SIGNED STAR 
\( (j, k) \)-DOMATIC NUMBER OF A GRAPH

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Abstract. Let \( G \) be a simple graph without isolated vertices with edge set \( E(G) \), and let \( j \) and \( k \) be two positive integers. A function \( f: E(G) \to \{-1, 1\} \) is said to be a signed star \( j \)-dominating function on \( G \) if \( \sum_{e \in E(v)} f(e) \geq j \) for every vertex \( v \) of \( G \), where \( E(v) = \{ uv \in E(G) \mid u \in N(v) \} \). A set \( \{ f_1, f_2, \ldots, f_d \} \) of distinct signed star \( j \)-dominating functions on \( G \) with the property that \( \sum_{i=1}^{d} f_i(e) \leq k \) for each \( e \in E(G) \), is called a signed star \( (j, k) \)-dominating family (of functions) on \( G \). The maximum number of functions in a signed star \( (j, k) \)-dominating family on \( G \) is the signed star \( (j, k) \)-domatic number of \( G \) denoted by \( d_{SS}^{(j,k)}(G) \).

In this paper we study properties of the signed star \( (j, k) \)-domatic number of a graph \( G \). In particular, we determine bounds on \( d_{SS}^{(j,k)}(G) \). Some of our results extend those ones given by Atapour, Sheikholeslami, Ghameslou and Volkmann [1] for the signed star domatic number, Sheikholeslami and Volkmann [5] for the signed star \((k, k)\)-domatic number and Sheikholeslami and Volkmann [4] for the signed star \( k \)-domatic number.

1. Introduction

Let \( G \) be a graph with vertex set \( V(G) \) and edge set \( E(G) \). We use [2] for terminology and notation which are not defined here and consider simple graphs without isolated vertices only. The integers \( n = |V(G)| \) and \( m = |E(G)| \) are the order and the size of the graph \( G \), respectively. For every vertex \( v \in V(G) \), the open neighborhood \( N(v) \) of \( v \) is the set \( \{ u \in V(G) \mid uv \in E(G) \} \), and the closed neighborhood of \( v \) is the set \( N[v] = N(v) \cup \{ v \} \). The degree of a vertex \( v \) is \( d(v) = |N(v)| \). The minimum and maximum degree of a graph \( G \) are denoted by \( \delta(G) \) and \( \Delta(G) \), respectively. The complement \( \overline{G} \) of a graph \( G \) is the graph with vertex set \( V(G) \) such that two vertices are adjacent in \( \overline{G} \) if and only if these vertices are not adjacent in \( G \).

The open neighborhood \( N_G(e) \) of an edge \( e \in E(G) \) is the set of all edges adjacent to \( e \). Its closed neighborhood is \( N_G[e] = N_G(e) \cup \{ e \} \). For a function \( f: E(G) \to \{-1, 1\} \) and a subset \( S \) of \( E(G) \), we define \( f(S) = \sum_{e \in S} f(e) \). The edge-neighborhood \( E_G(v) = E(v) \) of a vertex \( v \in V(G) \) is the set of all edges
incident with the vertex \(v\). For each vertex \(v \in V(G)\), we also define \(f(v) = \sum_{e \in E_G(v)} f(e)\).

Let \(j\) be a positive integer. A function \(f : E(G) \to \{-1, 1\}\) is called a signed star \(j\)-dominating function (SS\(j\)DF) on \(G\) if \(f(v) \geq j\) for every vertex \(v\) of \(G\). The signed star \(j\)-domination number of a graph \(G\) is \(\gamma_{jSS}(G) = \min\{\sum_{e \in E(G)} f(e) \mid f\) is a SS\(j\)DF on \(G\}\). The signed star \(j\)-dominating function \(f\) on \(G\) with \(f(E(G)) = \gamma_{jSS}(G)\) is called a \(\gamma_{jSS}(G)\)-function. As the assumption \(\delta(G) \geq j\) is clearly necessary, we will always assume that satisfy \(\delta(G) \geq j\) while discussing \(\gamma_{jSS}(G)\) all graphs involved. The signed star \(j\)-domination number was introduced by Xu and Li \([10]\) in 2009 and has been studied by several authors (see for instance, \([3, 4, 7]\)). The signed star 1-domination number is the usual signed star domination number, introduced in 2005 by Xu \([8]\). The signed star domination number was investigated for example, by \([3, 6, 9]\).

