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MEIR-KEELER TYPE CONTRACTION VIA RATIONAL EXPRESSION

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Abstract. In this paper, we establish a fixed point theorem for a Meir-Keeler type contraction via Gupta-Saxena's rational expression. This result extends and improves the corresponding results of B. Samet et al. [**6**], Najeh Redjel et al. [**5**], and Dass and Gupta [**2**].

1. Introduction

The Banach contraction principle is widely recognized in metric fixed point theory. This principle has been generalized in different direction in different spaces by many authors. Among them, one of the interesting generalization of Banach contraction principle was given in 1969, by Meir-Keeler [**4**].

Meir-Keeler proved the following theorem.

Theorem 1.1 ([**4**]). *Let (X, d) be a complete metric space and let S be a mapping from X into itself satisfying the following condition:*

$$\forall \varepsilon > 0,\ \exists \delta(\varepsilon) > 0 \text{ such that } \varepsilon \le d(x,y) < \varepsilon + \delta(\varepsilon) \implies d(Sx, Sy) < \varepsilon.$$

Then S has a unique fixed point $u \in X$. Moreover, for all $x \in X$, the sequence $\{S^n x\}$ converges to u.

In 1975, Das and Gupta [**2**] proved the following theorem.

Theorem 1.2 ([**2**]). *Let (X, d) be a complete metric space and let S be a mapping from X into itself satisfying*

$$d(S(x), S(y)) \le \alpha \frac{(1 + d(x, Sx))d(y, Sy)}{1 + d(x, y)} + \beta d(x, y)$$

for all $x, y \in X$, where α, β are constants with $\alpha, \beta > 0$ and $\alpha + \beta < 1$. Then S has a unique fixed point $u \in X$. Moreover, for all $x \in X$, the sequence $\{S^n x\}$ converges to u.

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Some generalizations of Theorem 1.2 exist in literature; see [**1, 6**].

In 2013, Samet et al. [**6**] proved a fixed point theorem of Meir-Keeler type that extends the above Theorem 1.2 as follows.

Theorem 1.3 ([**5**]). *Let (X,d) be a complete metric space and let $S\colon X \to X$ be a mapping. Assume that the following condition holds. For any $\varepsilon > 0$ such that for $x \neq y$ or $Sy \neq y$,*

$$\begin{aligned} &2\varepsilon \leq \Big(\frac{(1+d(x,Sx))d(y,Sy)}{1+d(x,y)} + d(x,y)\Big) < 2\varepsilon + \delta(\varepsilon) \\ &\implies \quad d(Sx,Sy) < \varepsilon \end{aligned} \tag{1.1}$$

for all $x, y, \in X$. Then S has a unique fixed point $\xi \in X$. The sequence $\{S^n x\}$ converges to ξ.

In 1984, Gupta and Saxena [**3**] proved the following theorem.

Theorem 1.4 ([**3**]). *Let S be a continuous selfmap on X, we assume that the following condition satisfies*

$$d(S(x),S(y)) \leq \alpha_1 \frac{(1+d(x,Sx))d(y,Sy)}{1+d(x,y)} + \alpha_2 \frac{d(x,Sx)d(y,Sy)}{d(x,y)} + \alpha_3\, d(x,y)$$

for all $x, y \in X$, $x \neq y$, where $\alpha_1, \alpha_2, \alpha_3$ are constants with $\alpha_1, \alpha_2, \alpha_3 > 0$ and $\alpha_1 + \alpha_2 + \alpha_3 < 1$. Then S has a unique fixed point $u \in X$. Moreover, for all $x \in X$, the sequence $\{S^n x\}$ converges to u.

In [**5**], Radjel et al. established a fixed point theorem of Meir-Keeler type involving Gupta-Saxena expression which extends Theorem 1.4 in case when α_i, $i = 1, 2, 3$ in $(0, \frac{1}{3})$.

Theorem 1.5 ([**5**]). *Let (X,d) be a complete metric space and let $S\colon X \to X$ be a continuous mapping. Assume that the following condition holds. For any $\varepsilon > 0$ such that for $x \neq y$ or $Sy \neq y$,*

$$\begin{aligned} &3\varepsilon \leq \Big(\frac{(1+d(x,Sx))d(y,Sy)}{1+d(x,y)} + \frac{d(x,Sx)d(y,Sy)}{d(x,y)} + d(x,y)\Big) < 3\varepsilon + \delta(\varepsilon) \\ &\implies \quad d(Sx,Sy) < \varepsilon \end{aligned} \tag{1.2}$$

for all $x, y \in X$. Then S has a unique fixed point $\xi \in X$. The sequence $\{S^n x\}$ converges to ξ.

