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A NOTE ON SPHERICAL F–TILINGS BY RIGHT TRIANGLES

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ABSTRACT. In this paper we present some spherical f-tilings by two (distinct) right triangles. We classify the group of symmetries of the presented tilings and the transitivity classes of isohedrality are also determined. The combinatorial structure is given in Table 1.

1. INTRODUCTION

Let S^2 be the Riemannian sphere of radius 1. By a dihedral *folding tiling* (*f-tiling* for short) of the sphere S^2 whose prototiles are spherical right triangles, T_1 and T_2, we mean a polygonal subdivision τ of S^2 such that each *cell* (*tile*) of τ is congruent to T_1 or T_2, and the vertices of τ satisfy the *angle-folding relation*, i.e., each vertex of τ is of even valency and the sums of alternating angles around each vertex are π. In this paper we shall discuss dihedral f-tilings by two spherical right triangles.

F-tilings are intrinsically related to the theory of isometric foldings of Riemannian manifolds introduced by S. A. Robertson [**4**] in 1977. In fact, the set of singularities of any spherical isometric folding corresponds to a folding tiling of the sphere.

We shall denote by $\Omega\,(T_1, T_2)$ the set, up to an isomorphism, of all dihedral f-tilings of S^2 whose prototiles are T_1 and T_2. From now T_1 is a spherical right triangle of internal angles $\frac{\pi}{2}$, α and β with edge lengths a (opposite to β), b (opposite to α) and c (opposite to $\frac{\pi}{2}$), and T_2 is a spherical right triangle of internal angles $\frac{\pi}{2}$, γ and δ with edge lengths d (opposite to δ), e (opposite to γ) and f (opposite to $\frac{\pi}{2}$) (see Figure 1). We will assume throughout the text that T_1 and T_2 are distinct triangles, i.e., $(\alpha, \beta) \neq (\gamma, \delta)$ and $(\alpha, \beta) \neq (\delta, \gamma)$. The case $\alpha = \beta$ or $\gamma = \delta$ was analyzed in [**2**], and so we will assume further that $\alpha \neq \beta$ and $\gamma \neq \delta$.

It follows straightway that

$$\alpha + \beta > \frac{\pi}{2} \text{ and } \gamma + \delta > \frac{\pi}{2}. \tag{1}$$

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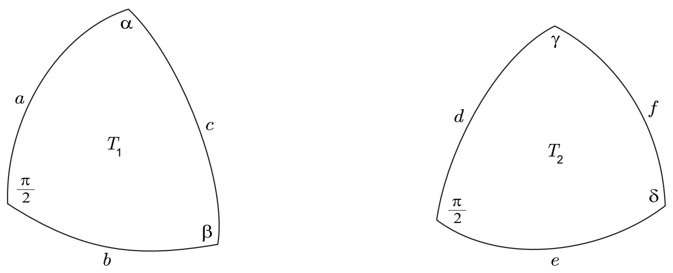


Figure 1. Prototiles: spherical right triangles T_1 and T_2.

A spherical isometry σ is a *symmetry* of a spherical tiling τ if σ maps every tile of τ into a tile of τ. The set of all symmetries of τ is a group under composition of maps denoted by $G(\tau)$. In this paper the group of symmetries of each f-tiling $\tau \in \Omega\,(T_1, T_2)$ will also be presented.

We say that the tiles T and T' of τ are in the same *transitivity class*, if the symmetry group $G(\tau)$ contains a transformation that maps T into T'. If all the tiles of τ form one transitivity class we say that τ is *tile-transitive* or *isohedral.* If there are k transitivity classes of tiles, then τ is k*-isohedral* or k*-tile-transitive.* Dihedral f-tilings are k-isohedral for $k \geq 2$. In this paper we also determine the transitivity classes of isohedrality of each presented tiling.

A *fundamental region* of τ is a part of S^2 as small or irredundant as possible which determines τ based on its symmetries. More precisely, the image of a point in S^2 under the symmetry group of τ forms an orbit of the action. A fundamental region of τ is a subset of S^2 which contains exactly one point from each orbit, therefore if $|G(\tau)| = n$, then the area of a fundamental region of τ is $\frac{4\pi}{n}$.

