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SIGNED STAR (j,k)-DOMATIC NUMBER OF A GRAPH

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Abstract. Let G be a simple graph without isolated vertices with edge set $E(G)$, and let j and k be two positive integers. A function $f\colon E(G) \to \{-1,1\}$ is said to be a signed star j-dominating function on G if $\sum_{e\in E(v)} f(e) \geq j$ for every vertex v of G, where $E(v) = \{uv \in E(G) \mid u \in N(v)\}$. A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed star j-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(e) \leq k$ for each $e \in E(G)$, is called a signed star (j,k)-dominating family (of functions) on G. The maximum number of functions in a signed star (j,k)-dominating family on G is the signed star (j,k)-domatic number of G denoted by $d_{SS}^{(j,k)}(G)$.

In this paper we study properties of the signed star (j,k)-domatic number of a graph G. In particular, we determine bounds on $d_{SS}^{(j,k)}(G)$. Some of our results extend those ones given by Atapour, Sheikholeslami, Ghameslou and Volkmann [**1**] for the signed star domatic number, Sheikholeslami and Volkmann [**5**] for the signed star (k,k)-domatic number and Sheikholeslami and Volkmann [**4**] for the signed star k-domatic number.

1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We use [**2**] for terminology and notation which are not defined here and consider simple graphs without isolated vertices only. The integers $n = |V(G)|$ and $m = |E(G)|$ are the *order* and the *size* of the graph G, respectively. For every vertex $v \in V(G)$, the *open neighborhood* $N(v)$ of v is the set $\{u \in V(G) \mid uv \in E(G)\}$, and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v is $d(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The *complement* $\overline{G}$ of a graph G is the graph with vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if these vertices are not adjacent in G.

The *open neighborhood* $N_G(e)$ of an edge $e \in E(G)$ is the set of all edges adjacent to e. Its *closed neighborhood* is $N_G[e] = N_G(e) \cup \{e\}$. For a function $f\colon E(G) \longrightarrow \{-1,1\}$ and a subset S of $E(G)$, we define $f(S) = \sum_{e\in S} f(e)$. The *edge-neighborhood* $E_G(v) = E(v)$ of a vertex $v \in V(G)$ is the set of all edges

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incident with the vertex v. For each vertex $v \in V(G)$, we also define $f(v) = \sum_{e \in E_G(v)} f(e)$.

Let j be a positive integer. A function $f\colon E(G) \longrightarrow \{-1, 1\}$ is called a *signed star j-dominating function* (SSjDF) on G if $f(v) \geq j$ for every vertex v of G. The *signed star j-domination number* of a graph G is $\gamma_{jSS}(G) = \min\{\sum_{e \in E(G)} f(e) \mid f \text{ is a SSjDF on } G\}$. The signed star j-dominating function f on G with $f(E(G)) = \gamma_{jSS}(G)$ is called a $\gamma_{jSS}(G)$-*function.* As the assumption $\delta(G) \geq j$ is clearly necessary, we will always assume that satisfy $\delta(G) \geq j$ while discussing $\gamma_{jSS}(G)$ all graphs involved. The signed star j-domination number was introduced by Xu and Li [**10**] in 2009 and has been studied by several authors (see for instance, [**3, 4, 7**]). The signed star 1-domination number is the usual signed star domination number, introduced in 2005 by Xu [**8**]. The signed star domination number was investigated for example, by [**3, 6, 9**].

Let k be a further positive integer. A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed star j-dominating functions on G with $\sum_{i=1}^{d} f_i(e) \leq k$ for each $e \in E(G)$, is called a *signed star (j,k)-dominating family* (SS(j,k)D family) (of functions) on G. The maximum number of functions in a signed star (j,k)-dominating family on G is the *signed star (j,k)-domatic number* of G denoted by $d_{SS}^{(j,k)}(G)$. The signed star (j,k)-domatic number is well-defined and

$$d_{SS}^{(j,k)}(G) \geq 1 \tag{1}$$

for all graphs G with $\delta(G) \geq j$, since the set consisting of any signed star j-dominating function forms a SS(j,k)D family on G. A $d_{SS}^{(j,k)}$-*family* of a graph G is a SS(j,k)D family containing exactly $d_{SS}^{(j,k)}(D)$ signed star j-dominating functions. The signed star (1,1)-domatic number $d_{SS}^{(1,1)}(G)$ is the usual signed star domatic number $d_{SS}(G)$ which was introduced by Atapour, Sheikholeslami, Ghameslou and Volkmann [**1**] in 2010.