In this paper, we obtain a new fixed point theorem of Meir-Keeler type contraction via Gupta-Saxena rational expression which extends Theorem 1.3 and Theorem 1.5. Throughout this paper, (X,d) is a complete metric space denoted by X.

2. Main Result

Our main result is following.

Theorem 2.1. *Let S be a continuous selfmap on X, we assume that the following condition satisfies*

$$\varepsilon \leq \phi\Big(\max\Big\{\frac{(1+d(x,Sx))d(y,Sy)}{1+d(x,y)}, \frac{d(x,Sx)d(y,Sy)}{d(x,y)}, d(x,y)\Big\}\Big) < \varepsilon + \lambda(\varepsilon) \implies d(Sx,Sy) < \varepsilon \tag{2.1}$$

for all $x, y \in X, x \neq y$ or $y \neq Sy$, where $\phi\colon R_+ \to R_+$ is a continuous monotonic increasing mapping, $\phi(t) < t$ for all $t > 0$ and $\phi(0) = 0$. Then S has a unique fixed point $\zeta \in X$. Moreover, for all $x \in X$, the sequence $\{S^n x\}$ converges to ζ.

Proof. The equation (2.1) trivially implies that

$$x \neq y \text{ or } y \neq Sy \text{ implies}$$
$$d(Sx,Sy) < \phi\Big(\max\Big\{\frac{(1+d(x,Sx))d(y,Sy)}{1+d(x,y)}, \frac{d(x,Sx)d(y,Sy)}{d(x,y)}, d(x,y)\Big\}\Big). \tag{2.2}$$

First, let $x_0 \in X$ and consider the sequence

$$x_{n+1} = Sx_n \qquad \text{for all } n \in N. \tag{2.3}$$

Now, we have to prove that the sequence $\{x_n\}$ is a Cauchy sequence in X. Suppose that $x_{n_0} = x_{n_0+1}$ for some $n_0 \in N$, then clearly x_{n_0} is a fixed point of S. Now, assume that $x_n \neq x_{n+1}$ for all $n \in N$. Let us denote $d_n = d(x_n, x_{n+1})$ for all $n \in N$, from (2.2) and (2.3), we have

$$\begin{aligned} d_n &= d(x_n,\ x_{n+1}) = d(Sx_{n-1},\ Sx_n) \\ &< \ \phi\Big(\max\Big\{\frac{(1+d(x_{n-1},\ Sx_{n-1}))\ d(x_n,\ Sx_n)}{1+d(x_{n-1},\ x_n)}, \\ &\qquad \frac{d(x_{n-1},\ Sx_{n-1})d(x_n,\ Sx_n)}{d(x_{n-1},x_n)},\ d(x_{n-1},\ x_n)\Big\}\Big) \\ &= \phi\Big(\max\Big\{\frac{(1+d(x_{n-1},x_n))d(x_n,x_{n+1})}{1+d(x_{n-1},x_n)}, \\ &\qquad \frac{d(x_{n-1},x_n)d(x_n,x_{n+1}),}{d(x_{n-1},x_n)}, d(x_{n-1},x_n)\Big\}\Big). \end{aligned} \tag{2.4}$$

Suppose that $\max\big\{d(x_n,\ x_{n+1}),\ d(x_{n-1},\ x_n)\big\} = d(x_{n+1},\ x_n)$.
Then from (2.4) and by the property of ϕ, we get

$$d(x_n,\ x_{n+1}) < \phi(d(x_n,\ x_{n+1})) < d(x_n,\ x_{n+1}),$$

a contradiction. Thus

$$d(x_n,\ x_{n+1}) < \phi(d(x_{n-1},\ x_n)) < d(x_{n-1},\ x_n). \tag{2.5}$$

Therefore, $d_n < d_{n-1}$ for all $n \in N$.

Suppose that d_n converges to some $\varepsilon \geq 0$ and moreover, $d_n \geq \varepsilon$ for all $n \geq 0$, we have $\phi\big(\max\{d_n,\ d_{n-1}\}\big) \to \varepsilon$ as $n \to \infty$.

From (2.1), if $\varepsilon > 0$, there exists $\lambda(\varepsilon) > 0$ such that

$$\varepsilon \leq \phi(\max\{d_n,\ d_{n-1}\}) < \varepsilon + \lambda(\varepsilon) \implies d(Sx_{n-1},\ Sx_n) = d(x_n,\ x_{n+1}) = d_n < \varepsilon,$$

a contradiction. Then we obtain that $d_n \to 0$ as $n \to \infty$.

