HARMONIC AND BIHARMONIC MAPS BETWEEN TANGENT BUNDLES

L. BELARBI and H. EL HENDI

Abstract. In this paper, we investigate the harmonicity and biharmonicity of a tangent map \( \phi: (TM, g^s) \to (TN, h^s) \) in the case where the tangent bundles \( TM, TN \) are endowed with Sasaki Riemannian metrics \( g^s, h^s \).

1. Introduction

Let \( \varphi: M \to N \) be a smooth map between the smooth manifolds \( M, N \). The map \( \varphi \) induces the tangent map \( \phi = d\varphi: TM \to TN \) between the tangent bundles of \( M \) and \( N \). In the case where \( M, N \) are Riemannian manifolds, their tangent bundles \( TM, TN \) may be endowed with the corresponding Sasaki metrics so that they become Riemannian manifolds. The motivation of this paper is to study harmonicity and biharmonicity of the tangent map \( \phi: (TM, g^s) \to (TN, h^s) \).

In this paper, we deal with these problems in the case where the tangent bundles \( TM, TN \) are endowed with Sasaki Riemannian metrics \( g^s, h^s \). We show that the map \( \phi \) is harmonic if \( \varphi \) is totally geodesic. Further, if \( TM \) is a compact tangent bundle, \( \phi \) is harmonic if and only if \( \varphi \) is totally geodesic.

In the biharmonicity, we show that if \( TM \) is a compact tangent bundle and \( \phi: (TM, g^s) \to (TN, h^s) \) the tangent map of harmonic map \( \varphi: (M, g) \to (N, h) \), then \( \phi \) is biharmonic if and only if \( \phi \) is harmonic. Furthermore, we calculate the bitension field of tangent map \( \phi \).

1.1. Harmonic maps

Consider a smooth map \( \phi: (M^n, g) \to (N^n, h) \) between two Riemannian manifolds, then the energy functional is defined by

\[
E(\phi) = \frac{1}{2} \int_M |d\phi|^2 dv_g
\]

(or over any compact subset \( K \subset M \)).

A map is called harmonic if it is a critical point of the energy functional \( E \) (or \( E(K) \) for all compact subsets \( K \subset M \)). For any smooth variation \( \{\phi_t\}_{t \in I} \) of \( \phi \)...
with $\phi_0 = \phi$ and $V = \frac{d\phi_t}{dt}|_{t=0}$, we have

$$\frac{d}{dt}E(\phi_t)|_{t=0} = -\int_M h(\tau(\phi), V)dv_g,$$

where

$$\tau(\phi) = \text{tr}_g \nabla d\phi$$

is the tension field of $\phi$. Then we have the following theorem.

**Theorem 1.1.** A smooth map $\phi: (M^m, g) \rightarrow (N^n, h)$ is harmonic if and only if

$$\tau(\phi) = 0.$$

If $(x^i)_{1 \leq i \leq m}$ and $(y^\alpha)_{1 \leq \alpha \leq n}$ denote local coordinates on $M$ and $N$, respectively, then equation 4 takes the form

$$\tau(\phi)^\alpha = \left(\Delta \phi^\alpha + g^{ij} \Gamma^N_{\beta \gamma} \frac{\partial \phi^\beta}{\partial x^i} \frac{\partial \phi^\gamma}{\partial x^j} \right) = 0,$$

where $\Delta \phi^\alpha = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} (\sqrt{|g|} g^{ij} \frac{\partial \phi^\alpha}{\partial x^j})$ is the Laplace operator on $(M^m, g)$ and $\Gamma^N_{\beta \gamma}$ are the Christoffel symbols on $N$.

### 1.2. Biharmonic maps

**Definition 1.1.** A map $\varphi: (M, g) \rightarrow (N, h)$ between Riemannian manifolds is called biharmonic if it is a critical point of bienergy functional

$$E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 dv^g,$$

where

$$\frac{d}{dt}E_2(\varphi_t)|_{t=0} = -\int_M h(\tau_2(\varphi), V)dv_g.$$

The Euler-Lagrange equation attached to bienergy is given by the vanishing of the bitension field

$$\tau_2(\varphi) = -J_\varphi(\tau(\varphi)) = -(\Delta \tau(\varphi) + \text{tr}_g R^N(\tau(\varphi), d\varphi)d\varphi),$$

where $J_\varphi$ is the Jacobi operator defined by

$$J_\varphi: \Gamma(\varphi^{-1}(TN)) \rightarrow \Gamma(\varphi^{-1}(TN)),$$

$$V \mapsto \Delta V + \text{tr}_g R^N(V, d\varphi)d\varphi.$$
2. Basic notions and definition on $TM$

Let $(M, g)$ be an $n$-dimensional Riemannian manifold and $(TM, \pi, M)$ be its tangent bundle.

A local chart $(U, x^i)_{i=1\ldots n}$ on $M$ induces a local chart $(\pi^{-1}(U), x^i, y^j)_{i=1\ldots n}$ on $TM$. By $\Gamma^k_{ij}$, denote the Christoffel symbols of $g$ and by $\nabla$, the Levi-Civita connection of $g$.