Our purpose in this paper is to initiate the study of the signed star (j,k)-domatic number in graphs. We study basic properties and bounds for the signed star (j,k)-domatic number $d_{SS}^{(j,k)}(G)$ of a graph G. In addition, we derive Nordhaus-Gaddum type results and bounds of the product and the sum of $\gamma_{jSS}(G)$ and $d_{SS}^{(j,k)}(G)$. Many of our results extend those given by Atapour, Sheikholeslami, Ghameslou and Volkmann [**1**] for the signed star domatic number, Sheikholeslami and Volkmann [**5**] for the signed star (k,k)-domatic number and Sheikholeslami and Volkmann [**4**] for the signed star k-domatic number.

Observation 1 ([**4**])**.** *Let G be a graph of size m with $\delta(G) \geq j$. Then $\gamma_{jSS}(G) = m$ if and only if each edge $e \in E(G)$ has an endpoint u such that $d(u) = j$ or $d(u) = j + 1$.*

2. Properties of the signed star (j,k)-domatic number

Theorem 2. *Let $j,k \geq 1$ be two integers. If G is a graph of minimum degree $\delta(G) \geq j$, then*

$$d_{SS}^{(j,k)}(G) \leq \frac{k\delta(G)}{j}.$$

Moreover, if $d_{SS}^{(j,k)}(G) = k\delta(G)/j$, then for each function of any signed star (j,k)-dominating family $\{f_1, f_2, \ldots, f_d\}$ with $d = d_{SS}^{(j,k)}(G)$ and for all vertices v of degree $\delta(G)$, $\sum_{e \in E_G(v)} f_i(e) = j$ and $\sum_{i=1}^{d} f_i(e) = k$ for every $e \in E_G(v)$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a signed star (j,k)-dominating family on G such that $d = d_{SS}^{(j,k)}(G)$. If $v \in V(G)$ is a vertex of minimum degree $\delta(G)$, then it follows that

$$\begin{aligned} d \cdot j = \sum_{i=1}^{d} j &\leq \sum_{i=1}^{d} \sum_{e \in E_G(v)} f_i(e) \\ &= \sum_{e \in E_G(v)} \sum_{i=1}^{d} f_i(e) \\ &\leq \sum_{e \in E_G(v)} k = k \cdot \delta(G), \end{aligned}$$

and this implies the desired upper bound on the signed star (j,k)-domatic number.

If $d_{SS}^{(j,k)}(G) = k\delta(G)/j$, then the two inequalities occurring in the proof become equalities, which leads to the two properties given in the statement. □

The special cases $j = k = 1$, $j = 1$ and $j = k$ in Theorem 2 can be found in [**1**], [**4**] and [**5**], respectively. As an application of Theorem 2, we will prove the following Nordhaus-Gaddum type result.

Corollary 3. *Let $j,k \geq 1$ be integers. If G is a graph of order n such that $\delta(G) \geq j$ and $\delta(\overline{G}) \geq j$, then*

$$d_{SS}^{(j,k)}(G) + d_{SS}^{(j,k)}(\overline{G}) \leq \frac{k}{j}(n-1).$$

If $d_{SS}^{(j,k)}(G) + d_{SS}^{(j,k)}(\overline{G}) = k(n-1)/j$, then G is regular.