Let \(k\) be a further positive integer. A set \(\{f_1, f_2, \ldots, f_d\}\) of distinct signed star \(j\)-dominating functions on \(G\) with \(\sum_{i=1}^d f_i(e) \leq k\) for each \(e \in E(G)\), is called a signed star \((j,k)\)-dominating family (SS\((j,k)D\) family) (of functions) on \(G\). The maximum number of functions in a signed star \((j,k)\)-dominating family on \(G\) is the signed star \((j,k)\)-domatic number of \(G\) denoted by \(\text{d}^{(j,k)}_{SS}(G)\). The signed star \((j,k)\)-domatic number is well-defined and

\[\text{d}^{(j,k)}_{SS}(G) \geq 1\]

for all graphs \(G\) with \(\delta(G) \geq j\), since the set consisting of any signed star \(j\)-dominating function forms a SS\((j,k)D\) family on \(G\). A \(\text{d}^{(j,k)}_{SS}\) family of a graph \(G\) is a SS\((j,k)D\) family containing exactly \(\text{d}^{(j,k)}_{SS}(D)\) signed star \(j\)-dominating functions.

The signed star \((1,1)\)-domatic number \(\text{d}^{(1,1)}_{SS}(G)\) is the usual signed star domatic number \(d_{SS}(G)\) which was introduced by Atapour, Sheikholeslami, Ghamelsou and Volkmann \([1]\) in 2010.

Our purpose in this paper is to initiate the study of the signed star \((j,k)\)-domatic number in graphs. We study basic properties and bounds for the signed star \((j,k)\)-domatic number \(d^{(j,k)}_{SS}(G)\) of a graph \(G\). In addition, we derive Nordhaus-Gaddum type results and bounds of the product and the sum of \(\gamma_{jSS}(G)\) and \(d^{(j,k)}_{SS}(G)\). Many of our results extend those given by Atapour, Sheikholeslami, Ghamelsou and Volkmann \([1]\) for the signed star domatic number, Sheikholeslami and Volkmann \([5]\) for the signed star \((k,k)\)-domatic number and Sheikholeslami and Volkmann \([4]\) for the signed star \(k\)-domatic number.

**Observation 1** \([4]\). Let \(G\) be a graph of size \(m\) with \(\delta(G) \geq j\). Then \(\gamma_{jSS}(G) = m\) if and only if each edge \(e \in E(G)\) has an endpoint \(u\) such that \(d(u) = j\) or \(d(u) = j + 1\).
2. Properties of the signed star \((j,k)\)-domatic number

**Theorem 2.** Let \(j, k \geq 1\) be two integers. If \(G\) is a graph of minimum degree \(\delta(G) \geq j\), then

\[
d_{SS}^{(j,k)}(G) \leq \frac{k\delta(G)}{j}.
\]

Moreover, if \(d_{SS}^{(j,k)}(G) = k\delta(G)/j\), then for each function of any signed star \((j,k)\)-dominating family \(\{f_1, f_2, \ldots, f_d\}\) with \(d = d_{SS}^{(j,k)}(G)\) and for all vertices \(v\) of degree \(\delta(G)\), \(\sum_{e \in E_G(v)} f_i(e) = j\) and \(\sum_{i=1}^{d} f_i(e) = k\) for every \(e \in E_G(v)\).

**Proof.** Let \(\{f_1, f_2, \ldots, f_d\}\) be a signed star \((j,k)\)-dominating family on \(G\) such that \(d = d_{SS}^{(j,k)}(G)\). If \(v \in V(G)\) is a vertex of minimum degree \(\delta(G)\), then it follows that

\[
d \cdot j = \sum_{i=1}^{d} j \leq \sum_{i=1}^{d} \sum_{e \in E_G(v)} f_i(e)
= \sum_{e \in E_G(v)} \sum_{i=1}^{d} f_i(e)
\leq \sum_{e \in E_G(v)} k = k \cdot \delta(G),
\]

and this implies the desired upper bound on the signed star \((j,k)\)-domatic number.

If \(d_{SS}^{(j,k)}(G) = k\delta(G)/j\), then the two inequalities occurring in the proof become equalities, which leads to the two properties given in the statement. \(\square\)

The special cases \(j = k = 1\), \(j = 1\) and \(j = k\) in Theorem 2 can be found in [1], [4] and [5], respectively. As an application of Theorem 2, we will prove the following Nordhaus-Gaddum type result.

