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A NOTE ON THE EQUIVALENCE OF MOTZKIN'S MAXIMAL DENSITY AND RUZSA'S MEASURES OF INTERSECTIVITY

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Abstract. In this short note, we see the equivalence of Motzkin's maximal density of integral sets whose no two elements are allowed to differ by an element of a given set M of positive integers and the measures of difference intersectivity defined by Ruzsa. Further more, the maximal density $\mu(M)$ has been determined for some infinite sets M and in a specific case of generalized arithmetic progression of dimension two a lower bound has been given for $\mu(M)$.

1. Introduction and the Equivalence

In an unpublished problem collection Motzkin [**12**] posed the problem of maximal density of integral sets defined as follows

Let S be a set of nonnegative integers and let $S(x)$ be the number of elements $n \in S$ such that $n \leq x$, $x \in \mathbb{R}$. The upper and lower densities of S (denoted by $\bar{d}(S)$ and $\underline{d}(S)$, respectively) are defined as follows

$$\bar{d}(S) := \limsup_{x\to\infty} \frac{S(x)}{x}, \qquad \underline{d}(S) := \liminf_{x\to\infty} \frac{S(x)}{x}.$$

If $\bar{d}(S) = \underline{d}(S)$, we denote the common value by $d(S)$, and say that S has density $d(S)$. Let M be a given set of positive integers. S is said to be an M-set if $a \in S, b \in S \Rightarrow a - b \notin M$. Motzkin asks to determine the maximal density $\mu(M)$ of M-sets, given by

$$\mu(M) := \sup_S \bar{d}(S),$$

where supremum is taken over all M-sets S. Almost all sets M for which $\mu(M)$ is determined exactly or the bounds of $\mu(M)$ have been obtained up to now are finite. For the complete survey on the problem see ([**1**], [**8**], [**7**], [**6**], [**10**], [**11**], [**13**], [**14**], [**15**]). Before we obtain $\mu(M)$ for some infinite sets M in the next section, we mention Ruzsa's "measures of intersectivity" below.

Define $S - S := \{a - b : a, b \in S\}$ and $S + a := \{x + a : x \in S\}$. A set M of positive integers is called (difference) intersective if $M \cap (S - S) \neq \phi$, whenever S

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has positive upper density. Instead of upper density one might equally write the lower density or just the natural density.

Define

$$\delta_1(M) := \sup\{d(S) : M \cap (S - S) = \phi\},$$

where the supremum is taken over all sets S having the natural density $d(S)$, and

$$\delta_2(M) := \sup\{\overline{d}(S) : d(S \cap (S + a)) = 0 \text{ for all } a \in M\}.$$

Clearly, we have $\delta_1(M) \le \mu(M) \le \delta_2(M)$.

Putting

$$D(M, n) = \max\{|T| : T \subset [1, n],\ M \cap (T - T) = \phi\},$$

and defining

$$\delta(M) := \lim_{n\to\infty} \frac{D(M, n)}{n} = \inf \frac{D(M, n)}{n},$$

we have the following theorem.

Theorem A (Ruzsa, [**17**]). *For each set M, $\delta_1(M) = \delta_2(M) = \delta(M)$.*

Consequently, Motzkin's maximal density and Ruzsa's measures of intersectivity are indeed the same.

