Acta Math. Univ. Comenianae
Vol. LXXXVIII, 2 (2019), pp. 187–199

HARMONIC AND BIHARMONIC MAPS BETWEEN TANGENT BUNDLES

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ABSTRACT. In this paper, we investigate the harmonicity and biharmonicity of a tangent map $\phi\colon (TM, g^s) \to (TN, h^s)$ in the case where the tangent bundles TM, TN are endowed with Sasaki Riemannian metrics g^s, h^s.

1. INTRODUCTION

Let $\varphi\colon M \to N$ be a smooth map between the smooth manifolds M, N. The map φ induces the tangent map $\phi = \mathrm{d}\varphi\colon TM \to TN$ between the tangent bundles of M and N. In the case where M, N are Riemannian manifolds, their tangent bundles TM, TN may be endowed with the corresponding Sasaki metrics so that they become Riemannian manifolds. The motivation of this paper is to study harmonicity and biharmonicity of the tangent map $\phi\colon (TM, g^s) \to (TN, h^s)$.

In this paper, we deal with these problems in the case where the tangent bundles TM, TN are endowed with Sasaki Riemannian metrics g^s, h^s. We show that the map ϕ is harmonic if φ is totally geodesic. Further, if TM is a compact tangent bundle, ϕ is harmonic if and only if φ is totally geodesic.

In the biharmonicity, we show that if TM is a compact tangent bundle and $\phi\colon (TM, g^s) \to (TN, h^s)$ the tangent map of harmonic map $\varphi\colon (M, g) \to (N, h)$, then ϕ is biharmonic if and only if ϕ is harmonic. Furthermore, we calculate the bitension field of tangent map ϕ.

1.1. Harmonic maps

Consider a smooth map $\phi\colon (M^n, g) \to (N^n, h)$ between two Riemannian manifolds, then the energy functional is defined by

$$E(\phi) = \frac{1}{2} \int_M |\mathrm{d}\phi|^2 \mathrm{d}v_g \tag{1}$$

(or over any compact subset $K \subset M$).

A map is called harmonic if it is a critical point of the energy functional E (or $E(K)$ for all compact subsets $K \subset M$). For any smooth variation $\{\phi\}_{t \in I}$ of ϕ

Received October 24, 2017; revised Januar 10, 2019.
2010 *Mathematics Subject Classification.* Primary 53A45; Secondary 53C20.
Key words and phrases. Horizontal lift; vertical lift; harmonic maps; biharmonic map.

with $\phi_0 = \phi$ and $V = \frac{\mathrm{d}\phi_t}{\mathrm{d}t}|_{t=0}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} E(\phi_t)|_{t=0} = -\int_M h(\tau(\phi), V)\mathrm{d}v_g, \tag{2}$$

where

$$\tau(\phi) = \mathrm{tr}_g \nabla \mathrm{d}\phi \tag{3}$$

is the tension field of ϕ. Then we have the following theorem.

Theorem 1.1. *A smooth map $\phi\colon (M^m, g) \to (N^n, h)$ is harmonic if and only if*

$$\tau(\phi) = 0. \tag{4}$$

If $(x^i)_{1\leq i\leq m}$ and $(y^\alpha)_{1\leq \alpha\leq n}$ denote local coordinates on M and N, respectively, then equation 4 takes the form

$$\tau(\phi)^\alpha = \Big(\Delta\phi^\alpha + g^{ij} \overset{N}{\Gamma}{}^\alpha_{\beta\gamma} \frac{\partial\phi^\beta}{\partial x^i}\frac{\partial\phi^\gamma}{\partial x^j}\Big) = 0, \tag{5}$$

where $\Delta\phi^\alpha = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}(\sqrt{|g|}g^{ij}\frac{\partial\phi^\alpha}{\partial x^j})$ is the Laplace operator on (M^m, g) and $\overset{N}{\Gamma}{}^\alpha_{\beta\gamma}$ are the Christoffel symbols on N.

1.2. Biharmonic maps

Definition 1.1. A map $\varphi\colon (M, g) \to (N, h)$ between Riemannian manifolds is called biharmonic it is a critical point of bienergy functional

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 \mathrm{d}v^g, \tag{6}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} E_2(\phi_t)|_{t=0} = -\int_M h(\tau_2(\phi), V)\mathrm{d}v_g. \tag{7}$$

The Euler-Lagrange equation attached to bienergy is given by the vanishing of the bitension field

$$\tau_2(\varphi) = -J_\varphi(\tau(\varphi)) = -(\Delta^\varphi\tau(\varphi) + \mathrm{tr}_g R^N(\tau(\varphi), \mathrm{d}\varphi)\mathrm{d}\varphi), \tag{8}$$

where J_φ is the Jacobi operator defined by

$$\begin{aligned} J_\varphi\colon \Gamma(\varphi^{-1}(TN)) &\to \Gamma(\varphi^{-1}(TN)),\\ V &\mapsto \Delta^\varphi V + \mathrm{tr}_g R^N(V, \mathrm{d}\varphi)\mathrm{d}\varphi. \end{aligned} \tag{9}$$

The biharmonic map introduced by Eelles and Sampson in 1964, is a generalization of harmonic maps. One can refer to [**9**], [**11**] and [**15**] for background on harmonic and biharmonic maps.

2. Basic notions and definition on TM

Let (M, g) be an n-dimensional Riemannian manifold and (TM, π, M) be its tangent bundle.

A local chart $(U, x^i)_{i=1\ldots n}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^i)_{i=1\ldots n}$ on TM. By Γ^k_{ij}, denote the Christoffel symbols of g and by ∇, the Levi-Civita connection of g.