Proof. Since $\delta(G) \geq j$ and $\delta(\overline{G}) \geq j$, it follows from Theorem 2 that

$$\begin{aligned} d_{SS}^{(j,k)}(G) + d_{SS}^{(j,k)}(\overline{G}) &\leq \frac{k\delta(G)}{j} + \frac{k\delta(\overline{G})}{j} \\ &= \frac{k}{j}(\delta(G) + (n - \Delta(G) - 1)) \leq \frac{k}{j}(n-1), \end{aligned}$$

and this is the desired Nordhaus-Gaddum inequality. If G is not regular, then $\Delta(G) - \delta(G) \geq 1$, and the above inequality chain leads to the better bound $d_{SS}^{(j,k)}(G) + d_{SS}^{(j,k)}(\overline{G}) \leq \frac{k}{j}(n-2)$. This completes the proof. □

Theorem 4. *Let $j, k \geq 1$ be integers. If v is a vertex of a graph G such that $d(v)$ is odd and j is even or $d(v)$ is even and j is odd, then*

$$d_{SS}^{(j,k)}(G) \leq \frac{k}{j+1} \cdot d(v).$$

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a signed star (j,k)-dominating family on G such that $d = d_{SS}^{(j,k)}(G)$. Assume first that $d(v)$ is odd and j is even. The definition yields to $\sum_{e \in E_G(v)} f_i(e) \geq j$ for each $i \in \{1, 2, \ldots, d\}$. On the left-hand side of this inequality a sum of an odd number of odd summands occurs. Therefore it is an odd number, and as j is even, we obtain $\sum_{e \in E_G(v)} f_i(e) \geq j+1$ for each $i \in \{1, 2, \ldots, d\}$. It follows that

$$\begin{aligned} k \cdot d(v) &= \sum_{e \in E_G(v)} k \geq \sum_{e \in E_G(v)} \sum_{i=1}^{d} f_i(e) \\ &= \sum_{i=1}^{d} \sum_{e \in E_G(v)} f_i(e) \geq \sum_{i=1}^{d} (j+1) = d(j+1), \end{aligned}$$

and this leads to the desired bound. Assume next that $d(v)$ is even and j is odd. Note that $\sum_{e \in E_G(v)} f_i(e) \geq j$ for each $i \in \{1, 2, \ldots, d\}$. On the left-hand side of this inequality a sum of an even number of odd summands occurs. Therefore it is an even number, and as j is odd, we obtain $\sum_{e \in E_G(v)} f_i(e) \geq j+1$ for each $i \in \{1, 2, \ldots, d\}$. Now the desired bound follows as above, and the proof is complete. $\square$

The next result is an immediate consequence of Theorem 4.

Corollary 5. *Let $j, k \geq 1$ be integers. If G is a graph such that $\delta(G)$ is odd and j is even or $\delta(G)$ is even and j is odd, then*

$$d_{SS}^{(j,k)}(G) \leq \frac{k}{j+1} \cdot \delta(G).$$

As an application of Corollary 5, we will improve the Nordhaus-Gaddum bound in Corollary 3 for many cases.

Theorem 6. *Let $j, k \geq 1$ be two integers and let G be a graph of order n such that $\delta(G) \geq j$ and $\delta(\overline{G}) \geq j$. If $\Delta(G) - \delta(G) \geq 1$ or j is odd or j is even and $\delta(G)$ is odd or j, $\delta(G)$ and n are even, then*

$$d_{SS}^{(j,k)}(G) + d_{SS}^{(j,k)}(\overline{G}) < \frac{k}{j}(n-1).$$

Proof. If $\Delta(G) - \delta(G) \geq 1$, then Corollary 3 implies the desired bound. Thus assume now that G is $\delta(G)$-regular.