Almost all sets M for which $\mu(M)$ has been determined exactly or some bounds have been given up to now are finite sets. The initial work on this problem was done by Cantor and Gordon [**1**], where they showed the existence of $\mu(M)$ for each set M of positive integers, and also determined $\mu(M)$ when M has one or two elements. They proved that if $|M| = 1$, then $\mu(M) = \frac{1}{2}$ and if $M = \{a, b\}$ with $\gcd(a, b) = 1$, then $\mu(M) = \frac{\lfloor \frac{a+b}{2} \rfloor}{a+b}$. By a result of Cantor and Gordon, it is sufficient to consider the problem only for those sets M whose elements are relatively prime. Later, Haralambis [**8**] gave some general estimates and expressions for $\mu(M)$ for most members of the families $\{1, a, b\}$ and $\{1, 2, a, b\}$. Gupta and Tripathi [**7**] obtained the value of $\mu(M)$, where M is finite and the elements of M are in arithmetic progression. Liu and Zhu [**10**] computed the values of $\mu(M)$ for $M = \{a, 2a, \ldots, (m-1)a, b\}$ and $M = \{a, b, a+b\}$, and they gave some bounds of $\mu(M)$ for $M = \{a, b, b-a, b+a\}$ using graph theoretic techniques. They further computed $\mu(M)$ for $M = [1, a] \cup [b, m+1]$, where $a < b$ in [**11**] using fractional chromatic number of distance graphs generated by the set M. Some more partial work on the problem can be found in ([**16**], [**4**], [**5**], [**9**], [**3**]) but all in the case where the given set M is finite. The present author together with Tripathi ([**13**], [**14**],[**15**]) have discussed the problem for the families $M = \{a, b, c\}$, where $a < b$, $c = nb$ or na and $M = \{a, b, n(a + b)\}$, and for the sets related to finite arithmetic progressions. In the next section, we obtain $\mu(M)$ for some infinite sets M out of which some sets are really interesting which were already discussed by Sàrközy ([**18**], [**19**], [**20**]) and Ruzsa [**17**]. In section 3, we discuss the maximal density of generalized arithmetic progression of dimension two in some specific cases and give some problems on this.

2. Maximal density of some infinite sets

It is straightforward from the definition that if $M_1 \subset M_2$, then $\mu(M_1) \geq \mu(M_2)$. Therefore, we have $0 \leq \mu(M) \leq 1/2$. Now a natural question arrives in whether that $\mu(M)$ can be zero for a finite set M. The answer is NO. Indeed, let the largest element in M be n, then clearly $M \subset [1, n]$, and hence $\mu(M) \geq \mu([1, n]) = \frac{1}{n+1} > 0$. So, we conclude that if $\mu(M) = 0$, then M is an infinite set. Below, we give some infinite sets M for which $\mu(M) = 0$. All non trivial examples are given by Sàrközy in a series of papers ([**18**], [**19**], [**20**]).

Example 1. If $M^+ = \{p + 1 : p \text{ is a prime}\}$ and $M^- = \{p - 1 : p \text{ is a prime}\}$ then $\mu(M^+) = 0 = \mu(M^-)$.

Example 2. If $M^{\square} = \{n^2 : n \text{ is a positive integer}\}$, then $\mu(M^{\square}) = 0$.

Example 3. If $M^{\boxplus} = \{n^2 + 1 : n \text{ is a positive integer}\}$ and $M^{\boxminus} = \{n^2 - 1 : n \text{ is a positive integer}\}$. then $\mu(M^{\boxplus}) = 0 = \mu(M^{\boxminus})$.

If $\mu(M) = 0$, we can always find M-sets S which may or may not be finite. Ruzsa [**17**] proved that there exists a set M for which $\mu(M) = 0$, but there does not exist any infinite M-set S. More generally, he proved the following theorem.

Theorem B. *Let f be any positive-valued function on natural numbers such that $\lim_{n\to\infty} f(n) = \infty$, but $\lim_{n\to\infty} \frac{f(n)}{n} = 0$. There is a set M such that $D(M, n) \ll f(n)$ and $f(n) \ll D(M, n)$, but there is no infinite set S for which $M \cap (S - S) = \phi$.*

As an example take $M = [a, \infty)$, where a is any natural number. We have $\mu(M) = 0$ for this M and there does not exist any infinite set S for which $M \cap (S - S) = \phi$.

For all above infinite sets M given so far, we have $\mu(M) = 0$. Below, we give some examples as theorems for which $|M| = \infty$, but $\mu(M) \neq 0$. We use the following result for the lower bound of $\mu(M)$.