**Corollary 3.** Let \(j, k \geq 1\) be integers. If \(G\) is a graph of order \(n\) such that \(\delta(G) \geq j\) and \(\delta(G) \geq j\), then

\[
d_{SS}^{(j,k)}(G) + d_{SS}^{(j,k)}(\overline{G}) \leq \frac{k}{j}(n-1).
\]

If \(d_{SS}^{(j,k)}(G) + d_{SS}^{(j,k)}(\overline{G}) = k(n-1)/j\), then \(G\) is regular.

**Proof.** Since \(\delta(G) \geq j\) and \(\delta(G) \geq j\), it follows from Theorem 2 that

\[
d_{SS}^{(j,k)}(G) + d_{SS}^{(j,k)}(\overline{G}) \leq \frac{k\delta(G)}{j} + \frac{k\delta(G)}{j}
= \frac{k}{j}(\delta(G) + (n - \Delta(G) - 1)) \leq \frac{k}{j}(n-1),
\]

and this is the desired Nordhaus-Gaddum inequality. If \(G\) is not regular, then \(\Delta(G) - \delta(G) \geq 1\), and the above inequality chain leads to the better bound \(d_{SS}^{(j,k)}(G) + d_{SS}^{(j,k)}(\overline{G}) \leq \frac{k}{j}(n-2)\). This completes the proof. \(\square\)
Theorem 4. Let $j, k \geq 1$ be integers. If $v$ is a vertex of a graph $G$ such that $d(v)$ is odd and $j$ is even or $d(v)$ is even and $j$ is odd, then

$$d_{SS}^{(j,k)}(G) \leq \frac{k}{j+1} \cdot d(v).$$

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a signed star $(j, k)$-dominating family on $G$ such that $d = d_{SS}^{(j,k)}(G)$. Assume first that $d(v)$ is odd and $j$ is even. The definition yields to $\sum_{e \in E_G(v)} f_i(e) \geq j$ for each $i \in \{1, 2, \ldots, d\}$. On the left-hand side of this inequality a sum of an odd number of odd summands occurs. Therefore it is an odd number, and as $j$ is even, we obtain $\sum_{e \in E_G(v)} f_i(e) \geq j + 1$ for each $i \in \{1, 2, \ldots, d\}$. It follows that $k \cdot d(v) = \sum_{e \in E_G(v)} k \geq \sum_{e \in E_G(v)} \sum_{i=1}^d f_i(e) = \sum_{i=1}^d \sum_{e \in E_G(v)} f_i(e) \geq \sum_{i=1}^d (j + 1) = d(j + 1)$,

and this leads to the desired bound. Assume next that $d(v)$ is even and $j$ is odd. Note that $\sum_{e \in E_G(v)} f_i(e) \geq j$ for each $i \in \{1, 2, \ldots, d\}$. On the left-hand side of this inequality a sum of an even number of odd summands occurs. Therefore it is an even number, and as $j$ is odd, we obtain $\sum_{e \in E_G(v)} f_i(e) \geq j + 1$ for each $i \in \{1, 2, \ldots, d\}$. Now the desired bound follows as above, and the proof is complete. $\square$

The next result is an immediate consequence of Theorem 4.

Corollary 5. Let $j, k \geq 1$ be integers. If $G$ is a graph such that $\delta(G)$ is odd and $j$ is even or $\delta(G)$ is even and $j$ is odd, then

$$d_{SS}^{(j,k)}(G) \leq \frac{k}{j+1} \cdot \delta(G).$$

As an application of Corollary 5, we will improve the Nordhaus-Gaddum bound in Corollary 3 for many cases.