It is well known that any spherical isometry is either a reflection, a rotation or a glide-reflection which consists of reflecting through some spherical great circle, and then rotating around the line orthogonal to the great circle and containing the origin.

Let v and v' be vertices of a spherical f-tiling τ and let σ be a symmetry of τ such that $\sigma(v) = v'$. Then every symmetry of τ that sends v into v' is composition of σ with a symmetry of τ fixing v'. On the other hand, the isometries that fix v' are exactly the rotations around the line containing $\pm v'$ and the reflections through the great circles by $\pm v'$.

In what follows, R^x_θ, R^y_θ and R^z_θ denote the rotations through an angle θ around the xx axis, yy axis and zz axis, respectively. The reflections on the coordinate planes xy, xz and yz are denoted, respectively, by ρ^{xy}, ρ^{xz} and ρ^{yz} with the notation used in [**1**] . For instance, it follows that $R^x_\theta \rho^{xy} = \rho^{xy} R^x_{-\theta}$, $R^x_\theta R^y_\pi = R^y_\pi R^x_{-\theta}$, $\rho^{xy} R^z_\theta = R^z_\theta \rho^{xy}$ and $\rho^{xy}\rho^{yz} = \rho^{yz}\rho^{xy} = R^y_\pi$. Besides, $2k$ is the smallest positive integer such that $(\rho^{xy} R^z_{\frac{\pi}{k}})^{2k} = id$.

The nth dihedral group D_n (group of symmetries of the planar regular n-gon) consists of n rotations and n symmetries (reflections). If a is a rotation of order n and b is a symmetry, then $\langle\, a, b \,:\, a^n = 1,\, b^2 = 1,\, ba = a^{n-1}b \,\rangle$ is a group presentation for D_n. Moreover, the elements 1, a, ..., a^{n-1}, b, ab, ..., $a^{n-1}b$ are pairwise disjoints.

In the next section we present some examples of f-tilings by right-triangles on a case of adjacency. The complete classification of all f-tilings by the considered prototiles is far from being achieved. We believe that this very hard study leads to infinite families of f-tilings (with discrete or continuous parameters) without no patterns, precisely due to the fact that both prototiles are right triangles. In contrast with other prototiles (consider, for instance, the rectangle illustrated in Figure 2(a) with $\alpha > \frac{\pi}{2}$ that cannot be subdivided in two tiles of the same family), the right triangles are special prototiles since they can be subdivided into two new right triangles (therefore within the same family of prototiles), and so on (see Figure 2(b)).

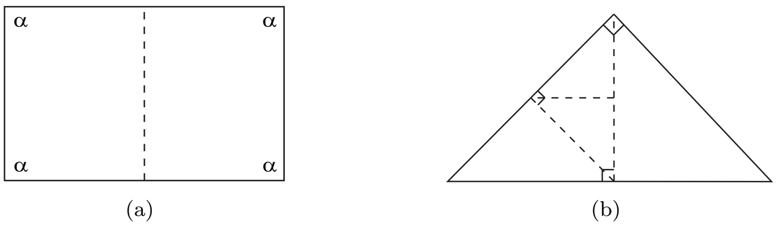


Figure 2. Prototiles subdivisions.

2. Some examples of f–tilings by right–triangles on a case of adjacency

We will suppose that any element of $\Omega(T_1, T_2)$ has at least two cells congruent, to T_1 and T_2, respectively, such that they are in adjacent positions as illustrated in Figure 3.

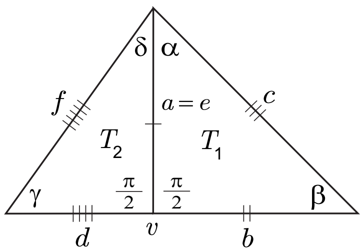


Figure 3. Case of adjacency.

As $a = e$, using spherical trigonometric formulas, we obtain

$$\frac{\cos\beta}{\sin\alpha} = \frac{\cos\gamma}{\sin\delta}. \tag{2}$$

We will assume that all the edges of T_1 and T_2 are pairwise distinct (except a and e). The study of right triangular spherical dihedral f–tilings on this case of adjacency, where the prototiles have two pairs of congruent sides, was already presented in [**3**].