Lemma 1 ([**1**]). *Let $M = \{m_1, m_2, m_3, \ldots\}$ and let c and m be positive integers such that* $\gcd(c, m) = 1$. *Then*

$$\mu(M) \geq \sup_{\gcd(c,m)=1} \frac{1}{m} \min_k |cm_k|_m,$$

where $|x|_m$ denotes the absolute value of the absolutely least remainder of $x \pmod m$.

Theorem 1. *Let $M = \{1, 3, 5, \ldots\}$. Then $\mu(M) = \frac{1}{2}$.*

Proof. Any set S of positive integers which does not contain integers of both parities will be an M-set. Clearly, for such a set S, $\overline{d}(S) \leq 1/2$. Now if the set $S = \{1, 3, 5, \ldots\}$, then equality holds. Therefore, $\mu(M) = 1/2$. □

Theorem 2. *Let $M = \{a, a+d, a+2d, \ldots\}$, where a and d are positive integers with* $\gcd(a, d) = 1$. *Then*

$$\mu(M) = \begin{cases} \frac{1}{2} & \text{if } d \text{ is even;} \\ \frac{d-1}{2d} & \text{if } d \text{ is odd.} \end{cases}$$

Proof. If d is even, then a is odd because $\gcd(a,d)=1$. Hence, $M \subset \{1,3,5,\ldots\}$. Therefore, $\mu(M) \geq \mu(\{1,3,5,\ldots\}) = \frac{1}{2}$. Conversely, we have $M \supset \{1\}$ and hence $\mu(M) \leq \mu(\{1\}) = \frac{1}{2}$. Thus $\mu(M) = \frac{1}{2}$. Now suppose that d is odd. It is known by Gupta and Tripathi [**7**] that

$$\lim_{n\to\infty} \mu(\{a, a+d, a+2d, \ldots, a+(n-1)d\}) = \frac{d-1}{2d}.$$

Therefore,

$$\mu(M) \leq \frac{d-1}{2d}.$$

Next, choose x such that

$$ax \equiv \frac{d-1}{2} \pmod{d}.$$

This gives

$$(a+kd)x \equiv \frac{d-1}{2} \pmod{d}$$

for each k. Therefore, by the Lemma 1, we have

$$\mu(M) \geq \frac{d-1}{2d}.$$

This proves the theorem. □

Remark 1. If $d=1$ in the above theorem, we get $\mu([a,\infty)) = 0$. On the other hand, if $d \neq 1$, then $\mu(M) \neq 0$.

Theorem 3. *Let $M = \{1, r, r^2, \ldots\}$, $r > 1$. Then* $\mu(M) = \dfrac{\lfloor \frac{r+1}{2} \rfloor}{r+1}$.

Proof. Clearly, $\mu(M) \leq \mu(\{1,r\}) = \frac{\lfloor \frac{r+1}{2} \rfloor}{r+1}$. If r is odd, then all integers in M are odd, and hence by the same argument as in the Theorem 2 we get $\mu(M) = \frac{1}{2} = \frac{\lfloor \frac{r+1}{2} \rfloor}{r+1}$. If r is even, then $\frac{\lfloor \frac{r+1}{2} \rfloor}{r+1} = \frac{r}{2(r+1)}$. Choose x such that

$$x \equiv \frac{r}{2} \pmod{r+1}.$$

Then

$$r^k x \equiv (-1)^k \frac{r}{2} \pmod{r+1}$$

for each $k \geq 0$. Therefore, by Lemma 1, we have $\mu(M) \geq \frac{r}{2(r+1)}$ and hence the theorem follows. □

Corollary 1. *Let $M = \{a, ar, ar^2, \ldots\}$, $a \geq 1$, and $r > 1$. Then* $\mu(M) = \frac{\lfloor \frac{r+1}{2} \rfloor}{r+1}$.