Case 1. Assume that j is odd. If $\delta(G)$ is even, then from Theorem 2 and Corollary 5 it follows that

$$\begin{aligned} d_{SS}^{(j,k)}(G) + d_{SS}^{(j,k)}(\overline{G}) &\leq \frac{k}{j+1}\delta(G) + \frac{k}{j}\delta(\overline{G}) \\ &< \frac{k}{j}(\delta(G) + (n - \delta(G) - 1)) \\ &= \frac{k}{j}(n-1). \end{aligned}$$

If $\delta(G)$ is odd, then n is even and thus $\delta(\overline{G}) = n - \delta(G) - 1$ is even. Combining Theorem 2 and Corollary 5, we find that

$$\begin{aligned} d_{SS}^{(j,k)}(G) + d_{SS}^{(j,k)}(\overline{G}) &\leq \frac{k}{j}\delta(G) + \frac{k}{j+1}\delta(\overline{G}) \\ &< \frac{k}{j}(\delta(G) + (n - \delta(G) - 1) \\ &= \frac{k}{j}(n-1), \end{aligned}$$

and this completes the proof of Case 1.

Case 2. Assume that j is even. If $\delta(G)$ is odd, then from Theorem 2 and Corollary 5 it follows that

$$d_{SS}^{(j,k)}(G) + d_{SS}^{(j,k)}(\overline{G}) \leq \frac{k}{j+1}\delta(G) + \frac{k}{j}(n - \delta(G) - 1) < \frac{k}{j}(n-1).$$

If $\delta(G)$ is even and n is even, then $\delta(\overline{G}) = n - \delta(G) - 1$ is odd, and we obtain the desired bound as above. □

Theorem 7. *Let $j, k \geq 1$ be integers. If G is a graph such that k is odd and $d_{SS}^{(j,k)}(G)$ is even or k is even and $d_{SS}^{(j,k)}(G)$ is odd, then*

$$d_{SS}^{(j,k)}(G) \leq \frac{k-1}{j} \cdot \delta(G).$$

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a signed star (j,k)-dominating family on G such that $d = d_{SS}^{(j,k)}(G)$. Assume first that k is odd and d is even. If $e \in E(G)$ is an arbitrary edge, then $\sum_{i=1}^{d} f_i(e) \leq k$. On the left-hand side of this inequality a sum of an even number of odd summands occurs. Therefore, it is an even number, and as k is odd, we obtain $\sum_{i=1}^{d} f_i(e) \leq k-1$ for each $e \in E(G)$. If v is a vertex of minimum degree, then it follows that

$$\begin{aligned} d \cdot j = \sum_{i=1}^{d} j &\leq \sum_{i=1}^{d} \sum_{e \in E_G(v)} f_i(e) \\ &= \sum_{e \in E_G(v)} \sum_{i=1}^{d} f_i(e) \leq \sum_{e \in E_G(v)} (k-1) = \delta(G)(k-1), \end{aligned}$$

and this yields to the desired bound. Assume second that k is even and d is odd. If $e \in E(G)$ is an arbitrary edge, then $\sum_{i=1}^{d} f_i(e) \le k$. On the left-hand side of this inequality a sum of an odd number of odd summands occurs. Therefore, it is an odd number and as k is even, we obtain $\sum_{i=1}^{d} f_i(e) \le k-1$ for each $e \in E(G)$. Now the desired bound follows as above, and the proof is complete. □

The special cases $j = k = 1$, $j = 1$ and $j = k$ of Theorem 4, Corollary 5 and Theorem 7 can be found in [**1**], [**4**] and [**5**], respectively. According to (1), $d_{SS}^{(j,k)}(G)$ is a positive integer. If we suppose in the case $j = k = 1$ that $d_{SS}(G) = d_{SS}^{(1,1)}(G)$ is an even integer, then Theorem 7 leads to the contradiction $d_{SS}(G) \le 0$. Consequently, we obtain the next known result.

Corollary 8 ([**1**]). *The signed star domatic number $d_{SS}(G)$ is an odd integer.*

Proposition 9. *Let j, k be two integers such that $j \ge 1$ and $k \ge 2$, and let G be a graph with minimum degree $\delta(G) \ge j$. Then $d_{SS}^{(j,k)}(G) = 1$ if and only if each edge $e \in E(G)$ has an endpoint u such that $d(u) = j$ or $d(u) = j+1$.*

Proof. Assume that each edge $e \in E(G)$ has an endpoint u such that $d(u) = j$ or $d(u) = j+1$. It follows from Observation 1 that $\gamma_{jSS}(G) = m$ and thus $d_{SS}^{(j,k)}(G) = 1$.