We have two complementary distributions on TM, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$, defined by

$$\mathcal{V}_{(x,u)} = \ker(d\pi_{(x,u)}) = \Big\{a^i \frac{\partial}{\partial y^i}_{|(x,u)} \ ; \ a^i \in \mathbb{R}\Big\},$$

$$\mathcal{H}_{(x,u)} = \Big\{\frac{\partial}{\partial x^i}_{|(x,u)} - a^i u^j \Gamma^k_{ij} \frac{\partial}{\partial y^k}_{|(x,u)} \ ; \ a^i \in \mathbb{R}\Big\},$$

where $(x, u) \in TM$ such that $T_{(x,u)}TM = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}$.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$X^V = X^i \frac{\partial}{\partial y^i}, \tag{10}$$

$$X^H = X^i \frac{\delta}{\delta x^i} = X^i \Big\{\frac{\partial}{\partial x^i} - y^{j\Gamma^k_{ij}\frac{\partial}{\partial y^k}}\Big\}. \tag{11}$$

For consequences, we have $(\frac{\partial}{\partial x^i})^H = \frac{\delta}{\delta x^i}$ and $(\frac{\partial}{\partial x^i})^V = \frac{\partial}{\partial y^i}$, then $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})_{i=1\ldots n}$ is a local adapted frame in TTM.

Definition 2.1. The Sasaki metric g^s on the tangent bundle TM of M is given by

1. $g^s(X^H, Y^H) = g(X, Y) \circ \pi$,
2. $g^s(X^H, Y^V) = 0$,
3. $g^s(X^V, Y^V) = g(X, Y) \circ \pi$

for all vector fields $X, Y \in \Gamma(TM)$.

Proposition 2.1 ([**10**])**.** *Let (M, g) be a Riemannian manifold and $\widehat{\nabla}$ be the Levi-Civita connection of the tangent bundle (TM, g^s) equipped with the Sasaki metric. Then*

$$(\widehat{\nabla}_{X^H} Y^H)_{(x,u)} = (\nabla_X Y)^H_{(x,u)} - \frac{1}{2}(R_x(X, Y)u)^V,$$

$$(\widehat{\nabla}_{X^H} Y^V)_{(x,u)} = (\nabla_X Y)^V_{(x,u)} + \frac{1}{2}(R_x(u, Y)X)^H,$$

$$(\widehat{\nabla}_{X^V} Y^H)_{(x,u)} = \frac{1}{2}(R_x(u, X)Y)^H,$$

$$(\widehat{\nabla}_{X^V} Y^V)_{(x,u)} = 0$$

for all vector fields $X, Y \in \Gamma(TM)$ and $(x, u) \in TM$.

Definition 2.2. Let (M,g) be a Riemannian manifold and $F \in \mathfrak{T}_1^1(M)$ be a tensor of type (1,1). Then we define a vertical and horizontal vector fields VF, HF on TM by

$$VF\colon TM \to TTM, \qquad (x,u) \mapsto (F(u))^V,$$
$$HF\colon TM \to TTM, \qquad (x,u) \mapsto (F(u))^H.$$

Proposition 2.2 ([**9**])**.** *Let (M,g) be a Riemannian manifold and $\widehat{\nabla}$ be the Levi-Civita connection of the tangent bundle (TM,g^s) equipped with the Sasaki metric. If $F \in \mathfrak{T}_1^1(M)$ is a tensor of type $(1,1)$, then*

$$(\widehat{\nabla}_{X^H} HF)_{(x,u)} = H(\nabla_X F)_{(x,u)} - \frac{1}{2}(R_x(X_x, F_x(u))u)^V,$$
$$(\widehat{\nabla}_{X^H} VF)_{(x,u)} = V(\nabla_X F)_{(x,u)}, + \frac{1}{2}(R_x(u, F_x(u))X_x)^H$$
$$(\widehat{\nabla}_{X^V} HF)_{(x,u)} = (F(X))^H_{(x,u)} + \frac{1}{2}(R_x(u, X_x)F(u))^H,$$
$$(\widehat{\nabla}_{X^V} VF)_{(x,u)} = (F(X))^V_{(x,u)},$$

where $(x,u) \in TM$ and $X \in \Gamma(TM)$.

Proposition 2.3 ([**9**])**.** *Let (M,g) be a Riemannian manifold and $\widehat{R}$ be the Riemann curvature tensor of the tangent bundle (TM,g^s) equipped with the Sasaki metric. The the following formulae hold:*

1. $\widehat{R}(X^V, Y^V)Z^V = 0,$
2. $\widehat{R}(X^V, Y^V)Z^H = \Big[R(X,Y)Z + \frac{1}{4}R(u,X)(R(u,Y)Z) - \frac{1}{4}5u,Y)(R(u,X)Z)\Big]^H_x,$
3. $\widehat{R}(X^H, Y^V)Z^V = -\Big[\frac{1}{2}R(Y,Z)X + \frac{1}{4}R(u,Y)(R(u,Z)X)\Big]^H_x,$
4. $\widehat{R}(X^H, Y^V)Z^H = \Big[\frac{1}{4}R(R(u,Y)Z,X)u + \frac{1}{2}R(X,Z)Y\Big]^V_x + \frac{1}{2}\Big[(\nabla_X R)(u,Y)Z\Big]^H_x,$
5. $\widehat{R}(X^H, Y^H)Z^V = \Big[R(X,Y)Z + \frac{1}{4}R(R(u,Z)Y,X)u - \frac{1}{4}R(R(u,Z)X,Y)u\Big]^V_x + \frac{1}{2}\Big[(\nabla_X R)(u,Z)Y - (\nabla_Y R)(u,Z)X\Big]^H_x,$
6. $\widehat{R}(X^H, Y^H)Z^H = \frac{1}{2}\Big[(\nabla_Z R)(X,Y)u\Big]^V_x + \Big[R(X,Y)Z + \frac{1}{4}R(u,R(Z,Y)u)X + \frac{1}{4}R(u,R(X,Z)u)Y + \frac{1}{2}R(u,R(X,Y)u)Z\Big]^H_x$

for all vectors $u, X, Y, Z \in T_xM$.