Proof. By a theorem of Cantor and Gordon [**1**], we have $\mu(\{a, ar, ar^2, \ldots\}) = \mu(\{1, r, r^2, \ldots\}) = \frac{\lfloor \frac{r+1}{2} \rfloor}{r+1}$. □

3. Maximal density of some specific sets of generalized arithmetic progression of dimension two

Theorem 4. *Let $M = \{a + x_1d_1 + x_2d_2 : 0 \leq x_1 \leq t_1, 0 \leq x_2 \leq t_2\}$, where a is an odd integer and d_1 is an even integer. Then $\mu(M) = 1/2$ if d_2 is even, and*

$$\mu(M) \geq d(M) \geq \frac{2a + t_1d_1 + t_2d_2 - t_2(a + t_1d_1)}{2(2a + t_1d_1 + t_2d_2)}$$

if d_2 is an odd integer.

Proof. If d_2 is even, then all elements of M are odd. Hence, the proof is the same as that one of the Theorem 1. So, assume that d_2 is odd. Let $m = 2a+t_1d_1+t_2d_2$. Clearly, m and t_2 have the same parity. Set $x = \frac{m-t_2}{2}$. Observe that for $0 \leq k \leq t_1$ and $0 \leq l \leq t_2$, we have

$$(a + kd_1 + ld_2)x \equiv -\big(a + (t_1 - k)d_1 + (t_2 - l)d_2\big)x \pmod m.$$

So, in order to use Lemma 1, we only need to consider the first congruences for which $0 \leq k \leq t_1$ and $0 \leq l \leq \lfloor \frac{t_2}{2} \rfloor$.
Case I: (l is even). Clearly, $a + kd_1 + ld_2$ is an odd integer. Hence, we have

$$\begin{aligned}(a + kd_1 + ld_2)x &\equiv \frac{m - t_2(a + kd_1 + ld_2)}{2} \pmod m\\ &= \frac{m - t_2(a + kd_1) - lt_2d_2}{2}\\ &= \frac{m - t_2(a + kd_1) - l(m - 2a - t_1d_1)}{2}\\ &\equiv \frac{m - t_2(a + kd_1) + l(2a + t_1d_1)}{2} \pmod m.\end{aligned}$$

Case II: (l is odd). Clearly, $a + kd_1 + ld_2$ is an even integer. Hence, we have

$$\begin{aligned}(a + kd_1 + ld_2)x &\equiv -\frac{t_2(a + kd_1 + ld_2)}{2} \pmod m\\ &= -\frac{t_2(a + kd_1) - lt_2d_2}{2}\\ &= -\frac{t_2(a + kd_1) - l(m - 2a - t_1d_1)}{2}\\ &\equiv \frac{m - t_2(a + kd_1) + l(2a + t_1d_1)}{2} \pmod m.\end{aligned}$$

Therefore, using Lemma 1, we have

$$\mu(M) \geq d(M) \geq \frac{m - t_2(a + t_1d_1)}{2m} = \frac{2a + t_1d_1 + t_2d_2 - t_2(a + t_1d_1)}{2(2a + t_1d_1 + t_2d_2)}.$$

This completes the proof of the theorem. □

Based on the numerous examples taken using computer programming, we have the following conjecture for this particular case of two-dimensional arithmetic progression.

Conjecture 1. Let $M = \{a + x_1d_1 + x_2d_2 : 0 \le x_1 \le t_1, 0 \le x_2 \le t_2\}$, where a and d_2 are odd integers and d_1 is an even integer. Then, there exists a positive integer d_0 such that for $d_2 \ge d_0$,

$$d(M) = \frac{2a + t_1d_1 + t_2d_2 - t_2(a + t_1d_1)}{2(2a + t_1d_1 + t_2d_2)}.$$

In both Theorem 4 and Conjecture 1, we can interchange the roles of the positive integers d_1 and d_2. We know from the definition of $d(M)$ that the denominator of $d(M)$ divides the sum of some two elements of M. In particular, we believe the following for generalized arithmetic progression of dimension two.

Conjecture 2. Let $M = \{a + x_1d_1 + x_2d_2 : 0 \le x_1 \le t_1, 0 \le x_2 \le t_2\}$. Then, the denominator of $d(M)$ divides $2a + t_1d_1 + t_2d_2$.

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