We have two complementary distributions on $TM$, the vertical distribution $V$ and the horizontal distribution $H$, defined by

$$V_{(x,u)} = \ker(d\pi_{(x,u)}) = \left\{ a^i \frac{\partial}{\partial y^i} \mid (x,u); a^i \in \mathbb{R} \right\},$$

$$H_{(x,u)} = \left\{ \frac{\partial}{\partial x^i} \mid (x,u) - a^i u^j \Gamma^k_{ij} \frac{\partial}{\partial y^k} 
\mid (x,u); a^i \in \mathbb{R} \right\},$$

where $(x,u) \in TM$ such that $T_{(x,u)}TM = H_{(x,u)} \oplus V_{(x,u)}$.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on $M$. The vertical and the horizontal lifts of $X$ are defined by

$$X^V = X^i \frac{\partial}{\partial y^i},$$

$$X^H = X^i \frac{\delta}{\delta x^i} = X^i \left\{ \frac{\partial}{\partial x^i} - g^{ij} \Gamma^k_{ij} \frac{\partial}{\partial y^k} \right\}.$$

For consequences, we have $(\frac{\delta}{\delta x^i})^H = \frac{\delta}{\delta x^i}$ and $(\frac{\delta}{\delta x^i})^V = \frac{\partial}{\partial y^i}$, then $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})_{i=1\ldots n}$ is a local adapted frame in $TTM$.

**Definition 2.1.** The Sasaki metric $g^s$ on the tangent bundle $TM$ of $M$ is given by

1. $g^s(X^H, Y^H) = g(X,Y) \circ \pi$,
2. $g^s(X^H, Y^V) = 0$,
3. $g^s(X^V, Y^V) = g(X,Y) \circ \pi$

for all vector fields $X, Y \in \Gamma(TM)$.

**Proposition 2.1 ([10]).** Let $(M, g)$ be a Riemannian manifold and $\nabla$ be the Levi-Civita connection of the tangent bundle $(TM, g^s)$ equipped with the Sasaki metric. Then

$$\left( \nabla_{X^V} Y^H \right)_{(x,u)} = (\nabla_X Y)^H_{(x,u)} - \frac{1}{2}(R_x(X,Y)u)^V,$$

$$\left( \nabla_{X^V} Y^V \right)_{(x,u)} = (\nabla_X Y)^V_{(x,u)} + \frac{1}{2}(R_x(u,Y)X)^H,$$

$$\left( \nabla_{X^H} Y^H \right)_{(x,u)} = \frac{1}{2}(R_x(u,X)Y)^H,$$

$$\left( \nabla_{X^H} Y^V \right)_{(x,u)} = 0$$

for all vector fields $X, Y \in \Gamma(TM)$ and $(x,u) \in TM$. 

Definition 2.2. Let \((M,g)\) be a Riemannian manifold and \(F \in \mathfrak{X}^1(M)\) be a tensor of type \((1,1)\). Then we define a vertical and horizontal vector fields \(VF, HF\) on \(TM\) by

\[
VF: TM \to TTM, \quad (x,u) \mapsto (F(u))^V,
\]

\[
HF: TM \to TTM, \quad (x,u) \mapsto (F(u))^H.
\]

Proposition 2.2 ([9]). Let \((M,g)\) be a Riemannian manifold and \(\hat{\nabla}\) be the Levi-Civita connection of the tangent bundle \((TM, g^*)\) equipped with the Sasaki metric. If \(F \in \mathfrak{X}^1(M)\) is a tensor of type \((1,1)\), then

\[
(\hat{\nabla}_X HF)(x,u) = H(\nabla_X F)(x,u) - \frac{1}{2}(R_x(X_x,F_x(u))u)^V,
\]

\[
(\hat{\nabla}_X VF)(x,u) = V(\nabla_X F)(x,u) + \frac{1}{2}(R_x(u,F_x(u))X_x)^H.
\]

Proposition 2.3 ([9]). Let \((M,g)\) be a Riemannian manifold and \(\hat{\mathcal{R}}\) be the Riemann curvature tensor of the tangent bundle \((TM, g^*)\) equipped with the Sasaki metric. The following formulae hold:

1. \(\hat{\mathcal{R}}(X^V,Y^V)Z^V = 0\),
2. \(\hat{\mathcal{R}}(X^V,Y^V)Z^H = \left[R(X,Y)Z + \frac{1}{4}R(u,X)(R(u,Y)Z) - \frac{1}{4}5u,Y)(R(u,X)Z\right]^H_x\),
3. \(\hat{\mathcal{R}}(X^H,Y^V)Z^V = -\left[\frac{1}{2}R(Y,Z)X + \frac{1}{4}R(u,Y)(R(u,Z)X)^H_x\right],
4. \(\hat{\mathcal{R}}(X^H,Y^V)Z^H = \left[\frac{1}{4}R(R(u,Y)Z,X)u + \frac{1}{2}R(X,Z)Y\right]^V_x + \frac{1}{2}\left[(\nabla_X R)(u,Y)Z\right]^H_x,
5. \(\hat{\mathcal{R}}(X^H,Y^H)Z^V = \left[R(X,Y)Z + \frac{1}{4}R(R(u,Z)X,Y)u - \frac{1}{4}R(R(u,Z)X,Y)u\right]^V_x
+ \frac{1}{2}\left[(\nabla_X R)(u,Z)Y - (\nabla_Y R)(u,Z)X\right]^H_x,
6. \(\hat{\mathcal{R}}(X^H,Y^H)Z^H = \frac{1}{2}\left[(\nabla_Z R)(X,Y)u\right]^V_x + \left[R(X,Y)Z + \frac{1}{4}R(u,R(Z,Y)u)X + \frac{1}{4}R(u,R(X,Z)u)Y
+ \frac{1}{2}R(u,R(X,Y)u)^Z^H\right]
\]

for all vectors \(u, X, Y, Z \in T_x M\).
3. Harmonic maps between two tangent bundles

**Lemma 3.1** ([17]). Let \( \varphi: (M^m, g) \rightarrow (N^n, h) \) be a smooth map between Riemannian manifolds. The map \( \varphi \) induces the tangent map

\[
\phi = d\varphi: (TM^m, g^s) \rightarrow (TN^n, h^s), \quad (x, y) \mapsto (\varphi(x), d\varphi(y)).
\]

For all vector field \( X \in \Gamma(TM) \), we have

\[
d\phi((X)^V) = (d\varphi(X))^V, \quad d\phi((X)^H) = (d\varphi(X))^H + (\nabla d\varphi(y, X))^V.
\]

**Theorem 3.1** ([17]). Let \( \phi: (TM^m, g^s) \rightarrow (TN^n, h^s) \) be the tangent map of \( \varphi: (M^m, g) \rightarrow (N^n, h) \), then the tension field associated with \( \phi \) is given by

\[
\tau(\phi) = (\tau(\varphi) + \text{tr}_h R^N (d\varphi(u), \nabla d\varphi(u, *))d\varphi(*))^H + (\text{div}(\nabla d\varphi)(u))^V.
\]

**Theorem 3.2.** Let \( \phi: (TM^m, g^s) \rightarrow (TN^n, h^s) \) be the tangent map of \( \varphi: (M^m, g) \rightarrow (N^n, h) \), then \( \phi \) is harmonic if and only if the following conditions are satisfied:

\[
\tau(\varphi) = 0, \quad \text{tr}_g R^N (d\varphi(u), \nabla d\varphi(u, *))d\varphi(*) = 0, \quad \text{div}(\nabla d\varphi) = 0.
\]

**Corollary 3.1.** Let \( \phi: (TM^m, g^s) \rightarrow (TN^n, h^s) \) be the tangent map of \( \varphi: (M^m, g) \rightarrow (N^n, h) \), if \( \varphi \) is totally geodesic, then \( \phi \) is harmonic.

**Lemma 3.2.** Let \( \phi: (TM^m, g^s) \rightarrow (TN^n, h^s) \) be the tangent map of \( \varphi: (M^m, g) \rightarrow (N^n, h) \), then the energy density associated \( \phi \) is given by

\[
e(\phi) = 2e(\varphi) + \frac{1}{2} \text{tr}_h |\nabla d\varphi(\cdot, u)|^2.
\]

**Proof.** Let \( (x, u) \in TM, (\varphi(x), d\varphi(u)) \in TN, \) and \( \{e^H_i, e^V_i\}_{i=1,..., m} \) be a local orthonormal frame on \( TM \), then

\[
2e(\phi)_{(\varphi(x), d\varphi(u))} = \sum_{i=1}^m (h^s_{(\varphi(x), d\varphi(u))}(d\phi(e^H_i), d\phi(e^H_i)) + h^s_{(\varphi(x), d\varphi(u))}(d\phi(e^V_i), d\phi(e^V_i))).
\]

Using Lemma 3.1, we obtain

\[
2e(\phi)_{(\varphi(x), d\varphi(u))} = \sum_{i=1}^m \left( h^s_{(\varphi(x), d\varphi(u))}((d\varphi(e_i))^H, (d\varphi(e_i))^H) + (d\varphi(e_i))^V, (d\varphi(e_i))^V) + (d\varphi(e_i))^V, (d\varphi(e_i))^V)\right)
\]

\[= 4e(\varphi) + \text{tr}_h |\nabla d\varphi(u, *)|^2. \]

\( \square \)
Theorem 3.3. Let $TM$ be a compact tangent bundle and $\phi: (TM^m, g^*) \to (TN^n, h^*)$ be its the tangent map of $\varphi: (M^m, g) \to (N^n, h)$. Then $\phi$ is a harmonic if and only if $\varphi$ is totally geodesic.

Proof. If $\varphi$ is totally geodesic, from Corollary 3.1, we deduce that $\phi$ is harmonic.