3. Harmonic maps between two tangent bundle

Lemma 3.1 ([**17**]). *Let $\varphi\colon (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds. The map φ induces the tangent map*

$$\phi = \mathrm{d}\varphi\colon (TM^m, g^s) \to (TN^n, h^s), \qquad (x, y) \mapsto (\varphi(x), \mathrm{d}\varphi(y)).$$

For all vector field $X \in \Gamma(TM)$, we have

$$\mathrm{d}\phi\big((X)^V\big) = \big(\mathrm{d}\varphi(X)\big)^V, \qquad \mathrm{d}\phi\big((X)^H\big) = \big(\mathrm{d}\varphi(X)\big)^H + \big(\nabla \mathrm{d}\varphi(y, X)\big)^V.$$

Theorem 3.1 ([**17**]). *Let $\phi\colon (TM^m, g^s) \to (TN^n, h^s)$ be a the tangent map of $\varphi\colon (M^m, g) \to (N^n, h)$, then the tension field associated with ϕ is*

$$\tau(\phi) = \big(\tau(\varphi) + \operatorname{tr}_h R^N(\mathrm{d}\varphi(u), \nabla \mathrm{d}\varphi(u, *))\mathrm{d}\varphi(*)\big)^H + \big(\operatorname{div}(\nabla \mathrm{d}\varphi))(u)\big)^V. \tag{12}$$

Theorem 3.2. *Let $\phi\colon (TM^m, g^s) \to (TN^n, h^s)$ be a the tangent map of $\varphi\colon (M^m, g) \to (N^n, h)$, then ϕ is harmonic if and only if the following conditions are verified:*

$$\tau(\varphi) = 0, \qquad \operatorname{tr}_g R^N(\mathrm{d}\varphi(u), \nabla \mathrm{d}\varphi(u, *))\mathrm{d}\varphi(*) = 0, \qquad \operatorname{div}(\nabla \mathrm{d}\varphi) = 0.$$

Corollary 3.1. *Let $\phi\colon (TM^m, g^s) \to (TN^n, h^s)$ be a the tangent map of $\varphi\colon (M^m, g) \to (N^n, h)$, if φ is totally geodesic, then ϕ is harmonic.*

Lemma 3.2. *Let $\phi\colon (TM^m, g^s) \to (TN^n, h^s)$ be a the tangent map of $\varphi\colon (M^m, g) \to (N^n, h)$, then the energy density associated ϕ is given by*

$$e(\phi) = 2e(\varphi) + \frac{1}{2} \operatorname{tr}_h |\nabla \mathrm{d}\varphi(*, u)|^2 \tag{13}$$

Proof. Let $(x, u) \in TM$, $(\varphi(x), \mathrm{d}\varphi(u)) \in TN$, and $\{e_i^H, e_i^V\}_{i=1\ldots m}$ be a local orthonormal frame on TM, then

$$\begin{aligned} 2e(\phi)_{(\varphi(x), \mathrm{d}\varphi(u))} = \sum_{i=1}^m \big(& h^s_{(\varphi(x), \mathrm{d}\varphi(u))}(\mathrm{d}\phi(e_i^H), \mathrm{d}\phi(e_i^H)) \\ &+ h^s_{(\varphi(x), \mathrm{d}\varphi(u))}(\mathrm{d}\phi(e_i^V), \mathrm{d}\phi(e_i^V))\big). \end{aligned}$$

Using Lemma 3.1, we obtain

$$\begin{aligned} 2e(\phi)_{(\varphi(x), \mathrm{d}\varphi(u))} &= \sum_{i=1}^m \Big(h^s_{(\varphi(x), \mathrm{d}\varphi(u))}((\mathrm{d}\varphi(e_i))^H, (\mathrm{d}\varphi(e_i))^H\Big) \\ &\quad + \sum_{i=1}^m \Big(h^s_{(\varphi(x), \mathrm{d}\varphi(u))}((\mathrm{d}\varphi(e_i))^V, (\mathrm{d}\varphi(e_i))^V\Big) \\ &\quad + \sum_{i=1}^m \Big(h^s_{(\varphi(x), \mathrm{d}\varphi(u))}((\nabla \mathrm{d}\varphi(u, e_i))^V, (\nabla \mathrm{d}\varphi(u, e_i))^V\Big) \\ &= 4e(\varphi) + \operatorname{tr}_h |\nabla \mathrm{d}\varphi(u, *)|^2. \end{aligned}$$

□

Theorem 3.3. *Let TM be a compact tangent bundle and $\phi\colon (TM^m, g^s) \to (TN^n, h^s)$ be its the tangent map of $\varphi\colon (M^m, g) \to (N^n, h)$. Then ϕ is a harmonic if and only if φ is totally geodesic.*

Proof. If φ is totally geodesic, from Corollary 3.1, we deduce that ϕ is harmonic. Inversely.

Let $\omega\colon I \times M \to N$ be a smooth map satisfying

$$\omega(t,x) = \varphi_t(x) = (1+t)\varphi(x) \qquad \text{and} \qquad \omega(0,x) = \varphi(x).$$

for all $t \in I = (-\varepsilon, \varepsilon)$, $\varepsilon > 0$ and all $x \in M$.

The variation vector field $v \in \Gamma(\varphi^{-1}TN)$ associated the variation $\{\varphi_t\}_{t\in I}$ is given for all $x \in M$, by

$$v(x) = \mathrm{d}_{(0,x)}\omega\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big).$$

From Lemma 3.2, we have

$$e(\phi_t) = 2e(\varphi_t) + (1+t)^2 |\nabla \mathrm{d}\varphi(*, u)|^2 \mathrm{d}v_{h^S}.$$

If ϕ is a critical point of the energy functional, from equation 2, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} E(\phi_t)_{t=0} = 0 = -2\int_{TM} h(v, \tau(\varphi))\mathrm{d}v_{g^S} + \int_{TM} |\nabla \mathrm{d}\varphi(*, u)|^2 \mathrm{d}v_{h^S}$$

If ϕ is harmonic, hence $\tau(\varphi) = 0$, then $\nabla \mathrm{d}\varphi = 0$. □

4. Biharmonic maps between two tangent bundle

In the section, we denote

(14) $$\tau^h(\phi) = \tau(\varphi) + \operatorname{tr}_h R^N(\mathrm{d}\varphi(u), \nabla \mathrm{d}\varphi(u, *))\mathrm{d}\varphi(*)$$