Theorem 6. Let $j, k \geq 1$ be two integers and let $G$ be a graph of order $n$ such that $\delta(G) \geq j$ and $\delta(G) \geq j$. If $\Delta(G) - \delta(G) \geq 1$ or $j$ is odd or $j$ is even and $\delta(G)$ is odd or $j, \delta(G)$ and $n$ are even, then

$$d_{SS}^{(j,k)}(G) + d_{SS}^{(j,k)}(\overline{G}) < \frac{k}{j}(n - 1).$$

Proof. If $\Delta(G) - \delta(G) \geq 1$, then Corollary 3 implies the desired bound. Thus assume now that $G$ is $\delta(G)$-regular.
Case 1. Assume that \( j \) is odd. If \( \delta(G) \) is even, then from Theorem 2 and Corollary 5 it follows that
\[
d^{(j,k)}(G) + d^{(j,k)}(\overline{G}) \leq \frac{k}{j+1} \delta(G) + \frac{k}{j} \delta(\overline{G})
\]
\[
< \frac{k}{j} (\delta(G) + (n - \delta(G) - 1))
\]
\[
= \frac{k}{j} (n - 1).
\]
If \( \delta(G) \) is odd, then \( n \) is even and thus \( \delta(\overline{G}) = n - \delta(G) - 1 \) is even. Combining Theorem 2 and Corollary 5, we find that
\[
d^{(j,k)}(G) + d^{(j,k)}(\overline{G}) \leq \frac{k}{j} \delta(G) + \frac{k}{j} (n - \delta(G) - 1)
\]
\[
= \frac{k}{j} (n - 1),
\]
and this completes the proof of Case 1.

Case 2. Assume that \( j \) is even. If \( \delta(G) \) is odd, then from Theorem 2 and Corollary 5 it follows that
\[
d^{(j,k)}(G) + d^{(j,k)}(\overline{G}) \leq \frac{k}{j+1} \delta(G) + \frac{k}{j} \delta(\overline{G})
\]
\[
< \frac{k}{j} (\delta(G) + (n - \delta(G) - 1))
\]
\[
= \frac{k}{j} (n - 1).
\]
If \( \delta(G) \) is even and \( n \) is even, then \( \delta(\overline{G}) = n - \delta(G) - 1 \) is odd, and we obtain the desired bound as above. \( \square \)

**Theorem 7.** Let \( j, k \geq 1 \) be integers. If \( G \) is a graph such that \( k \) is odd and \( d^{(j,k)}(G) \) is even or \( k \) is even and \( d^{(j,k)}(G) \) is odd, then
\[
d^{(j,k)}(G) \leq \frac{k-1}{j} \cdot \delta(G).
\]

**Proof.** Let \( \{f_1, f_2, \ldots, f_d\} \) be a signed star \((j,k)\)-dominating family on \( G \) such that \( d = d^{(j,k)}(G) \). Assume first that \( k \) is odd and \( d \) is even. If \( e \in E(G) \) is an arbitrary edge, then \( \sum_{i=1}^{d} f_i(e) \leq k \). On the left-hand side of this inequality a sum of an even number of odd summands occurs. Therefore, it is an even number, and as \( k \) is odd, we obtain \( \sum_{i=1}^{d} f_i(e) \leq k - 1 \) for each \( e \in E(G) \). If \( v \) is a vertex of minimum degree, then it follows that
\[
d \cdot j = \sum_{i=1}^{d} j \leq \sum_{i=1}^{d} \sum_{e \in E_G(v)} f_i(e)
\]
\[
= \sum_{e \in E_G(v)} \sum_{i=1}^{d} f_i(e) \leq \sum_{e \in E_G(v)} (k - 1) = \delta(G)(k - 1),
\]
and this yields to the desired bound. Assume second that $k$ is even and $d$ is odd. If $e \in E(G)$ is an arbitrary edge, then $\sum_{i=1}^{d} f_i(e) \leq k$. On the left-hand side of this inequality a sum of an odd number of odd summands occurs. Therefore, it is an odd number and as $k$ is even, we obtain $\sum_{i=1}^{d} f_i(e) \leq k - 1$ for each $e \in E(G)$. Now the desired bound follows as above, and the proof is complete. \hfill \Box

The special cases $j = k = 1$, $j = 1$ and $j = k$ of Theorem 4, Corollary 5 and Theorem 7 can be found in [1], [4] and [5], respectively. According to (1), $d_{SS}^{(j,k)}(G)$ is a positive integer. If we suppose in the case $j = k = 1$ that $d_{SS}(G) = d_{SS}^{(1,1)}(G)$ is an even integer, then Theorem 7 leads to the contradiction $d_{SS}(G) \leq 0$. Consequently, we obtain the next known result.