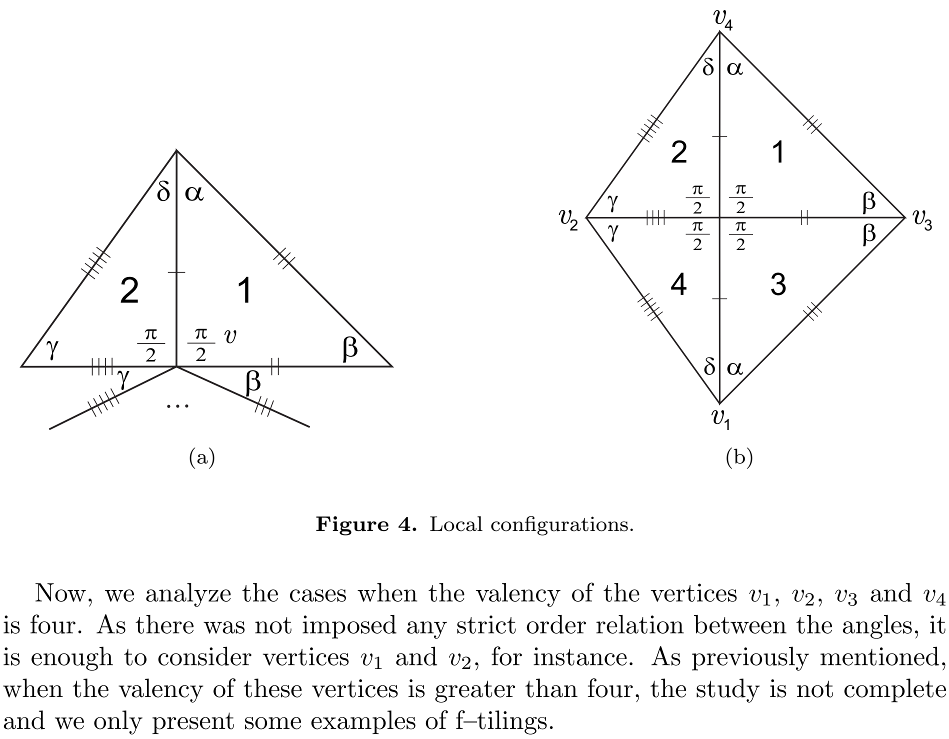
In order to pursue any dihedral f-tiling $\tau \in \Omega(T_1, T_2)$, we start by considering one of its *local configurations*, beginning with a common vertex to two tiles of τ in adjacent positions, and then enumerating the following tiles according to the angles and edges relations until a complete f-tiling or an impossibility is achieved.

Lemma 2.1. *With the previous assumptions vertex v (Figure 3) has valency four.*

Proof. Suppose that vertex v has valency greater than four (Figure 4(a)). Then, if

- $\beta > \alpha$, we must have $\frac{\pi}{2} + \beta + k\gamma = \pi$ or $\frac{\pi}{2} + \beta + k\delta = \pi$ for some $k \geq 1$. In the first case an incompatibility between sides cannot be avoided around vertex v. In the last case we obtain $\frac{\pi}{2} + \beta + k\delta = \pi = \frac{\pi}{2} + \alpha + \gamma + (k-1)\delta$ which is not possible (observe that $\frac{\pi}{2} + \frac{\pi}{2} + \alpha + \beta + \gamma + \delta > 2\pi$).
- $\beta < \alpha$, one gets $\frac{\pi}{2} + k_1\beta + k_2\gamma = \pi$ or $\frac{\pi}{2} + k_1\beta + k_2\delta = \pi$ for some $k_1, k_2 \geq 1$. Analogously, in the first case an incompatibility between sides cannot be avoided around vertex v. In the last case we obtain $\frac{\pi}{2} + k_1\beta + k_2\delta = \pi = \frac{\pi}{2} + \alpha + \gamma + (k_1 - 1)\beta + (k_2 - 1)\delta$ which is not possible.

Therefore, vertex v has valency four as illustrated in Figure 4(b). □



(a) (b)

Figure 4. Local configurations.