Now, let $\varepsilon > 0$, condition (2.1) remains true with $\lambda(\varepsilon)$ replaced by $\lambda^1(\varepsilon) = \min\{\lambda(\varepsilon),\ \varepsilon,\ 1\}$. By the convergence of sequence $d_n \to 0$ as $n \to \infty$, there exists $k_0 \in N$ such that

$$d(x_p,\ x_{p+1}) < \frac{\lambda^1(\varepsilon)}{2} \qquad \text{for all } p \geq k_0. \tag{2.6}$$

Now, we consider the set

$$\Xi = \big\{x_p \mid p \geq k_0,\ d(x_p,\ x_{k_0}) < \varepsilon + \lambda^1(\varepsilon)\big\}. \tag{2.7}$$

First, we prove that $S(\Xi) \subset \Xi$. Let $\alpha \in \Xi$, there exists $p \geq k_0$ such that $\alpha = x_p$ and $d(x_p,\ x_{k_0}) < \varepsilon + \lambda^1(\varepsilon)$. If $p = k_0$, we have $S(\alpha) = x_{k_0+1} \in \Xi$ by (2.6). Then if we take $p > k_0$, we get the following two cases.

Case 1:

$$\varepsilon \leq d(x_p,\ x_{k_0}) < \varepsilon + \lambda^1(\varepsilon). \tag{2.8}$$

Now using (2.2) and (2.3), we have

$$d(Sx_p, Sx_{k_0}) < \phi\Big(\max\Big\{\frac{(1 + (x_p, x_{p+1}))d(x_{k_0}, x_{k_0+1})}{1 + d(x_p, x_{k_0})}, \frac{d(x_p, x_{p+1})d(x_{k_0}, x_{k_0+1})}{d(x_p, x_{k_0})}, d(x_p, x_{k_0})\Big\}\Big).$$

Using (2.6), (2.8) in RHS term of the above equation, we get
RHS term of the above equation

$$= \phi\Big(\max\Big\{\frac{d(x_{k_0}, x_{k_0+1})}{1 + d(x_p, x_{k_0})} + \frac{d(x_p, x_{p+1})d(x_{k_0}, x_{k_0+1})}{1 + d(x_p, x_{k_0})}, \frac{d(x_p, x_{p+1})d(x_{k_0}, x_{k_0+1})}{d(x_p, x_{k_0})}, d(x_p, x_{k_0})\Big\}\Big).$$

Since

$$\begin{aligned}\frac{d(x_{k_0},\ x_{k_0+1})}{1 + d(x_p,\ x_{k_0})} &\leq d(x_{k_0},\ x_{k_0+1}) < \frac{\lambda^1(\varepsilon)}{2} < 1\\ &< \phi\Big(\max\Big\{\frac{\lambda^1(\varepsilon)}{2} + \frac{\lambda^1(\varepsilon)}{2}, \frac{\lambda^1(\varepsilon)}{2}, \varepsilon + \lambda^1(\varepsilon)\Big\}\Big)\\ &< \phi\Big(\max\Big\{\lambda^1(\varepsilon), \frac{\lambda^1(\varepsilon)}{2}, \varepsilon + \lambda^1(\varepsilon)\Big\}\Big) < \varepsilon + \lambda^1(\varepsilon),\end{aligned}$$

by (2.1), we have

$$d(Sx_p,\ Sx_{k_0}) < \varepsilon. \tag{2.9}$$

Using triangular inequality, and (2.6), (2.9), we have

$$d(Sx_p,\ x_{k_0}) \le d(Sx_p,\ Sx_{k_0}) + d(Sx_{k_0},\ x_{k_0}) < \varepsilon + \frac{\lambda^1(\varepsilon)}{2} < \varepsilon + \lambda^1(\varepsilon). \tag{2.10}$$

It follows that $S(\Xi) \subset \Xi$.

Case 2:

$$d(x_p,\ x_{k_0}) < \varepsilon \tag{2.11}$$

From (2.2) and (2.3), we get

(2.12)

$$\begin{aligned} d(Sx_p,\ x_{k_0}) &\le d(Sx_p,\ Sx_{k_0}) + d(Sx_{k_0},\ x_{k_0}) \\ &< \phi\Big(\max\Big\{\frac{(1+d(x_p,x_{p+1}))d(x_{k_0},\ x_{k_0+1})}{1+d(x_p,\ x_{k_0})}, \\ &\qquad \frac{d(x_p,x_{p+1})\ d(x_{k_0},\ x_{k_0+1})}{d(x_p,\ x_{k_0})},\ d(x_p,\ x_{k_0})\Big\}\Big) + d(x_{k_0+1},\ x_{k_0}), \end{aligned}$$

since $\frac{d(x_{k_0},\ x_{k_0+1})}{1+d(x_p,\ x_{k_0})} \le d(x_{k_0},\ x_{k_0+1}) < \frac{\lambda^1(\varepsilon)}{2} < 1$.