Inversely. Let $\omega: I \times M \to N$ be a smooth map satisfying

$$\omega(t, x) = \varphi_t(x) = (1 + t)\varphi(x)$$

for all $t \in I = (-\epsilon, \epsilon), \epsilon > 0$ and all $x \in M$.

The variation vector field $v \in \Gamma(\varphi^{-1}TN)$ associated the variation $\{\varphi_t\}_{t \in I}$ is given for all $x \in M$, by

$$v(x) = d(0, x)\omega\left(\frac{d}{dt}\right).$$

From Lemma 3.2, we have

$$\tau^h(\varphi_t) = (1 + t)^3\tau^h(\varphi),$$

$$\tau^v(\varphi_t) = (1 + t)^3\tau^v(\varphi).$$

Equation 2, we have

$$\frac{d}{dt}E(\varphi_t)_{t=0} = -2\int_{TM} h(v, \tau(\varphi))dv_{gh} + \int_{TM} |\nabla d\varphi(\ast, u)|^2 dv_{gh}$$

If $\phi$ is harmonic, hence $\tau(\varphi) = 0$, then $\nabla d\varphi = 0$. $\Box$

4. Biharmonic maps between two tangent bundle

In the section, we denote

(14) \[ \tau^h(\phi) = \tau(\varphi) + \text{tr}_h R^N(d\varphi(u), \nabla d\varphi(u, \ast))d\varphi(\ast) \]

(15) \[ \tau^v(\phi) = \text{div}(\nabla d\varphi)(u). \]

Theorem 4.1. Let $TM$ be a compact tangent bundle and $\phi: (TM^m, g^*) \to (TN^n, h^*)$ the tangent map of harmonic map $\varphi: (M, g) \to (N, h)$, then $\phi$ is biharmonic if and only if $\phi$ is harmonic.

Proof. Let $\varphi_t$ be a compactly supported variation of $\varphi$ defined by $\varphi_t(x) = (1 + t)\varphi(x)$, and $\varphi_0(x) = \varphi(x)$.

From the formulas (14) and (15), we have

$$\tau^h(\phi_t) = (1 + t)^3\tau^h(\phi),$$

$$\tau^v(\phi_t) = (1 + t)^3\tau^v(\phi),$$

$$E_2(\phi_t) = \frac{1}{2} \int |\tau^h(\phi_t)|^2_{h^*} dv_{h^*}$$

$$= \frac{1}{2} \int |\tau^h(\phi_t)|^2_{h} dv_{h} + \frac{1}{2} \int |\tau^v(\phi_t)|^2_{h} dv_{h}$$

$$= (1 + t)^6 \frac{1}{2} \int |\tau^h(\phi)|^2_{h} dv_{h} + (1 + t)^2 \frac{1}{2} \int |\tau^v(\phi)|^2_{h} dv_{h}.$$
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Since the section $\phi$ is biharmonic, then for variation $\phi_t$, we have

$$0 = \frac{d}{dt} E_2(\phi_t)|_{t=0} = 3 \int |\tau^h(\phi)|_h^2 dv + \int |\tau^v(\phi)|_v^2 dv$$

Hence $\tau^h(\phi) = 0$ and $\tau^v(\phi) = 0$, then $\tau(\phi) = 0$. 

**Lemma 4.1.** Let $\phi: (TM^m, g^*) \to (TN^n, h^*)$ be a the tangent map of $\varphi: (M^m, g) \to (N^n, h)$, then the Jacobi tensor $J_{\phi}(\tau^v(\phi)V)$ is given by

$$J_{\phi}(\tau^v(\phi)V)_{(\varphi(x), d\varphi(u))} = \left\{ \text{tr}_h (\nabla^2 \tau^v(\phi)\nabla^2 \tau^v(\phi))_{(\varphi(x), d\varphi(u))} + \frac{1}{2} R(d\varphi(u), \tau^v(\phi)) d\varphi(u) \right\}^H_{(\varphi(x), d\varphi(u))}$$

for all $(\varphi(x), d\varphi(u)) \in TN$.

**Proof.** Let $(\varphi(x), d\varphi(u)) \in TN$, and $\{e_i^H, e_i^V\}_{i=1}^m$ be a local orthonormal frame on $TM$ such that $(\nabla e_i^H)_x = 0$, denoted by $F_i = \frac{1}{2} R(\cdot, \tau^v(\phi)) d\varphi(e_i)$, we have

$$\left( \nabla^H_{e_i^H} \tau^v(\phi) + \nabla^V_{e_i^H} \tau^v(\phi) \right)_{(\varphi(x), d\varphi(u))} = \left( \begin{array}{c} \nabla^H_{d\varphi(e_i)} \tau^v(\phi) + (\nabla d\varphi(u, e_i)) \nabla^H \tau^v(\phi) \\ \nabla^V_{d\varphi(e_i)} \tau^v(\phi) \end{array} \right)_{(\varphi(x), d\varphi(u))} = \left( \begin{array}{c} \nabla d\varphi(e_i) \tau^v(\phi) \\ H F_i \end{array} \right)_{(\varphi(x), d\varphi(u))}$$

