) \Leftrightarrow v_3(n) + v_3(n + 3) + v_3(j) \geq 3$. So

(i) if $n \equiv 0, 6 \pmod 9$, then $z(b) = a$,

(ii) if $n \equiv 3 \pmod 9$, then $z(b) = 3a$.

Case 2.3: $n \equiv 1 \pmod 4$ and $n \equiv 1 \pmod 3$. Then $v_3(F_{aj}) \geq v_3(b) \Leftrightarrow v_3(n+2)+v_3(j) \geq 2$. So

(i) if $n + 2 \equiv 0 \pmod 9$, then $z(b) = a$,

(ii) if $n + 2 \equiv 3, 6 \pmod 9$, then $z(b) = 3a$.

Case 2.4: $n \equiv 1 \pmod 4$ and $n \equiv 2 \pmod 3$. Then $v_3(F_{aj}) \geq v_3(b) \Leftrightarrow v_3(n+4)+v_3(j) \geq 2$. So

(i) if $n + 4 \equiv 0 \pmod 9$, then $z(b) = a$,

(ii) if $n + 4 \equiv 3, 6 \pmod 9$, then $z(b) = 3a$.

Case 2.5: $n \equiv 2 \pmod 4$ and $n \equiv 0 \pmod 3$. Then $v_3(F_{aj}) \geq v_3(b) \Leftrightarrow v_3(n+3)+v_3(n + 6) + v_3(j) \geq 3$. So

(i) if $n \equiv 3, 6 \pmod 9$, then $z(b) = a$,

(ii) if $n \equiv 0 \pmod 9$, then $z(b) = 3a$.

Case 2.6: $n \equiv 2 \pmod 4$ and $n \equiv 2 \pmod 3$. Then $v_3(F_{aj}) \geq v_3(b) \Leftrightarrow v_3(n+1)+v_3(j) \geq 2$. So

(i) if $n + 1 \equiv 0 \pmod 9$, then $z(b) = a$,

(ii) if $n + 1 \equiv 3, 6 \pmod 9$, then $z(b) = 3a$.

This completes the proof. □

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