Now, we analyze the cases when the valency of the vertices v_1, v_2, v_3 and v_4 is four. As there was not imposed any strict order relation between the angles, it is enough to consider vertices v_1 and v_2, for instance. As previously mentioned, when the valency of these vertices is greater than four, the study is not complete and we only present some examples of f–tilings.

Proposition 2.2. *Let T_1 and T_2 be spherical right triangles such that they are in adjacent positions as illustrated in Figure 4(b). If at least one of the vertices v_1 and v_2 has valency four, then $\Omega(Q,T) \neq \emptyset$ iff $\alpha+\delta=\pi$ and $\beta=\gamma=\frac{\pi}{k}$ for some $k \geq 3$. In this case, for each $k \geq 3$, there is a family of f-tilings denoted by $\mathcal{R}_{\alpha}^{k}$ with $\alpha \in \left(\frac{(k-2)\pi}{2k}, \frac{(k+2)\pi}{2k}\right)$. Planar and 3D representations are given in Figure 6.*

Proof. Suppose that any element of $\Omega(T_1, T_2)$, has at least two cells congruent, respectively, to T_1 and T_2, such that they are in adjacent positions as illustrated in Figure 4(b).

1. If v_1 has valency four, taking into account the edge lengths, we must have $\alpha+\delta=\pi$ and the last configuration is extended to the one illustrated in Figure 5. Repeating the same argument, we get $\beta=\gamma=\frac{\pi}{k}$ with $k \geq 3$. The extension of the last configuration gives rise to the "closed" planar representation, see Figure 6(a). We denote this family of f–tilings by $\mathcal{R}_{\alpha}^{k}$ with $\alpha \in \left(\frac{(k-2)\pi}{2k}, \frac{(k+2)\pi}{2k}\right)$ and $k \geq 3$. A 3D representation of $\mathcal{R}_{\alpha}^{k}$ for $k=3$, is given in Figure 6(b).

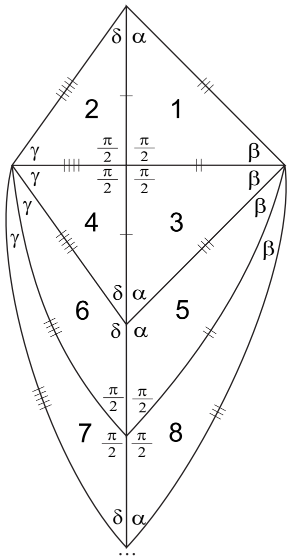
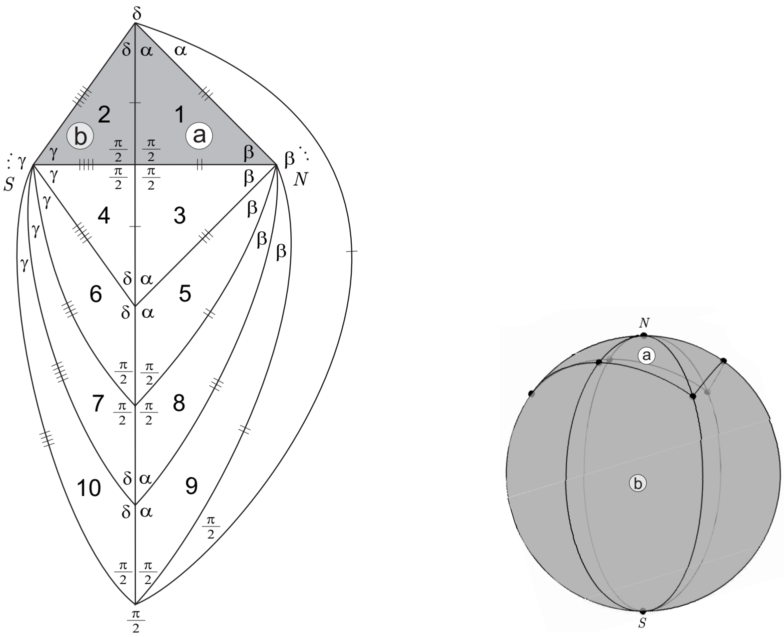


Figure 5. Local configuration.