Conversely, assume that $d_{SS}^{(j,k)}(G) = 1$. If G contains an edge $e = uv$ such that $d(u) \ge j+2$ and $d(v) \ge j+2$, then the functions $f_i\colon E(G) \to \{-1, 1\}$ such that $f_1(x) = 1$ for each $x \in E(G)$ and $f_2(e) = -1$ and $f_2(x) = 1$ for each edge $x \in E(G) \setminus \{e\}$ are signed star j-dominating functions on G such that $f_1(x) + f_2(x) \le 2 \le k$ for each edge $x \in E(G)$. Thus $\{f_1, f_2\}$ is a signed star (j,k)-dominating family on G, a contradiction to $d_{SS}^{(j,k)}(G) = 1$. □

The next result is an immediate consequence of Observation 1 and Proposition 9.

Corollary 10. *Let j, k be two integers such that $j \ge 1$ and $k \ge 2$, and let G be a graph with minimum degree $\delta(G) \ge j$. Then $d_{SS}^{(j,k)}(G) = 1$ if and only if $\gamma_{jSS}(G) = m$.*

Next we present a lower bound on the signed star (j,k)-domatic number.

Proposition 11. *Let j, k be two integers such that $k \ge j \ge 1$, and let G be a graph with minimum degree $\delta(G) \ge j$. If G contains a vertex $v \in V(G)$ such that all vertices of $N[N[v]]$ have degree at least $j+2$, then $d_{SS}^{(j,k)}(G) \ge j$.*

Proof. Let $\{u_1, u_2, \ldots, u_j\} \subset N(v)$. The hypothesis that all vertices of $N[N[v]]$ have degree at least $j+2$ implies that the functions $f_i\colon E(G) \to \{-1, 1\}$ such that $f_i(vu_i) = -1$ and $f_i(x) = 1$ for each edge $x \in E(G) \setminus \{vu_i\}$ are signed star j-dominating functions on G for $i \in \{1, 2, \ldots, j\}$. Since $f_1(x) + f_2(x) + \ldots + f_j(x) \le j \le k$ for each edge $x \in E(G)$, we observe that $\{f_1, f_2, \ldots, f_j\}$ is a signed star (j,k)-dominating family on G, and Proposition 11 is proved. □

Corollary 12. *Let j, k be two integers such that $k \ge j \ge 1$. If G is a graph of minimum degree $\delta(G) \ge j+2$, then $d_{SS}^{(j,k)}(G) \ge j$.*

Corollary 13. *Let $j,k \geq 1$ be integers, and let G be an r-regular graph with $r \geq j$.*

(1) *If $j \leq r \leq j+1$, then $d_{SS}^{(j,k)}(G) = 1$.*

(2) *If $r = j+2p+1$ with an integer $p \geq 1$ and $k \geq j$, then $j \leq d_{SS}^{(j,k)}(G) \leq \frac{kr}{j+1}$.*

(3) *If $r = j+2p$ with an integer $p \geq 1$ and $k \geq j$, then $j \leq d_{SS}^{(j,k)}(G) \leq \frac{kr}{j}$.*

Proof. (1) Assume that $j \leq r \leq j+1$. According to Observation 1, $\gamma_{jSS}(G) = m$ and thus $d_{SS}^{(j,k)}(G) = 1$.

(2) Assume that $r = j+2p+1$ with $p \geq 1$. The condition $k \geq j$ and Corollary 12 imply that $j \leq d_{SS}^{(j,k)}(G)$. If j is even, then $r = j+2p+1$ is odd, and if j is odd, then $r = j+2p+1$ is even, Therefore, Corollary 5 leads to the desired upper bound of $d_{S,S}^{(j,k)}(G)$.