If $d(x_{k_0},\ x_{k_0+1}) \le d(x_p,\ x_{k_0})$, then (2.12) becomes

$$\begin{aligned} d(Sx_p,\ x_{k_0}) &< \phi\Big(\max\Big\{d(x_{k_0},\ x_{k_0+1}) + \frac{d(x_p,\ x_{p+1})\ d(x_{k_0},\ x_{k_0+1})}{1+d(x_p,\ x_{k_0})} \\ &\qquad \frac{d(x_p,\ x_{p+1})d(x_{k_0},\ x_{k_0+1})}{d(x_p,\ x_{k_0})},\ d(x_p,\ x_{k_0})\Big\}\Big) + d(x_{k_0+1},\ x_{k_0}) \\ &< \phi\Big(\max\Big\{\frac{\lambda^1(\varepsilon)}{2} + \frac{\lambda^1(\varepsilon)}{2}, \frac{\lambda^1(\varepsilon)}{2}, \varepsilon\Big\}\Big) + \frac{\lambda^1(\varepsilon)}{2} < \varepsilon + \frac{\lambda^1(\varepsilon)}{2}. \end{aligned}$$

If $d(x_{k_0},\ x_{k_0+1}) > d(x_p,\ x_{k_0})$, then

$$\begin{aligned} d(Sx_p, x_{k_0}) &\le d(Sx_p, x_p) + d(x_p, x_{k_0}) \\ &= d(x_{p+1}, x_p) + d(x_p, x_{k_0}) \\ &< d(x_{p+1}, x_p) + d(x_{k_0}, x_{k_0+1}) \\ &< \frac{\lambda^1(\varepsilon)}{2} + \frac{\lambda^1(\varepsilon)}{2} = \lambda^1(\varepsilon). \end{aligned}$$

In both Cases, we have $S(\alpha) = S(x_p) = x_{p+1} \in \Xi$. Thus $d(x_m, x_n) < \varepsilon + \lambda^1(\varepsilon)$ for all $m > k_0$.

If $m, n \in N$ satisfying $m > n > k_0$, then

$$d(x_m, x_n) \le d(x_m, x_{k_0}) + d(x_n, x_{k_0}) < 2\varepsilon + \lambda^1(\varepsilon) = 3\varepsilon'.$$

This implies that the sequence $\{x_n\}$ is a Cauchy in X. Since X is complete, then it follows that there exists $\zeta \in X$ such that x_n converges to ζ as $n \to \infty$. In fact, from (2.3) and continuity of S, we get $S\zeta = \zeta$. Thus ζ is a fixed point of S.

Suppose now that η is another fixed point of S. Using our inequality, we get

$$\begin{aligned} d(\eta,\zeta) = d(S\eta, S\zeta) &< \phi\Big(\max\Big\{ \frac{[1+d(\eta,S\eta)]d(\zeta,S\zeta)}{1+d(\eta,\zeta)},\ \frac{d(\eta,S\eta)d(\zeta,S\zeta)}{d(\eta,\zeta)}, d(\eta,\zeta)\Big\}\Big) \\ &= \phi\big(\max\{0,0,(d(\eta,\zeta\}\big) \\ &< \phi(d(\eta,\zeta) < d(\eta,\zeta), \end{aligned}$$

a contradiction. This proves the uniqueness of the fixed point and completes the proof of the theorem. □

Corollary 2.1. *Let S be a continuous selfmap on X, we assume that the following condition satisfies*

$$d(Sx,Sy) \le \phi\Big(\frac{(1+d(x,Sx))d(y,Sy)}{1+d(x,y)} + \frac{d(x,Sx)d(y,Sy)}{d(x,y)} + d(x,y)\Big)$$

for all $x, y \in X$, $x \neq y$ or $y \neq Sy$, where $\phi\colon R_+ \to R_+$ is a continuous monotonic increasing mapping, $\phi(t) < t$ for all $t > 0$ and $\phi(0) = 0$. Then S has a unique fixed point $\zeta \in X$. Moreover, for all $x \in X$, the sequence $\{S^n x\}$ converges to ζ.

The following corollary is a corollary of Corollary 2.1 and Theorem 1.4.