In the planar representation, the dark region corresponds to a fundamental region of $\mathcal{R}_{\alpha}^{3}$. In fact, any symmetry of this f-tiling fixes the north vertex N; the symmetries that fix N are the rotations $id = R_0^z$, $R_{\frac{2\pi}{3}}^z$, $R_{\frac{4\pi}{3}}^z$, and the reflections ρ^{yz}, $\rho_1 = \rho^{yz} \circ R_{\frac{2\pi}{3}}^z$, $\rho_2 = \rho^{yz} \circ R_{\frac{4\pi}{3}}^z$, and so the symmetry group of $\mathcal{R}_{\alpha}^{3}$ is isomorphic to D_3 generated by $R_{\frac{2\pi}{3}}^z$ and ρ^{yz}. Similarly, we prove that $G\left(\mathcal{R}_{\alpha}^{k}\right)$ is isomorphic



(a) Planar representation of $\mathcal{R}_\alpha^k$ (b) 3D representation of $\mathcal{R}_\alpha^3$

Figure 6. f-tilings $\mathcal{R}_\alpha^k$, with $\alpha \in \left(\frac{(k-2)\pi}{2k}, \frac{(k+2)\pi}{2k}\right)$ and $k \geq 3$.

to D_k. It follows immediately that $\mathcal{R}_\alpha^k$ is 2-isohedral with respect to the symmetry group.

2. Suppose now that v_2 has valency four. As $\gamma \neq \frac{\pi}{2}$ (otherwise $a = f$), it follows that $\delta + \gamma = \pi$ (see Figure 7). At vertex v_1 we must have $\gamma + k\alpha = \pi$, $k \geq 1$. However, an incompatibility between sides cannot be avoided around this vertex for all k. □

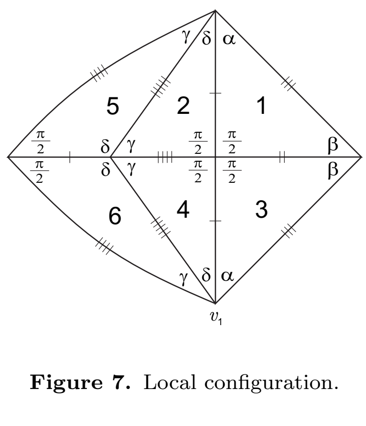
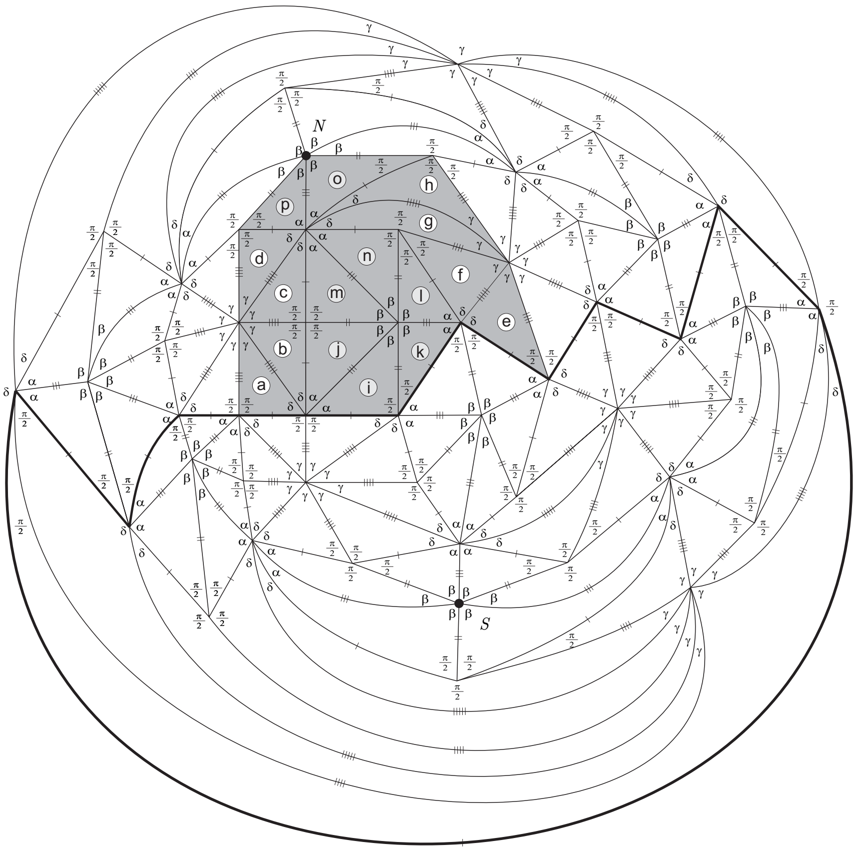
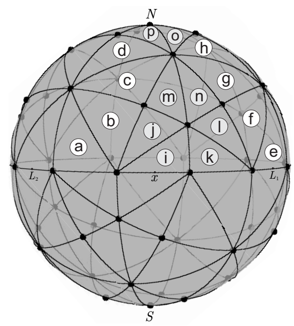