**Corollary 8** ([1]). The signed star domatic number $d_{SS}(G)$ is an odd integer.

**Proposition 9.** Let $j,k$ be two integers such that $j \geq 1$ and $k \geq 2$, and let $G$ be a graph with minimum degree $\delta(G) \geq j$. Then $d_{SS}^{(j,k)}(G) = 1$ if and only if each edge $e \in E(G)$ has an endpoint $u$ such that $d(u) = j$ or $d(u) = j + 1$.

Proof. Assume that each edge $e \in E(G)$ has an endpoint $u$ such that $d(u) = j$ or $d(u) = j + 1$. It follows from Observation 1 that $\gamma_{j,SS}(G) = m$ and thus $d_{SS}^{(j,k)}(G) = 1$.

Conversely, assume that $d_{SS}^{(j,k)}(G) = 1$. If $G$ contains an edge $e = uv$ such that $d(u) \geq j + 2$ and $d(v) \geq j + 2$, then the functions $f_1 : E(G) \to \{-1,1\}$ such that $f_1(x) = 1$ for each $x \in E(G)$ and $f_2(e) = -1$ and $f_2(x) = 1$ for each edge $x \in E(G) \setminus \{e\}$ are signed star $j$-dominating functions on $G$ such that $f_1(x) + f_2(x) \leq 2 \leq k$ for each edge $x \in E(G)$. Thus $\{f_1, f_2\}$ is a signed star $(j,k)$-dominating family on $G$, a contradiction to $d_{SS}^{(j,k)}(G) = 1$.

The next result is an immediate consequence of Observation 1 and Proposition 9.

**Corollary 10.** Let $j,k$ be two integers such that $j \geq 1$ and $k \geq 2$, and let $G$ be a graph with minimum degree $\delta(G) \geq j$. Then $d_{SS}^{(j,k)}(G) = 1$ if and only if $\gamma_{SS}(G) = m$.

Next we present a lower bound on the signed star $(j,k)$-dominant number.

**Proposition 11.** Let $j,k$ be two integers such that $k \geq j \geq 1$, and let $G$ be a graph with minimum degree $\delta(G) \geq j$. If $G$ contains a vertex $v \in V(G)$ such that all vertices of $N[N[v]]$ have degree at least $j + 2$, then $d_{SS}^{(j,k)}(G) \geq j$.

Proof. Let $\{u_1, u_2, \ldots, u_j\} \subseteq N(v)$. The hypothesis that all vertices of $N[N[v]]$ have degree at least $j + 2$ implies that the functions $f_i : E(G) \to \{-1,1\}$ such that $f_1(\{vu_i\}) = -1$ and $f_i(x) = 1$ for each edge $x \in E(G) \setminus \{vu_i\}$ are signed star $j$-dominating functions on $G$ for $i \in \{1,2,\ldots,j\}$. Since $f_1(x) + f_2(x) + \ldots + f_j(x) \leq j \leq k$ for each edge $x \in E(G)$, we observe that $\{f_1, f_2, \ldots, f_j\}$ is a signed star $(j,k)$-dominating family on $G$, and Proposition 11 is proved. \hfill \Box

**Corollary 12.** Let $j,k$ be two integers such that $k \geq j \geq 1$. If $G$ is a graph of minimum degree $\delta(G) \geq j + 2$, then $d_{SS}^{(j,k)}(G) \geq j$. 
Corollary 13. Let $j,k \geq 1$ be integers, and let $G$ be an $r$-regular graph with $r \geq j$.

1. If $j \leq r \leq j+1$, then $d^{(j,k)}_{SS}(G) = 1$.
2. If $r = j + 2p + 1$ with an integer $p \geq 1$ and $k \geq j$, then $j \leq d^{(j,k)}_{SS}(G) \leq \frac{kr}{j+1}$.
3. If $r = j + 2p$ with an integer $p \geq 1$ and $k \geq j$, then $j \leq d^{(j,k)}_{SS}(G) \leq \frac{kr}{j}$.

Proof. (1) Assume that $j \leq r \leq j+1$. According to Observation 1, $\gamma_{jSS}(G) = m$ and thus $d^{(j,k)}_{SS}(G) = 1$.

(2) Assume that $r = j + 2p + 1$ with $p \geq 1$. The condition $k \geq j$ and Corollary 12 imply that $j \leq d^{(j,k)}_{SS}(G)$. If $j$ is even, then $r = j + 2p + 1$ is odd, and if $j$ is odd, then $r = j + 2p + 1$ is even. Therefore, Corollary 5 leads to the desired upper bound of $d^{(j,k)}_{SS}(G)$.