Then

$$\text{tr}_g \nabla^2 (\tau^v(\phi)V)_{(\varphi(x), d\varphi(u))} = \sum_{i=1}^m \left( \nabla^H_{e_i^H} \tau^v(\phi) \right)_{(\varphi(x), d\varphi(u))} + \sum_{i=1}^m \left( \nabla^V_{e_i^H} \tau^v(\phi) \right)_{(\varphi(x), d\varphi(u))} = \sum_{i=1}^m \left( \nabla^H_{d\varphi(e_i)} \tau^v(\phi) + (\nabla d\varphi(u, e_i)) \nabla^H \tau^v(\phi) \right)_{(\varphi(x), d\varphi(u))} + \sum_{i=1}^m \left( \nabla d\varphi(e_i) \tau^v(\phi) \right)_{(\varphi(x), d\varphi(u))} = \sum_{i=1}^m \left( \nabla^H_{d\varphi(e_i)} \tau^v(\phi) \right)_{(\varphi(x), d\varphi(u))} + \sum_{i=1}^m \left( \nabla^V_{d\varphi(e_i)} \tau^v(\phi) \right)_{(\varphi(x), d\varphi(u))} + \sum_{i=1}^m \left( \nabla d\varphi(e_i) \tau^v(\phi) \right)_{(\varphi(x), d\varphi(u))} = \sum_{i=1}^m \left( \nabla^H_{d\varphi(e_i)} \tau^v(\phi) \right)_{(\varphi(x), d\varphi(u))} + \sum_{i=1}^m \left( \nabla^V_{d\varphi(e_i)} \tau^v(\phi) \right)_{(\varphi(x), d\varphi(u))} + \sum_{i=1}^m \left( \nabla d\varphi(e_i) \tau^v(\phi) \right)_{(\varphi(x), d\varphi(u))}.$$
Using Proposition 2.2, we obtain
\[\begin{align*}
\text{tr}_g \nabla^2 (\tau^v(\phi))^{V}_{(\varphi(x),d\varphi(u))} &= \sum_{i=1}^{m} \left\{ (\nabla_{d\varphi(e_i)} \nabla_{d\varphi(e_i)} \tau^v(\phi))(d\varphi(x),d\varphi(u)) \\
&\quad - \frac{1}{4} R(d\varphi(e_i), R(d\varphi(u), \tau^v(\phi))d\varphi(e_i))d\varphi(u) \right\}^V \\
&+ \sum_{i=1}^{m} \left\{ \frac{1}{2} R(d\varphi(u), \nabla_{d\varphi(e_i)} \tau^v(\phi))d\varphi(e_i) + \frac{1}{2} (\nabla_{d\varphi(e_i)}) R(d\varphi(u), \tau^v(\phi))d\varphi(e_i) \\
&\quad + \frac{1}{4} R(d\varphi(u), \nabla d\varphi(u, e_i)) R(d\varphi(u), \tau^v(\phi))d\varphi(e_i) \\
&\quad + \frac{1}{2} R(\nabla_{d\varphi(u, e_i)}, \tau^v(\phi))d\varphi(e_i) \right\}^H \\
&\quad + \sum_{i=1}^{m} \left\{ (\nabla_{d\varphi(u, e_i)} \tau^v(\phi))d\varphi(e_i) \right\}^H \\
&= \sum_{i=1}^{m} \left\{ R^{TN}(\tau^v(\phi))^V, (d\varphi(e_i))^H + R^{TN}((\tau^v(\phi))^V, d\varphi(e_i)^V)d\varphi(e_i)^V \right\} \\
&= \sum_{i=1}^{m} \left\{ R^{TN}((\tau^v(\phi))^V, (d\varphi(e_i))^H)(d\varphi(e_i))^H \\
&\quad + R^{TN}(\tau^v(\phi))^V, (d\varphi(e_i))^H)(\nabla d\varphi(u, e_i))^V \\
&\quad + R^{TN}((\tau^v(\phi))^V, (\nabla d\varphi(u, e_i))^V)(d\varphi(e_i))^H \right\}.
\end{align*}\]