Corollary 2.2 ([**5**])**.** *Let S be a continuous selfmap on X, we assume that for $x \neq y$ or $S(y) \neq y$, S satisfies*

$$d(S(x),S(y)) \le k\Big(\frac{(1+d(x,Sx))d(y,Sy)}{1+d(x,y)} + \frac{d(x,Sx)d(y,Sy)}{d(x,y)} + d(x,y)\Big),$$

where $k \in]0, \frac{1}{3}[$ is a constant. Then S has a unique fixed point $u \in X$. Moreover, for all $x \in X$, the sequence $\{S^n(x)\}$ converges to u.

Proof. We get it by taking $\phi(t) = k\,t$ in Corollary 2.1. □

Example 2.2. Let $X = [0,1]$ and define a mapping S from X into itself. We define a function $Sx = \frac{1+x}{4}$, where $x \in X$, and $\phi\colon R_+ \to R_+$ by $\phi(t) = \frac{t}{\sqrt{2}}$ for all $t \in R_+$. The function ϕ is continuous monotonic increasing, $\phi(t) < t$, for all $t > 0$, $\phi(0) = 0$. Now, we verify the condition (2.2). Assume that if $x < y$, then

$$\Big|\frac{x-y}{4}\Big| = d(Sx,Sy) < \phi(d(x,y)) = \Big|\frac{x-y}{\sqrt{2}}\Big|.$$

It leads to the result that condition (2.2) holds and $\frac{1}{3}$ is a unique fixed point of S.

In particular, at $x = 1$ and $y = 0$, Corollary 2.2 does not hold.

Corollary 2.3. *Let S be a continuous selfmap on X, we assume that the following condition satisfies*

$$d(Sx,Sy) \le \phi\Big(\frac{(1+d(x,Sx))d(y,Sy)}{1+d(x,y)} + d(x,y)\Big)$$

for all $x, y \in X$, $x \neq y$ or $y \neq Sy$, where $\phi\colon R_+ \to R_+$ is a continuous monotonic increasing mapping, $\phi(t) < t$ for all $t > 0$ and $\phi(0) = 0$. Then S has a unique fixed point $\zeta \in X$. Moreover, for all $x \in X$, the sequence $\{S^n x\}$ converges to ζ.

Corollary 2.4. *Let S be a continuous selfmap on X, we assume that the following condition satisfies*

$$d(Sx, Sy) \leq \phi\Big(\max\Big\{\frac{(1 + d(x, Sx))d(y, Sy)}{1 + d(x, y)}, d(x, y)\Big\}\Big)$$

for all $x, y \in X$, $x \neq y$ or $y \neq Sy$, where $\phi\colon R_+ \to R_+$ is a continuous monotonic increasing mapping, $\phi(t) < t$ for all $t > 0$ and $\phi(0) = 0$. Then S has a unique fixed point $\zeta \in X$. Moreover, for all $x \in X$, the sequence $\{S^n x\}$ converges to ζ.

The following corollary is a corollary of Corollary 2.4.

Corollary 2.5. *Let S be a continuous selfmap on X, we assume that for $x \neq y$ or $S(y) \neq y$, S satisfies*

$$d(S(x), S(y)) \leq k\Big(\frac{(1 + d(x, Sx))d(y, Sy)}{1 + d(x, y)} + d(x, y)\Big),$$

where $k \in]0, \frac{1}{2}[$ is a constant. Then S has a unique fixed point $\zeta \in X$. Moreover, for all $x \in X$, the sequence $\{S^n(x)\}$ converges to ζ.

Proof. By taking $\phi(t) = k\, t$ in Corollary 2.3. □

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References

1. Chatterjee H., *On generalization of Banach contraction principle*, Indian J. Pure Appl. Math. **10**(4) (1979), 400–403.
2. Dass B. K. and Gupta S., *An extension of Banach contraction principle through rational expression*, Indian J. Pure Appl. Math. **6** (1975), 1455–1458.
3. Gupta A. N. and Saxena A., *A unique fixed point in metric spaces*, Math. Stud. **52** (1984), 156–158.
4. Meir A. and Keeler E., *A theorem on contraction mappings*, J. Math. Anal. Appl. **28** (1969), 326–329.
5. Redjel N., Dehici A. and Erhan I. M., *A fixed point theorem for Meir-Keeler type contraction via Gupta-Saxena expression*, Fixed Point Theory Appl. **2015** (2015), #115.
6. Samet B., Vetro C. and Yazidi H., *A fixed point theorem for a Meir-Keeler type contraction through rational expression*, J. Nonlinear Sci. Appl. **6** (2013), 162–169.

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