(3) Assume that $r = j + 2p$ with $p \geq 1$. The condition $k \geq j$ and Corollary 12 imply that $j \leq d^{(j,k)}_{SS}(G)$. In addition, Theorem 2 yields the desired upper bound of $d^{(j,k)}_{SS}(G)$. \qed

3. Bounds on the product and the sum of $\gamma_{jSS}(G)$ and $d^{(j,k)}_{SS}(G)$

Note that $\gamma_{jSS}(G) = m$ implies immediately $d^{(j,k)}_{SS}(G) = 1$, and so $\gamma_{jSS}(G) \cdot d^{(j,k)}_{SS}(G) = m$ and $\gamma_{jSS}(G) + d^{(j,k)}_{SS}(G) = m + 1$. In this section, we present general bounds on the product and the sum of $\gamma_{jSS}(G)$ and $d^{(j,k)}_{SS}(G)$.

Theorem 14. Let $j,k \geq 1$ be integers. If $G$ is a graph of size $m$ and minimum degree $\delta(G) \geq j$, then

$$\gamma_{jSS}(G) \cdot d^{(j,k)}_{SS}(G) \leq mk.$$ 

Moreover, if $\gamma_{jSS}(G) \cdot d^{(j,k)}_{SS}(G) = mk$, then for each $d^{(j,k)}_{SS}$-family $\{f_1, f_2, \ldots, f_d\}$ of $G$, each function $f_i$ is a $\gamma_{jSS}(G)$-function and $\sum_{i=1}^{d} f_i(e) = k$ for all $e \in E(G)$.

Proof. If $\{f_1, f_2, \ldots, f_d\}$ is a signed star $(j,k)$-dominating family on $G$ such that $d = d^{(j,k)}_{SS}(G)$, then the definitions imply

$$d \cdot \gamma_{jSS}(G) = \sum_{i=1}^{d} \gamma_{jSS}(G) \leq \sum_{i=1}^{d} \sum_{e \in E(G)} f_i(e) = \sum_{e \in E(G)} \sum_{i=1}^{d} f_i(e) \leq \sum_{e \in E(G)} k = mk$$

as desired.

If $\gamma_{jSS}(G) \cdot d^{(j,k)}_{SS}(G) = mk$, then the two inequalities occurring in the proof become equalities. Hence for the $d^{(j,k)}_{SS}$-family $\{f_1, f_2, \ldots, f_d\}$ of $G$ and for each $i$, $\sum_{e \in E(G)} f_i(e) = \gamma_{jSS}(G)$, thus each function $f_i$ is a $\gamma_{jSS}(G)$-function and $\sum_{i=1}^{d} f_i(e) = k$ for all $e \in E(G)$. \qed
Theorem 15. Let \( j, k \geq 1 \) be integers. If \( G \) is a graph of size \( m \) and minimum degree \( \delta(G) \geq j \), then
\[
d_{SS}^{(j,k)}(G) + \gamma_{jSS}(G) \leq mk + 1.
\]

Proof. According to Theorem 14, we have
\[
d_{SS}^{(j,k)}(G) + \gamma_{jSS}(G) \leq d_{SS}^{(j,k)}(G) + \frac{km}{d_{SS}^{(j,k)}(G)}.
\]
Using the fact that the function \( g(x) = x + (km)/x \) is decreasing for \( 1 \leq x \leq \sqrt{km} \) and increasing for \( \sqrt{km} \leq x \leq km \), we obtain
\[
d_{SS}^{(j,k)}(G) + \gamma_{jSS}(G) \leq \max \left\{ 1 + mk, mk + \frac{km}{km} \right\} = mk + 1.
\]
Next we improve Theorem 15 considerably.