(3) Assume that $r = j+2p$ with $p \geq 1$. The condition $k \geq j$ and Corollary 12 imply that $j \leq d_{SS}^{(j,k)}(G)$. In addition, Theorem 2 yields the desired upper bound of $d_{S,S}^{(j,k)}(G)$. □

3. Bounds on the product and the sum of $\gamma_{jSS}(G)$ and $d_{SS}^{(j,k)}(G)$

Note that $\gamma_{jSS}(G) = m$ implies immediately $d_{SS}^{(j,k)}(G) = 1$, and so $\gamma_{jSS}(G) \cdot d_{SS}^{(j,k)}(G) = m$ and $\gamma_{jSS}(G) + d_{SS}^{(j,k)}(G) = m+1$. In this section, we present general bounds of the product and the sum of $\gamma_{jSS}(G)$ and $d_{SS}^{(j,k)}(G)$.

Theorem 14. *Let $j,k \geq 1$ be integers. If G is a graph of size m and minimum degree $\delta(G) \geq j$, then*

$$\gamma_{jSS}(G) \cdot d_{SS}^{(j,k)}(G) \leq mk.$$

Moreover, if $\gamma_{jSS}(G) \cdot d_{SS}^{(j,k)}(G) = mk$, then for each $d_{SS}^{(j,k)}$-family $\{f_1, f_2, \cdots, f_d\}$ of G, each function f_i is a $\gamma_{jSS}(G)$-function and $\sum_{i=1}^{d} f_i(e) = k$ for all $e \in E(G)$.

Proof. If $\{f_1, f_2, \ldots, f_d\}$ is a signed star (j,k)-dominating family on G such that $d = d_{SS}^{(j,k)}(G)$, then the definitions imply

$$\begin{aligned} d \cdot \gamma_{jSS}(G) &= \sum_{i=1}^{d} \gamma_{jSS}(G) \leq \sum_{i=1}^{d} \sum_{e \in E(G)} f_i(e) \\ &= \sum_{e \in E(G)} \sum_{i=1}^{d} f_i(e) \leq \sum_{e \in E(G)} k = mk \end{aligned}$$

as desired.

If $\gamma_{jSS}(G) \cdot d_{SS}^{(j,k)}(G) = mk$, then the two inequalities occurring in the proof become equalities. Hence for the $d_{SS}^{(j,k)}$-family $\{f_1, f_2, \ldots, f_d\}$ of G and for each i, $\sum_{e \in E(G)} f_i(e) = \gamma_{jSS}(G)$, thus each function f_i is a $\gamma_{jSS}(G)$-function and $\sum_{i=1}^{d} f_i(e) = k$ for all $e \in E(G)$. □

Theorem 15. *Let $j, k \geq 1$ be integers. If G is a graph of size m and minimum degree $\delta(G) \geq j$, then*

$$d_{SS}^{(j,k)}(G) + \gamma_{jSS}(G) \leq mk + 1.$$

Proof. According to Theorem 14, we have

$$d_{SS}^{(j,k)}(G) + \gamma_{jSS}(G) \leq d_{SS}^{(j,k)}(G) + \frac{km}{d_{SS}^{(j,k)}(G)}.$$

Using the fact that the function $g(x) = x + (km)/x$ is decreasing for $1 \leq x \leq \sqrt{km}$ and increasing for $\sqrt{km} \leq x \leq km$, we obtain

$$d_{SS}^{(j,k)}(G) + \gamma_{jSS}(G) \leq \max\left\{1 + mk, mk + \frac{km}{km}\right\} = mk + 1.$$

$\square$

Next we improve Theorem 15 considerably.