By calculating at \((\varphi(x),d\varphi(u))\), we obtain
\[\begin{align*}
\text{tr}_g \left\{ R^{TN}(\tau^v(\phi))^V, d\varphi \right\} &= \sum_{i=1}^{m} \left\{ - \frac{1}{4} R(d\varphi(u), \tau^v(\phi))d\varphi(e_i), d\varphi(e_i))d\varphi(u) \\
&\quad + \frac{1}{2} R(d\varphi(e_i), d\varphi(e_i))\tau^v(\phi) \right\}^V \\
&\quad + \sum_{i=1}^{m} \left\{ R(\tau^v(\phi), \nabla d\varphi(u, e_i))d\varphi(e_i) \\
&\quad + \frac{1}{4} R(d\varphi(u), \tau^v(\phi))R(d\varphi(u), \nabla d\varphi(u, e_i))d\varphi(e_i) \\
&\quad - \frac{1}{4} R(d\varphi(u), \nabla d\varphi(u, e_i))) R(d\varphi(u), \tau^v(\phi))d\varphi(e_i)) \\
&\quad + \frac{1}{2} R(\tau^v(\phi)), \nabla d\varphi(u, e_i))d\varphi(e_i) \\
&\quad + \frac{1}{4} R(d\varphi(u), \tau^v(\phi))R(d\varphi(u), \nabla d\varphi(u, e_i))d\varphi(e_i) \\
&\quad - \frac{1}{2} (\nabla d\varphi(e_i)) R(d\varphi(u), \tau^v(\phi))d\varphi(e_i) \right\}^H.
\end{align*}\]
Considering the formula (9), we deduce
\[ J_{\phi}((\tau^v(\phi))^V)_{(\varphi(x),d\varphi(u))} \]
\[ = \left\{ \begin{array}{c}
\text{tr}_h(\nabla^v)^2\tau^v(\phi) \\
\left( R(d\varphi(u),\nabla^x\tau^v(\phi))d\varphi(e_i) \\
+ R(\tau^v(\phi)\nabla d\varphi(u,e_i))d\varphi(e_i) \\
+ \frac{1}{2} R(d\varphi(u),\tau^v(\phi))R(d\varphi(u),\nabla d\varphi(u,e_i))d\varphi(e_i) \right) V \\
\end{array} \right\}^H_{(\varphi(x),d\varphi(u))}. \]

From the following equality
\[ \nabla_{d\varphi(e_i)}R(d\varphi(u),\tau^v(\phi))d\varphi(e_i) \]
\[ = (\nabla_{d\varphi(e_i)}R)(d\varphi(u),\tau^v(\phi))d\varphi(e_i) + R(\nabla_{d\varphi(e_i)}d\varphi(u),\tau^v(\phi))d\varphi(e_i) \]
\[ + R(d\varphi(u),\nabla_{d\varphi(e_i)}\tau^v(\phi))d\varphi(e_i), \]
the proof of Lemma 4.1 is completed. \( \square \)

**Lemma 4.2.** Let \( \phi: (TM^m,g^*) \to (TN^n,h^*) \) be a the tangent map of \( \varphi: (M^m,g) \to (N^n,h) \), then the Jacobi tensor \( J_{\phi}((\tau^h(\phi))^H) \) is given by
\[ J_{\phi}((\tau^h(\phi))^H)_{(\varphi(x),d\varphi(u))} \]
\[ = \text{tr}_h \left\{ \begin{array}{c}
2 R(\tau^h(\phi),d\varphi(*))\nabla d\varphi(u,*) \\
+ \frac{1}{2} R(\nabla_{d\varphi(u)}d\varphi(*))d\varphi(*),\tau^h(\phi))d\varphi(u) \\
- R(d\varphi(*),\nabla^x\tau^h(\phi))d\varphi(u) \end{array} \right\}^V_{(\varphi(x),d\varphi(u))} \]
\[ + \text{tr}_h \left\{ \begin{array}{c}
\nabla^x\nabla^x\tau^h(\phi) + R(\nabla d\varphi(u),\nabla d\varphi(u,*)\nabla^x\nabla^x\tau^h(\phi) \\
+ \frac{1}{2} R(d\varphi(u),\nabla^x\nabla d\varphi(u,*)\tau^h(\phi) \\
+ R(d\varphi(u),R(\tau^h(\phi),d\varphi(*))d\varphi(*) \\
+ (\nabla_{\tau^h(\phi)}R)(d\varphi(u),\nabla d\varphi(u,*)d\varphi(*))d\varphi(*) \end{array} \right\}^H_{(\varphi(x),d\varphi(u))} \]

for all \( (\varphi(x),d\varphi(u)) \in TN \).

**Proof.** Let \( (\varphi(x),d\varphi(u)) \in TN \) and \( \{e_i^H,e_i^V\}_{i=1}^m \) be a local orthonormal frame on \( TM \) such that \( (\nabla e_i,e_i)_x = 0 \), we denote by
\[ F_i = \frac{1}{2} R(d\varphi(e_i),\tau^h(\phi)) \]
and
\[ G_i = \frac{1}{2} R(*,\nabla d\varphi(u,e_i))\tau^h(\phi), \]
then
\[ L_i = \frac{1}{2} R(*,d\varphi(e_i))\tau^h(\phi). \]
First, using Proposition 2.1, we calculate
\[ \text{tr}_g \nabla^2 (\tau^h(\phi)^H)_{(\varphi(x), d\varphi(u))} \]
\[ = \sum_{i=1}^{m} \left\{ \nabla^\phi \nabla^\phi_i \nabla^\phi_i \tau^h(\phi)^H \right\}_{(\varphi(x), d\varphi(u))} + \sum_{i=1}^{m} \left\{ \nabla^\phi_i \nabla^\phi_i \nabla^\phi \tau^h(\phi)^H \right\}_{(\varphi(x), d\varphi(u))} \]
\[ = \sum_{i=1}^{m} \left\{ \nabla (\nabla \tau(u)) (\nabla \tau(u))^T \right\}_{(\varphi(x), d\varphi(u))} \]
\[ + \sum_{i=1}^{m} \nabla \tau(u) \nabla \tau(u)^T \left( \nabla \tau(u) \nabla \tau(u)^T \right)_{(\varphi(x), d\varphi(u))}. \]