(15) $$\tau^v(\phi) = \operatorname{div}(\nabla \mathrm{d}\varphi))(u).$$

Theorem 4.1. *Let TM be a compact tangent bundle and $\phi\colon (TM^m, g^s) \to (TN^n, h^s)$ the tangent map of harmonic map $\varphi\colon (M, g) \to (N, h)$, then ϕ is biharmonic if and only if ϕ is harmonic.*

Proof. Let φ_t be a compactly supported variation of φ defined by

$$\varphi_t(x) = (1+t)\varphi(x), \qquad \text{and} \qquad \varphi_0(x) = \varphi(x).$$

From the formulas (14) and (15), we have

$$\tau^h(\phi_t) = (1+t)^3 \tau^h(\phi),$$
$$\tau^v(\phi_t) = (1+t)\tau^v(\phi),$$

$$\begin{aligned} E_2(\phi_t) &= \frac{1}{2}\int |\tau(\phi_t)|^2_{h^s} \mathrm{d}v_{h^S} \\ &= \frac{1}{2}\int |\tau^h(\phi_t)|^2_h \mathrm{d}v_h + \frac{1}{2}\int |\tau^v(\phi_t)|^2_h \mathrm{d}v_h \\ &= \frac{(1+t)^6}{2}\int |\tau^h(\phi)|^2_h \mathrm{d}v_{h^S} + \frac{(1+t)^2}{2}\int |\tau^v(\phi)|^2_h \mathrm{d}v_{h^S}. \end{aligned}$$

Since the section ϕ is biharmonic, then for variation ϕ_t, we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} E_2(\phi_t)|_{t=0} = 3\int |\tau^h(\phi)|_h^2 \mathrm{d}v_h + \int |\tau^v(\phi)|_h^2 \mathrm{d}v_h$$

Hence $\tau^h(\phi) = 0$ and $\tau^v(\phi) = 0$, then $\tau(\phi) = 0$. □

Lemma 4.1. *Let $\phi\colon (TM^m, g^s) \to (TN^n, h^s)$ be a the tangent map of $\varphi\colon (M^m, g) \to (N^n, h)$, then the Jacobi tensor $J_\phi((\tau^v(\phi))^V)$ is given by*

$$\begin{aligned}
&J_\phi((\tau^v(\phi))^V)_{(\varphi(x),\mathrm{d}\varphi(u))}\\
&= \Big\{ \mathrm{tr}_h(\nabla^\varphi)^2 \tau^v(\phi) \Big\}^V_{(\varphi(x),\mathrm{d}\varphi(u))}\\
&\quad + \Big\{ \mathrm{tr}_h \Big(R(\mathrm{d}\varphi(u), \nabla^\varphi_* \tau^v(\phi))\mathrm{d}\varphi(*) + R(\tau^v(\phi), \nabla \mathrm{d}\varphi(u,*))\mathrm{d}\varphi(*)\\
&\quad + \frac{1}{2} R(\mathrm{d}\varphi(u), \tau^v(\phi)) R(\mathrm{d}\varphi(u),, \nabla \mathrm{d}\varphi(u,*))\mathrm{d}\varphi(*) \Big) \Big\}^H_{(\varphi(x),\mathrm{d}\varphi(u))}
\end{aligned}$$

for all $(\varphi(x), \mathrm{d}\varphi(u)) \in TN$.

Proof. Let $(\varphi(x), \mathrm{d}\varphi(u)) \in TN$, and $\{e_i^H, e_i^V\}_{i=1}^m$ be a local orthonormal frame on TM such that $(\nabla_{e_i} e_i)_x = 0$, denoted by $F_i = \frac{1}{2} R(*, \tau^v(\phi))\mathrm{d}\varphi(e_i)$, we have

$$\begin{aligned}
&\Big(\nabla^\phi_{e_i^H} \tau^v(\phi)^V + \nabla^\phi_{e_i^V} \tau^v(\phi)^V \Big)_{(\varphi(x),\mathrm{d}\varphi(u))}\\
&= \nabla^{TN}_{(\mathrm{d}\varphi(e_i))^H + (\nabla \mathrm{d}\varphi(u,e_i))^V} \tau^v(\phi)^V |_{(\varphi(x),\mathrm{d}\varphi(u))}\\
&= (\nabla_{\mathrm{d}\varphi(e_i)} \tau^v(\phi))^V_{(\varphi(x),\mathrm{d}\varphi(u))} + \frac{1}{2} (R(\mathrm{d}\varphi(u), \tau^v(\phi))\mathrm{d}\varphi(e_i))^H\\
&= (\nabla_{\mathrm{d}\varphi(e_i)} \tau^v(\phi))^V_{(\varphi(x),\mathrm{d}\varphi(u))} + HF_i(\varphi(x), \mathrm{d}\varphi(u)).
\end{aligned}$$

Then

$$\begin{aligned}
&\mathrm{tr}_{g^S} \nabla^2 (\tau^v(\phi)^V)_{(\varphi(x),\mathrm{d}\varphi(u))}\\
&= \sum_{i=1}^m \Big\{ \nabla^\phi_{e_i^H} \nabla^\phi_{e_i^H} \tau^v(\phi)^V \Big\}_{(\varphi(x),\mathrm{d}\varphi(u))} + \sum_{i=1}^m \Big\{ \nabla^\phi_{e_i^V} \nabla^\phi_{e_i^V} \tau^v(\phi)^V \Big\}_{(\varphi(x),\mathrm{d}\varphi(u))}\\
&= \sum_{i=1}^m \Big\{ \nabla^{TN}_{(\mathrm{d}\varphi(e_i))^H + (\nabla \mathrm{d}\varphi(u,e_i))^V} (\nabla_{\mathrm{d}\varphi(e_i)} \tau^v(\phi))^V_{(\mathrm{d}\varphi(x),\mathrm{d}\varphi(u))} + HF_i \Big\}_{(\varphi(x),\mathrm{d}\varphi(u))}\\
&= \sum_{i=1}^m \Big\{ \nabla^{TN}_{(\mathrm{d}\varphi(e_i))^H} (\nabla_{\mathrm{d}\varphi(e_i)} \tau^v(\phi))^V\\
&\quad + \nabla^{TN}_{(\mathrm{d}\varphi(e_i))^H} HF_i + \nabla^{TN}_{(\nabla \mathrm{d}\varphi(u,e_i))^V} HF_i \Big\}_{(\varphi(x),\mathrm{d}\varphi(u))}.
\end{aligned}$$