Figure 7. Local configuration.



(a) Planar representation of $\mathcal{D}_a$



(b) 3D representation of $\mathcal{D}_a$

Figure 8. f-tiling $\mathcal{D}_a$.

Given the reasons referred earlier, when the valency of all the vertices v_i, $i = 1, 2, 3, 4$, is greater than four, we present examples of f–tilings satisfying:

(i) $\alpha + \delta = \frac{\pi}{2}$, $\beta = \frac{\pi}{3}$, $\gamma = \frac{\pi}{4}$ and $\delta = \arctan\sqrt{2}$,

(ii) $\alpha + 2\delta = \pi$, $\gamma = \frac{\pi}{2k}$ and $\beta = \frac{\pi}{3}$, $k = 2, 3$,

(iii) $\delta + \alpha + \gamma = \pi$, $\beta = \frac{\pi}{3}$ and $\alpha \in \left(\frac{\pi}{6}, \frac{\pi}{3}\right)$,

(iv) $\alpha + 3\delta = \pi$, $\gamma = \frac{\pi}{4}$ and $\beta = \frac{\pi}{3}$.

In the first case we consider three f-tilings, say $\mathcal{D}_a$, $\mathcal{D}_b$ and $\mathcal{D}_c$.

In Figure 8 a planar and 3D representations of $\mathcal{D}_a$ are given. The dark region in the planar representation corresponds to a fundamental region of $\mathcal{D}_a$. In fact, the dark line corresponding to a great circle composed by 12 segments of length $\frac{\pi}{6}$ is invariant under any symmetry. Thus, any symmetry of $\mathcal{D}_a$ fixes N or maps N into S. The symmetries that fix N are generated by the rotation $R^z_{\frac{2\pi}{3}}$ and the symmetries that send N into S are R^x_π, $R^{L_1}_\pi = R^z_{\frac{2\pi}{3}} \circ R^x_\pi$ and $R^{L_2}_\pi = R^z_{\frac{4\pi}{3}} \circ R^x_\pi$. Note that $\left(R^{L_i}_\pi\right)^2 = id$, $i = 1, 2$, and the symmetry group of $\mathcal{D}_a$ is $G(\mathcal{D}_a) = \langle R^z_{\frac{2\pi}{3}}, R^x_\pi \rangle \simeq D_3$; $\mathcal{D}_a$ is 16-isohedral (8 transitivity classes of triangles T_1 and 8 transitivity classes of triangles T_2).