From Proposition 2.2, we have
\[ (\mathbf{19}) \]
\[ \text{tr}_g \nabla^2 (\tau^h(\phi)^H)_{(\varphi(x), d\varphi(u))} \]
\[ = \sum_{i=1}^{m} \left\{ (\nabla \tau(u)) (\nabla \tau(u))^T \right\}_{(\varphi(x), d\varphi(u))} + \frac{1}{2} R(d\varphi(u), \nabla d\varphi(u,e_i)) \nabla d\varphi(u,e_i) \tau^h(\phi)^H \]
\[ - V(\nabla \tau(u)) (\nabla \tau(u))^T \right\}_{(\varphi(x), d\varphi(u))} \]
\[ - \frac{1}{2} R(d\varphi(u), \nabla d\varphi(u,e_i)) \tau^h(\phi) d\varphi(u) )^T \right\}_{(\varphi(x), d\varphi(u))} \]
\[ - \left( \frac{1}{2} R(d\varphi(u), F_1(\nabla d\varphi(u,e_i))) \right)^H + H(\nabla d\varphi(u,e_i)) \left( \frac{1}{2} R(d\varphi(u), G(d\varphi(u))) d\varphi(u) )^T \right\}_{(\varphi(x), d\varphi(u))} \]
\[ + \left( \frac{1}{2} R(d\varphi(u), \nabla d\varphi(u,e_i)) G_1(d\varphi(u))) \right)^H + L_1(d\varphi(u)) \]
\[ + \frac{1}{2} R(d\varphi(u), d\varphi(u)) L_1(d\varphi(u))^H \right\}_{(\varphi(x), d\varphi(u))}. \]

By substituting (16), (17), and (18) in (19), we arrive at
\[ \text{tr}_g \nabla^2 (\tau^h(\phi)^H)_{(\varphi(x), d\varphi(u))} \]
\[ = \sum_{i=1}^{m} \left\{ \nabla \tau(u) \nabla d\varphi(u) + R(d\varphi(u), \nabla d\varphi(u,e_i)) \nabla \tau^h(\phi) \right\}_{(\varphi(x), d\varphi(u))} \]
\[ + \frac{1}{2} R(d\varphi(u), \nabla d\varphi(u,e_i)) \nabla \tau^h(\phi) \]
\[ + \frac{1}{4} R(d\varphi(u), \nabla d\varphi(u,e_i)) R(d\varphi(u), \nabla d\varphi(u,e_i)) \nabla \tau^h(\phi) \]
\[ + \frac{1}{4} (\nabla \tau(u,e_i) R(d\varphi(u,e_i)) \nabla \tau^h(\phi) d\varphi(u) \]
\[ + \sum_{i=1}^{m} \left\{ \frac{1}{2} R(d\varphi(u,e_i), \nabla \tau(u,e_i)) \nabla \tau(u,e_i) + R(d\varphi(u,e_i), \nabla \tau(u,e_i)) \nabla \tau^h(\phi) \right\}_{(\varphi(x), d\varphi(u))} \]
\[ + \nabla \tau(u,e_i) R(d\varphi(u,e_i)) \nabla \tau^h(\phi) d\varphi(u) \]
\[ + \frac{1}{4} R(d\varphi(u,e_i), \nabla \tau(u,e_i)) \nabla \tau^h(\phi) d\varphi(u) \right\}_{(\varphi(x), d\varphi(u))}. \]
On the other hand, we have

$$\text{tr}_{\varphi^*} (R^{TN}((\tau^h(\phi))^H, d\phi) d\phi)_{(\varphi(x), d\varphi(u))}$$

$$= \sum_{i=1}^{m} \left\{ \left[ R(\tau^h(\phi), d\varphi(e_i)) d\varphi(e_i) + \frac{3}{4} R(d\varphi(u), R(\tau^h(\phi), d\varphi(e_i)) d\varphi(u)) d\varphi(e_i) \right. \right.$$  

$$+ \left( \nabla_{\tau^h(\phi)} R \right)(d\varphi(u), \nabla d\varphi(u, e_i)) d\varphi(e_i)$$

$$- \frac{1}{2} \left( \nabla_{d\varphi(u, e_i)} R \right)(d\varphi(u), \nabla d\varphi(u, e_i)) \tau^h(\phi)$$

$$- \frac{1}{4} R(d\varphi(u), \nabla d\varphi(u, e_i)) R(d\varphi(u), \nabla d\varphi(u, e_i)) \tau^h(\phi)$$