Using Proposition 2.2, we obtain

$$\begin{aligned}
&\operatorname{tr}_{g^S} \nabla^2(\tau^v(\phi)^V)_{(\varphi(x),\mathrm{d}\varphi(u))}\\
&= \sum_{i=1}^{m} \Big\{ (\nabla_{\mathrm{d}\varphi(e_i)}\nabla_{\mathrm{d}\varphi(e_i)}\tau^v(\phi))_{(\mathrm{d}\varphi(x),\mathrm{d}\varphi(u))}\\
&\quad - \frac{1}{4}R(\mathrm{d}\varphi(e_i), R(\mathrm{d}\varphi(u),\tau^v(\phi))\mathrm{d}\varphi(e_i))\mathrm{d}\varphi(u)\Big\}^V_{(\varphi(x),\mathrm{d}\varphi(u))}\\
&\quad + \sum_{i=1}^{m}\Big\{\frac{1}{2}R(\mathrm{d}\varphi(u),\nabla_{\mathrm{d}\varphi(e_i)}\tau^v(\phi)\mathrm{d}\varphi(e_i) + \frac{1}{2}(\nabla_{\mathrm{d}\varphi(e_i))})R(\mathrm{d}\varphi(u),\tau^v(\phi))\mathrm{d}\varphi(e_i)\\
&\quad + \frac{1}{4}R(\mathrm{d}\varphi(u),\nabla\mathrm{d}\varphi(u,e_i))R(\mathrm{d}\varphi(u),\tau^v(\phi))\mathrm{d}\varphi(e_i)\\
&\quad + \frac{1}{2}R(\nabla\mathrm{d}\varphi(u,e_i),\tau^v(\phi))\mathrm{d}\varphi(e_i)\Big\}^H_{(\varphi(x),\mathrm{d}\varphi(u))}.
\end{aligned}$$

From Lemma 3.1, we have

$$\begin{aligned}
&\operatorname{tr}_{g^S} R^{TN}((\tau^v(\phi))^V,\mathrm{d}\phi)\mathrm{d}\phi\\
&= \sum_{i=1}^{m}\Big\{R^{TN}((\tau^v(\phi))^V,\mathrm{d}\phi(e_i^H))\mathrm{d}\phi(e_i^H) + R^{TN}((\tau^v(\phi))^V,\mathrm{d}\phi(e_i^V))\mathrm{d}\phi(e_i^V)\Big\}\\
&= \sum_{i=1}^{m}\Big\{R^{TN}((\tau^v(\phi))^V,(\mathrm{d}\varphi(e_i))^H)(\mathrm{d}\varphi(e_i))^H\\
&\quad + R^{TN}((\tau^v(\phi))^V,(\mathrm{d}\varphi(e_i))^H)(\nabla\mathrm{d}\varphi(u,e_i))^V\\
&\quad + R^{TN}((\tau^v(\phi))^V,(\nabla\mathrm{d}\varphi(u,e_i))^V)(\mathrm{d}\varphi(e_i))^H\Big\}.
\end{aligned}$$

By calculating at $(\varphi(x),\mathrm{d}\varphi(u))$, we obtain

$$\begin{aligned}
&\operatorname{tr}_{g^S}(R^{TN}((\tau^v(\phi))^V,\mathrm{d}\phi)\mathrm{d}\phi)_{(\mathrm{d}\varphi(x),\mathrm{d}\varphi(u))}\\
&= \sum_{i=1}^{m}\Big\{-\frac{1}{4}R(R(\mathrm{d}\varphi(u),\tau^v(\phi))\mathrm{d}\varphi(e_i),\mathrm{d}\varphi(e_i))\mathrm{d}\varphi(u)\\
&\quad + \frac{1}{2}R(\mathrm{d}\varphi(e_i),\mathrm{d}\varphi(e_i))\tau^v(\phi)\Big\}^V + \sum_{i=1}^{m}\Big\{R(\tau^v(\phi),\nabla\mathrm{d}\varphi(u,e_i))\mathrm{d}\varphi(e_i)\\
&\quad + \frac{1}{4}R(\mathrm{d}\varphi(u),\tau^v(\phi))R(\mathrm{d}\varphi(u),\nabla\mathrm{d}\varphi(u,e_i))\mathrm{d}\varphi(e_i)\\
&\quad - \frac{1}{4}R(\mathrm{d}\varphi(u),\nabla\mathrm{d}\varphi(u,e_i)))R(\mathrm{d}\varphi(u),\tau^v(\phi))\mathrm{d}\varphi(e_i))\\
&\quad + \frac{1}{2}R(\tau^v(\phi)),\nabla\mathrm{d}\varphi(u,e_i))\mathrm{d}\varphi(e_i)\\
&\quad + \frac{1}{4}R(\mathrm{d}\varphi(u),\tau^v(\phi))R(\mathrm{d}\varphi(u),\nabla\mathrm{d}\varphi(u,e_i))\mathrm{d}\varphi(e_i)\\
&\quad - \frac{1}{2}(\nabla_{\mathrm{d}\varphi(e_i)}R)(\mathrm{d}\varphi(u),\tau^v(\phi))\mathrm{d}\varphi(e_i)\Big\}^H.
\end{aligned}$$