Theorem 16. *Let $j, k \geq 1$ be two integers. If G is a graph of size m and minimum degree $\delta(G) \geq j$, then*

$$\gamma_{jSS}(G) + d_{SS}^{(j,k)}(G) \leq \begin{cases} m+1 & \text{if } k = 1, \\ \frac{mk}{2} + 2 & \text{if } k \geq 2. \end{cases}$$

Proof. If $k = 1$, then Theorem 15 leads to the desired bound. Therefore we assume next that $k \geq 2$. If the order $n = 2$, then $\gamma_{jSS}(G) = m = 1$ and $d_{SS}^{(j,k)}(G) = 1$ and hence the desired bound is valid. Now we assume that $n \geq 3$. Let f be a SSjDF on G. Since $\sum_{e \in E_G(v)} f(e) \geq j$ for every vertex v of G, it follows that

$$2 \sum_{e \in E(G)} f(e) = \sum_{v \in V(G)} \sum_{e \in E_G(v)} f(e) \geq \sum_{v \in V(G)} j = nj.$$

This implies $\gamma_{jSS}(G) \geq nj/2$. As $n \geq 3$ and $j \geq 1$, we obtain $\gamma_{jSS}(G) \geq 2$. Theorem 14 implies that

$$\gamma_{jSS}(G) + d_{SS}^{(j,k)}(G) \leq \gamma_{jSS}(G) + \frac{mk}{\gamma_{jSS}(G)}.$$

If we define $x = \gamma_{jSS}(G)$ and $g(x) = x + (mk)/x$ for $x > 0$, then because $2 \leq \gamma_{jSS}(G) \leq m$, we have to determine the maximum of the function g in the interval $I : 2 \leq x \leq m$. Using the condition $k \geq 2$ and the fact that $m \geq 2$, it is easy to see that

$$\begin{aligned} \max_{x \in I}\{g(x)\} &= \max\{g(2), g(m)\} \\ &= \max\left\{2 + \frac{mk}{2}, m + \frac{mk}{m}\right\} \\ &= \frac{mk}{2} + 2, \end{aligned}$$

and the proof is complete. $\square$

Theorem 17. *Let $j, k \geq 1$ be two integers. If G is a graph of size m, minimum degree $\delta(G) \geq j$ and order $n \geq 2p+1$ for an integer $p \geq 1$, then*

$$\gamma_{jSS}(G) + d_{SS}^{(j,k)}(G) \leq \begin{cases} m+k & \text{if } 1 \leq k \leq p, \\ \frac{mk}{p+1} + p + 1 & \text{if } k \geq p+1. \end{cases}$$

Proof. We proceed by induction on p. Theorem 16 shows that the statement is valid for $p = 1$. Now let $p \geq 2$ and assume that the statement is true for all integers $1 \leq i \leq p-1$. Then the induction hypothesis implies that $\gamma_{jSS}(G) + d_{SS}^{(j,k)}(G) \leq m + k$ for $1 \leq k \leq p-1$. Thus assume next that $k \geq p$. The hypothesis $n \geq 2p+1$ leads as in the proof of Theorem 16 to

$$\gamma_{jSS}(G) \geq \frac{nj}{2} \geq \frac{(2p+1)j}{2} \geq \frac{2p+1}{2}$$

and thus $p + 1 \leq \gamma_{jSS}(G) \leq m$. Therefore, it follows from Theorem 14 that

$$\begin{aligned} \gamma_{jSS}(G) + d_{SS}^{(j,k)}(G) &\leq \gamma_{jSS}(G) + \frac{mk}{\gamma_{jSS}(G)} \\ &\leq \max\left\{p+1+\frac{mk}{p+1}, m+k\right\}. \end{aligned} \tag{2}$$

Note that the hypothesis $n \geq 2p+1$ yields to $m \geq p+1$.

If $k = p$, then we deduce from the inequality $m \geq p+1$ that

$$\max\left\{p+1+\frac{mk}{p+1}, m+k\right\} = \max\left\{p+1+\frac{mp}{p+1}, m+p\right\} = m+p.$$

If $k \geq p+1$, then

$$p+1+\frac{mk}{p+1} \geq m+k$$

is equivalent with $m(k-p-1) \geq (p+1)(k-p-1)$, and this inequality is valid since $k \geq p+1$ and $m \geq p+1$. Hence the desired result follows from (2), and the proof is complete. □

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