$$\left. + \frac{1}{2} R(d\varphi(e_i), d\varphi(e_i)) \tau^h(\phi) \right) \right\}_{(\varphi(x), d\varphi(u))}^H$$

$$+ \sum_{i=1}^{m} \left\{ \frac{1}{2} \left( \nabla_{d\varphi(u, e_i)} R \right)(\tau^h(\phi), d\varphi(e_i)) d\varphi(u) + \frac{3}{2} R(\tau^h(\phi), d\varphi(e_i)) \nabla d\varphi(u, e_i) \right.$$

$$+ \frac{1}{2} R(d\varphi(u), \nabla d\varphi(u, e_i)) d\varphi(e_i), \tau^h(\phi)) d\varphi(u)$$

$$\left. - \frac{1}{4} R(d\varphi(u), \nabla d\varphi(u, e_i)) \tau^h(\phi), d\varphi(e_i)) d\varphi(u) \right\}_{(\varphi(x), d\varphi(u))}^V.$$  

By summing (20) and (21), we obtain the Lemma 4.2. \hfill \square

From Lemma 4.1 and Lemma 4.2, we deduce the next theorem.

**Theorem 4.2.** Let \( \phi: (TM^m, g^*) \rightarrow (TN^n, h^*) \) be a the tangent map of \( \varphi: (M^m, g) \rightarrow (N^n, h) \), then the bitension field of \( \phi \) is given by

\( \tau_2(\phi)_{(\varphi(x), d\varphi(u))} \)

$$= \text{tr}_h \left( \left\{ \nabla^{\varphi^*} \tau^h(\phi) + 2 R(\tau^h(\phi), d\varphi(*)) \nabla d\varphi(u, *) \right. \right.$$  

$$+ \frac{1}{2} R(d\varphi(u), \nabla d\varphi(u, *)) d\varphi(*), \tau^h(\phi)) d\varphi(u)$$

$$- R(d\varphi(*), \nabla^{\varphi^*} \tau^h(\phi)) d\varphi(u) \right\}_{(\varphi(x), d\varphi(u))}^V$$

$$+ \text{tr}_h \left\{ \left( \nabla^{\varphi^*} \nabla^{\varphi^*} \tau^h(\phi) + R(\tau^h(\phi), d\varphi(*)) d\varphi(*) \right. \right.$$  

$$+ \frac{1}{2} R(d\varphi(u), \tau^h(\phi)) R(d\varphi(u), \nabla d\varphi(u, *)) d\varphi(*)$$

$$+ R(d\varphi(u), \nabla^{\varphi^*} \tau^h(\phi)) d\varphi(*) + R(\tau^h(\phi), \nabla d\varphi(u, *)) d\varphi(*)$$

$$\left. + \frac{1}{2} R(d\varphi(u), \nabla^{\varphi^*} d\varphi(u, *)) \tau^h(\phi) \right) + R(d\varphi(u), \nabla d\varphi(u, *)) \nabla^{\varphi^*} \tau^h(\phi)$$

$$+ R(d\varphi(u), \nabla d\varphi(u, *)) \nabla^{\varphi^*} \tau^h(\phi) + R(d\varphi(u), R(\tau^h(\phi), d\varphi(*)) d\varphi(u)) d\varphi(*)$$

$$+ (\nabla_{\tau^h(\phi)} R)(d\varphi(u), \nabla d\varphi(u, *)) d\varphi(*) \right\}^H_{(\varphi(x), d\varphi(u))}.$$  

for all \((\varphi(x), d\varphi(u)) \in TN \).
By Theorem 4.2, we have the following theorem.

**Theorem 4.3.** The tangent map \( \phi: (TM^m, g^*) \rightarrow (TN^n, h^*) \) of \( \varphi: (M^m, g) \rightarrow (N^n, h) \) is biharmonic if and only if the following conditions are verified

\[
0 = \mathrm{tr}_n \left\{ (\nabla^\phi)^2 \tau^\phi (\varphi) + 2R(\tau^h(\phi), d\varphi(u, *))\nabla d\varphi(u, *) \right. \\
+ \frac{1}{2} R(d\varphi(u), d\varphi(u, *)) R(d\varphi(u, \nabla d\varphi(u, *)) d\varphi(\cdot) + R(\tau^\phi, d\varphi(u, *)) d\varphi(*) \\
- R(d\varphi(\cdot), \nabla^\phi(\nabla^\phi d\varphi(u))) \bigg\}_{\varphi(x)},
\]

for all \( (\varphi(x), d\varphi(u)) \in TN \).

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**References**


L. Belarbi, Department of Mathematics, Laboratory of Pure and Applied Mathematics, University of Mostaganem (U.M.A.B.), B.P.227,27000, Mostaganem, Algeria, *e-mail: lakehalbelarbi@gmail.com*

H. El Hendi, Department of Mathematics, University of Bechar, 08000, Bechar, Algeria, *e-mail: elhendihichem@yahoo.fr*