Considering the formula (9), we deduce

$$\begin{aligned}
&J_\phi((\tau^v(\phi))^V)_{(\varphi(x),\mathrm{d}\varphi(u))}\\
&= \Big\{\operatorname{tr}_h(\nabla^\varphi)^2\tau^v(\phi)\Big\}^V_{(\varphi(x),\mathrm{d}\varphi(u))} + \Big\{\Big(R(\mathrm{d}\varphi(u),\nabla^\varphi_{e_i}\tau^v(\phi))\mathrm{d}\varphi(e_i)\\
&\quad + R(\tau^v(\phi),\nabla\mathrm{d}\varphi(u,e_i))\mathrm{d}\varphi(e_i)\\
&\quad + \frac{1}{2}R(\mathrm{d}\varphi(u),\tau^v(\phi))R(\mathrm{d}\varphi(u),,\nabla\mathrm{d}\varphi(u,e_i))\mathrm{d}\varphi(e_i)\Big)\Big\}^H_{(\varphi(x),\mathrm{d}\varphi(u))}.
\end{aligned}$$

From the following equality

$$\begin{aligned}
&\nabla_{\mathrm{d}\varphi(e_i)}R(\mathrm{d}\varphi(u),\tau^v(\phi))\mathrm{d}\varphi(e_i)\\
&= (\nabla_{\mathrm{d}\varphi(e_i)}R)(\mathrm{d}\varphi(u),\tau^v(\phi))\mathrm{d}\varphi(e_i) + R(\nabla_{\mathrm{d}\varphi(e_i)}\mathrm{d}\varphi(u),\tau^v(\phi))\mathrm{d}\varphi(e_i)\\
&\quad + R(\mathrm{d}\varphi(u),\nabla_{\mathrm{d}\varphi(e_i)}\tau^v(\phi))\mathrm{d}\varphi(e_i),
\end{aligned}$$

the proof of Lemma 4.1 is completed. □

Lemma 4.2. *Let $\phi\colon (TM^m,g^s)\to(TN^n,h^s)$ be a the tangent map of $\varphi\colon (M^m,g)\to(N^n,h)$, then the Jacobi tensor $J_\phi((\tau^h(\phi))^H)$ is given by*

$$\begin{aligned}
&J_\phi((\tau^h(\phi))^H)_{(\varphi(x),\mathrm{d}\varphi(u))}\\
&= \operatorname{tr}_h\Big\{2R(\tau^h(\phi),\mathrm{d}\varphi(*))\nabla\mathrm{d}\varphi(u,*)\\
&\quad + \frac{1}{2}R(R(\mathrm{d}\varphi(u),\nabla\mathrm{d}\varphi(u,*))\mathrm{d}\varphi(*),\tau^h(\phi))\mathrm{d}\varphi(u)\\
&\quad - R(\mathrm{d}\varphi(*),\nabla^\varphi_*\tau^h(\phi))\mathrm{d}\varphi(u)\Big\}^V_{(\varphi(x),\mathrm{d}\varphi(u))}\\
&\quad + \operatorname{tr}_h\Big\{\nabla^\varphi_*\nabla^\varphi_*\tau^h(\phi) + R(\mathrm{d}\varphi(u),\nabla\mathrm{d}\varphi(u,*))\nabla^\varphi_*\nabla^\varphi_*\tau^h(\phi)\\
&\quad + \frac{1}{2}R(\mathrm{d}\varphi(u),\nabla^\varphi_*\nabla\mathrm{d}\varphi(u,*))\tau^h(\phi)\\
&\quad + R(\mathrm{d}\varphi(u),R(\tau^h(\phi),\mathrm{d}\varphi(*))\mathrm{d}\varphi(*)\\
&\quad + (\nabla_{\tau^h(\phi)}R)(\mathrm{d}\varphi(u),\nabla\mathrm{d}\varphi(u,*))\mathrm{d}\varphi(*)\Big\}^H_{(\varphi(x),\mathrm{d}\varphi(u))}
\end{aligned}$$

for all $(\varphi(x),\mathrm{d}\varphi(u))\in TN$.

Proof. Let $(\varphi(x),\mathrm{d}\varphi(u))\in TN$ and $\{e_i^H,e_i^V\}_{i=1}^m$ be a local orthonormale frame on TM such that $(\nabla_{e_i}e_i)_x=0$, we denote by

$$F_i = \frac{1}{2}R(\mathrm{d}\varphi(e_i),\tau^h(\phi))* \tag{16}$$

and

$$G_i = \frac{1}{2}R(*,\nabla\mathrm{d}\varphi(u,e_i))\tau^h(\phi), \tag{17}$$

then

$$L_i = \frac{1}{2}R(*,\mathrm{d}\varphi(e_i))\tau^h(\phi) \tag{18}$$

First, using Proposition 2.1, we calculate

$$\begin{aligned}
&\operatorname{tr}_{g^S} \nabla^2(\tau^h(\phi)^H)_{(\varphi(x),\mathrm{d}\varphi(u))} \\
&= \sum_{i=1}^m \Big\{\nabla^\phi_{e_i^H}\nabla^\phi_{e_i^H}\tau^h(\phi)^H\Big\}_{(\varphi(x),\mathrm{d}\varphi(u))} + \sum_{i=1}^m \Big\{\nabla^\phi_{e_i^V}\nabla^\phi_{e_i^V}\tau^h(\phi)^H\Big\}_{(\varphi(x),\mathrm{d}\varphi(u))} \\
&= \sum_{i=1}^m \Big\{\nabla_{(\mathrm{d}\varphi(e_i))^H}\Big((\nabla_{\mathrm{d}\varphi(e_i)}\tau^h(\phi))^H - VF_i + HG_i + HL_i\Big)\Big\}_{(\varphi(x),\mathrm{d}\varphi(u))} \\
&\quad + \sum_{i=1}^m \Big\{\nabla_{(\nabla\mathrm{d}\varphi(u,e_i))^V}\Big((\nabla_{\mathrm{d}\varphi(e_i)}\tau^h(\phi))^H - VF_i + HG_i + HL_i\Big)\Big\}_{(\varphi(x),\mathrm{d}\varphi(u))}.
\end{aligned}$$