Theorem 16. Let \( j, k \geq 1 \) be two integers. If \( G \) is a graph of size \( m \) and minimum degree \( \delta(G) \geq j \), then
\[
\gamma_{jSS}(G) + d_{SS}^{(j,k)}(G) \leq \begin{cases} m + 1 & \text{if } k = 1, \\ \frac{mk}{2} + 2 & \text{if } k \geq 2. \end{cases}
\]

Proof. If \( k = 1 \), then Theorem 15 leads to the desired bound. Therefore we assume next that \( k \geq 2 \). If the order \( n = 2 \), then \( \gamma_{jSS}(G) = m = 1 \) and \( d_{SS}^{(j,k)}(G) = 1 \) and hence the desired bound is valid. Now we assume that \( n \geq 3 \). Let \( f \) be a \( SSjDF \) on \( G \). Since \( \sum_{e \in E_G(v)} f(e) \geq j \) for every vertex \( v \) of \( G \), it follows that
\[
2 \sum_{e \in E_G(v)} f(e) = \sum_{v \in V(G)} \sum_{e \in E_G(v)} f(e) \geq \sum_{v \in V(G)} j = nj.
\]
This implies \( \gamma_{jSS}(G) \geq nj/2 \). As \( n \geq 3 \) and \( j \geq 1 \), we obtain \( \gamma_{jSS}(G) \geq 2 \). Theorem 14 implies that
\[
\gamma_{jSS}(G) + d_{SS}^{(j,k)}(G) \leq \frac{mk}{\gamma_{jSS}(G)}.
\]
If we define \( x = \gamma_{jSS}(G) \) and \( g(x) = x + (mk)/x \) for \( x > 0 \), then because \( 2 \leq \gamma_{jSS}(G) \leq m \), we have to determine the maximum of the function \( g \) in the interval \( I : 2 \leq x \leq m \). Using the condition \( k \geq 2 \) and the fact that \( m \geq 2 \), it is easy to see that
\[
\max_{x \in I}\{g(x)\} = \max\{g(2), g(m)\} = \max\left\{2 + \frac{mk}{2}, m + \frac{mk}{m}\right\} = \frac{mk}{2} + 2,
\]
and the proof is complete. \( \square \)
**Theorem 17.** Let \( j, k \geq 1 \) be two integers. If \( G \) is a graph of size \( m \), minimum degree \( \delta(G) \geq j \) and order \( n \geq 2p+1 \) for an integer \( p \geq 1 \), then

\[
\gamma_{jSS}(G) + d_{SS}^{(j,k)}(G) \leq \begin{cases} 
  m + k & \text{if } 1 \leq k \leq p, \\
  \frac{mk}{p+1} + p + 1 & \text{if } k \geq p + 1.
\end{cases}
\]

**Proof.** We proceed by induction on \( p \). Theorem 16 shows that the statement is valid for \( p = 1 \). Now let \( p \geq 2 \) and assume that the statement is true for all integers \( 1 \leq i \leq p - 1 \). Then the induction hypothesis implies that \( \gamma_{jSS}(G) + d_{SS}^{(j,k)}(G) \leq m + k \) for \( 1 \leq k \leq p - 1 \). Thus assume next that \( k \geq p \). The hypothesis \( n \geq 2p+1 \) leads as in the proof of Theorem 16 to

\[
\gamma_{jSS}(G) \geq \frac{n}{2} \geq \frac{(2p+1)j}{2} \geq \frac{2p+1}{2}.
\]

and thus \( p + 1 \leq \gamma_{jSS}(G) \leq m \). Therefore, it follows from Theorem 14 that

\[
\gamma_{jSS}(G) + d_{SS}^{(j,k)}(G) \leq \gamma_{jSS}(G) + \frac{mk}{\gamma_{jSS}(G)} \leq \max \left\{ p + 1 + \frac{mk}{p+1}, m + k \right\}.
\]

Note that the hypothesis \( n \geq 2p+1 \) yields to \( m \geq p + 1 \).

If \( k = p \), then we deduce from the inequality \( m \geq p + 1 \) that

\[
\max \left\{ p + 1 + \frac{mk}{p+1}, m + k \right\} = \max \left\{ p + 1 + \frac{mp}{p+1}, m + p \right\} = m + p.
\]

If \( k \geq p + 1 \), then

\[
p + 1 + \frac{mk}{p+1} \geq m + k
\]

is equivalent with \( m(k - p - 1) \geq (p + 1)(k - p - 1) \), and this inequality is valid since \( k \geq p + 1 \) and \( m \geq p + 1 \). Hence the desired result follows from (2), and the proof is complete. \( \square \)

**References**

5. , *Signed star (k,k)-domatic number of a graph*, submitted.


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