From Proposition 2.2, we have

$$\begin{aligned}
&\operatorname{tr}_{g^S} \nabla^2(\tau^h(\phi)^H)_{(\varphi(x),\mathrm{d}\varphi(u))} \\
&= \sum_{i=1}^m \Big\{(\nabla_{\mathrm{d}\varphi(e_i)}\nabla_{\mathrm{d}\varphi(e_i)}\tau^h(\phi))^H + (\frac{1}{2}R(\mathrm{d}\varphi(u),\nabla\mathrm{d}\varphi(u,e_i))\nabla_{\mathrm{d}\varphi(e_i)}\tau^h(\phi))^H \\
&\quad - V(\nabla_{\mathrm{d}\varphi(e_i)}F_i) - (F_i(\nabla\mathrm{d}\varphi(u,e_i)))^V - (\frac{1}{2}R(\mathrm{d}\varphi(e_i),\nabla_{\mathrm{d}\varphi(e_i)}\tau^h(\phi))\mathrm{d}\varphi(u))^V \\
&\quad - (\frac{1}{2}R(\mathrm{d}\varphi(u),F_i(\mathrm{d}\varphi(u))\mathrm{d}\varphi(e_i)))^H + H(\nabla_{\mathrm{d}\varphi(e_i)}G_i) - \frac{1}{2}(R(\mathrm{d}\varphi(e_i),G(\mathrm{d}\varphi(u)))\mathrm{d}\varphi(u))^V \\
&\quad + (G_i(\nabla\mathrm{d}\varphi(u,e_i)))^H \frac{1}{2}(R(\mathrm{d}\varphi(u),\nabla\mathrm{d}\varphi(u,e_i))G_i(\mathrm{d}\varphi(u)))^H + H(L_i(\mathrm{d}\varphi(e_i))) \\
&\quad + \frac{1}{2}(R(\mathrm{d}\varphi(u),\mathrm{d}\varphi(e_i))L_i(\mathrm{d}\varphi(u)))^H\Big\}_{(\varphi(x),\mathrm{d}\varphi(u))}.
\end{aligned} \tag{19}$$

By substituting (16), (17), and (18) in (19), we arrive at

$$\begin{aligned}
&\operatorname{tr}_{g^S} \nabla^2(\tau^h(\phi)^H)_{(\varphi(x),\mathrm{d}\varphi(u))} \\
&= \sum_{i=1}^m \Big\{\nabla_{\mathrm{d}\varphi(e_i)}\nabla_{\mathrm{d}\varphi(e_i)}\tau^h(\phi) + R(\mathrm{d}\varphi(u),\nabla\mathrm{d}\varphi(u,e_i))\nabla_{\mathrm{d}\varphi(e_i)}\tau^h(\phi) \\
&\quad + \frac{1}{2}R(\mathrm{d}\varphi(u),\nabla_{\mathrm{d}\varphi(e_i)}\nabla\mathrm{d}\varphi(u,e_i))\tau^h(\phi) \\
&\quad + \frac{1}{4}R(\mathrm{d}\varphi(u),\nabla\mathrm{d}\varphi(u,e_i))R(\mathrm{d}\varphi(u),\nabla\mathrm{d}\varphi(u,e_i))\tau^h(\phi) \\
&\quad + \frac{1}{2}(\nabla_{\mathrm{d}\varphi(e_i)}R)(\mathrm{d}\varphi(u),\nabla\mathrm{d}\varphi(u,e_i))\tau^h(\phi) \\
&\quad - \frac{1}{4}R(\mathrm{d}\varphi(u),R(\mathrm{d}\varphi(e_i),\tau^h(\phi))\mathrm{d}\varphi(u))\mathrm{d}\varphi(e_i) + \frac{1}{2}R(\mathrm{d}\varphi(e_i),\mathrm{d}\varphi(e_i))\tau^h(\phi) \\
&\quad + \frac{1}{4}R(\mathrm{d}\varphi(u),\mathrm{d}\varphi(e_i))R(\mathrm{d}\varphi(u),\mathrm{d}\varphi(e_i))\tau^h(\phi)\Big\}^H_{(\varphi(x),\mathrm{d}\varphi(u))} \\
&\quad + \sum_{i=1}^m \Big\{\frac{1}{2}R(\mathrm{d}\varphi(e_i),\tau^h(\phi))\nabla\mathrm{d}\varphi(u,e_i) + R(\mathrm{d}\varphi(e_i),\nabla_{\mathrm{d}\varphi(e_i)}\tau^h(\phi))\mathrm{d}\varphi(u) \\
&\quad + (\nabla_{\mathrm{d}\varphi(e_i)}R)(\mathrm{d}\varphi(e_i),\tau^h(\phi))\mathrm{d}\varphi(u) \\
&\quad + \frac{1}{4}R(\mathrm{d}\varphi(e_i),R(\mathrm{d}\varphi(u),\nabla\mathrm{d}\varphi(u,e_i))\tau^h(\phi))\mathrm{d}\varphi(u)\Big\}^V_{(\varphi(x),\mathrm{d}\varphi(u))}.
\end{aligned} \tag{20}$$

On the other hand, we have

$$
\begin{aligned}
&\operatorname{tr}_{g^S}(R^{TN}((\tau^h(\phi))^H, d\phi)d\phi)_{(\varphi(x),d\varphi(u))}\\
&= \sum_{i=1}^{m}\Big\{R(\tau^h(\phi), d\varphi(e_i))d\varphi(e_i)\\
&\quad + \frac{3}{4}R(d\varphi(u), R(\tau^h(\phi), d\varphi(e_i))d\varphi(u))d\varphi(e_i)\\
&\quad + (\nabla_{\tau^h(\phi)}R)(d\varphi(u), \nabla d\varphi(u,e_i))d\varphi(e_i)\\
&\quad - \frac{1}{2}(\nabla_{d\varphi(e_i)}R)(d\varphi(u), \nabla d\varphi(u,e_i))\tau^h(\phi)\\
&\quad - \frac{1}{4}R(d\varphi(u), \nabla d\varphi(u,e_i))R(d\varphi(u), \nabla d\varphi(u,e_i)\tau^h(\phi)\\
&\quad - \frac{1}{2}R(d\varphi(e_i), d\varphi(e_i))\tau^h(\phi)\\
&\quad - \frac{1}{4}R(d\varphi(u), d\varphi(e_i))R(d\varphi(u), d\varphi(e_i))\tau^h(\phi)\Big\}^H_{(\varphi(x),d\varphi(u))}\\
&\quad + \sum_{i=1}^{m}\Big\{\frac{1}{2}(\nabla_{d\varphi(e_i)}R)(\tau^h(\phi), d\varphi(e_i))d\varphi(u) + \frac{3}{2}R(\tau^h(\phi), d\varphi(e_i))\nabla d\varphi(u,e_i)\\
&\quad + \frac{1}{2}R(R(d\varphi(u), \nabla d\varphi(u,e_i))d\varphi(e_i), \tau^h(\phi))d\varphi(u)\\
&\quad - \frac{1}{4}R(R(d\varphi(u), \nabla d\varphi(u,e_i))\tau^h(\phi), d\varphi(e_i))d\varphi(u)\Big\}^V_{(\varphi(x),d\varphi(u))}.
\end{aligned}
\tag{21}
$$

By summing (20) and (21), we obtain the Lemma 4.2. □

From Lemma 4.1 and Lemma 4.2, we deduce the next theorem.

Theorem 4.2. *Let $\phi\colon (TM^m, g^s) \to (TN^n, h^s)$ be a the tangent map of $\varphi\colon (M^m, g) \to (N^n, h)$, then the bitension field of ϕ is given by*

$$
\begin{aligned}
&\tau_2(\phi)_{(\varphi(x),d\varphi(u))}\\
&= \operatorname{tr}_h\Big\{(\nabla^{\varphi})^2\tau^v(\phi) + 2R(\tau^h(\phi), d\varphi(*))\nabla d\varphi(u,*)\\
&\quad + \frac{1}{2}R(R(d\varphi(u), \nabla d\varphi(u,*))d\varphi(*), \tau^h(\phi))d\varphi(u)\\
&\quad - R(d\varphi(*), \nabla^{\varphi}{}_*\tau^h(\phi))d\varphi(u)\Big\}^V_{(\varphi(x),d\varphi(u))}\\
&\quad + \operatorname{tr}_h\Big\{\nabla^{\varphi}{}_*\nabla^{\varphi}{}_*\tau^h(\phi) + R(\tau^h(\phi), d\varphi(*))d\varphi(*)\\
&\quad + \frac{1}{2}R(d\varphi(u), \tau^v(\phi))R(d\varphi(u), \nabla d\varphi(u,*)d\varphi(*)\\
&\quad + R(d\varphi(u), \nabla^{\varphi}_*\tau^v(\phi))d\varphi(*) + R(\tau^v(\phi), \nabla d\varphi(u,*))d\varphi(*)\\
&\quad + \frac{1}{2}R(d\varphi(u), \nabla^{\varphi}_*\nabla d\varphi(u,*))\tau^h(\phi)\\
&\quad + R(d\varphi(u), \nabla d\varphi(u,*))\nabla^{\varphi}_*\tau^h(\phi) + R(d\varphi(u), R(\tau^h(\phi), d\varphi(*))d\varphi(u))d\varphi(*)\\
&\quad + (\nabla_{\tau^h(\phi)}R)(d\varphi(u), \nabla d\varphi(u,*)d\varphi(*)\Big\}^H_{(\varphi(x),d\varphi(u))}
\end{aligned}
$$

for all $(\varphi(x), d\varphi(u)) \in TN$.

By Theorem 4.2, we have the following theorem.

Theorem 4.3. *The tangent map* $\phi\colon (TM^m, g^s) \to (TN^n, h^s)$ *of* $\varphi\colon (M^m, g) \to (N^n, h)$ *is biharmonic if and only if the following conditions are verified*

$$\begin{aligned}0 = \operatorname{tr}_h \Big\{&(\nabla^{\varphi})^2\tau^v(\phi) + 2R(\tau^h(\phi), \mathrm{d}\varphi(*))\nabla\mathrm{d}\varphi(u,*)\\&+ \frac{1}{2}R(R(\mathrm{d}\varphi(u), \nabla\mathrm{d}\varphi(u,*))\mathrm{d}\varphi(*), \tau^h(\phi))\mathrm{d}\varphi(u)\\&- R(\mathrm{d}\varphi(*), \nabla^{\varphi_*}\tau^h(\phi))\mathrm{d}\varphi(u)\Big\}^V_{(\varphi(x)},\end{aligned}$$

$$\begin{aligned}0 = \operatorname{tr}_h \Big\{&\nabla^{\varphi_*}\nabla^{\varphi_*}\tau^h(\phi) + R(\tau^h(\phi), \mathrm{d}\varphi(*))\mathrm{d}\varphi(*)\\&+ \frac{1}{2}R(\mathrm{d}\varphi(u), \tau^v(\phi))R(\mathrm{d}\varphi(u), \nabla\mathrm{d}\varphi(u,*)\mathrm{d}\varphi(*)\\&+ R(\mathrm{d}\varphi(u), \nabla^{\varphi}_*\tau^v(\phi))\mathrm{d}\varphi(*) + R(\tau^v(\phi), \nabla\mathrm{d}\varphi(u,*))\mathrm{d}\varphi(*)\\&+ \frac{1}{2}R(\mathrm{d}\varphi(u), \nabla^{\varphi}_*\nabla\mathrm{d}\varphi(u,*))\tau^h(\phi)\\&+ R(\mathrm{d}\varphi(u), \nabla\mathrm{d}\varphi(u,*))\nabla^{\varphi}_*\tau^h(\phi) + R(\mathrm{d}\varphi(u), R(\tau^h(\phi), \mathrm{d}\varphi(*))\mathrm{d}\varphi(u))\mathrm{d}\varphi(*)\\&+ (\nabla_{\tau^h(\phi)}R)(\mathrm{d}\varphi(u), \nabla\mathrm{d}\varphi(u,*)\mathrm{d}\varphi(*)\Big\}^H_{(\varphi(x)}\end{aligned}$$

for all $(\varphi(x), \mathrm{d}\varphi(u)) \in TN$.

Acknowledgment. The authors would like to thank the referees for all helpful comments and suggestions that have improved the quality of our initial manuscript.

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