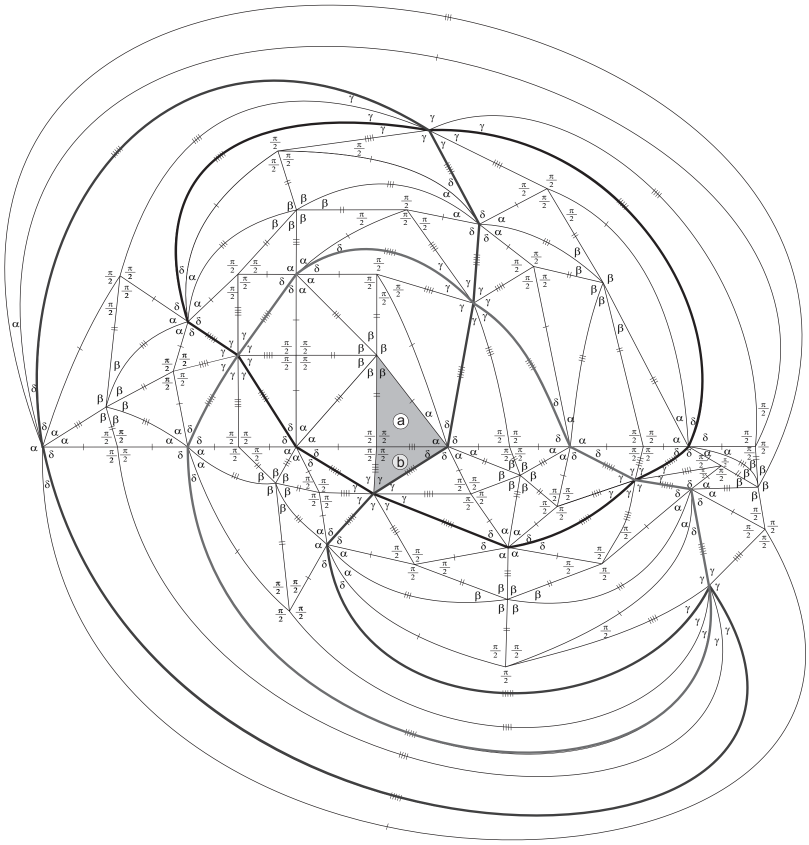
The f-tilings $\mathcal{D}_b$ and $\mathcal{D}_c$ (Figure 9 and Figure 10) are obtained from $\mathcal{D}_a$ rotating the southern hemisphere $\frac{\pi}{6}$ for the left and the right, respectively.

A planar and 3D representations of $\mathcal{D}_b$ are illustrated in Figure 9. The three great circles $x = 0$, $y = 0$ and $z = 0$ depicted in the planar representation have 8 segments of length $\frac{\pi}{4}$ and there exist exactly six vertices surrounded by 8 angles γ (say N, S, W, E, C, L). The symmetries that fix vertex N are generated by $R^z_{\frac{\pi}{2}}$ and ρ^{yz} with 8 symmetries (for the other vertices the analysis is similar). And so the symmetry group contains 48 symmetries. It follows immediately that $G(\mathcal{D}_b)$ is isomorphic to $C_2 \times S_4$, the octahedral group, and $\mathcal{D}_b$ is 2-isohedral.

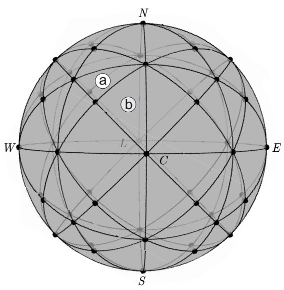
Concerning to the tiling $\mathcal{D}_c$, the dark line (corresponding to the equator) is invariant under any symmetry. Thus, any symmetry of $\mathcal{D}_c$ fixes N (and S) or maps N into S (and vice-versa). The symmetries that fix this points are generated by the rotation $R^z_{\frac{2\pi}{3}}$ and the reflection ρ^{yz}. On the other hand, the reflection ρ^{xy} sends N into S and commutes with the previous symmetries. It follows that $G(\mathcal{D}_c)$ is isomorphic to $C_2 \times D_3$ and $\mathcal{D}_c$ is 8-isohedral. Finally, the dark region in the planar representation of $\mathcal{D}_c$ corresponds to a fundamental region of this tiling.
In the case (ii) we consider four f-tilings denoted by $\mathcal{L}$, $\mathcal{M}$, $\mathcal{Q}^2$ and $\mathcal{Q}^3$.

A planar representation of $\mathcal{L}$ is illustrated in Figure 11(a). One has $\beta = \frac{\pi}{3}$, $\gamma = \frac{\pi}{4}$, $\delta = \arccos\frac{\sqrt{2}}{4} \approx 69.3°$ and $\alpha = \pi - 2\delta \approx 41.4°$. Its 3D representation is given in Figure 11(b).

Similarly to the previous case, any symmetry of $\mathcal{L}$ fixes N or maps N into S. The symmetries that fix N (and S) are generated, for instance, by the rotation $R^z_{\frac{2\pi}{3}}$ and the reflection ρ^{yz} giving rise to a subgroup G of $G(\mathcal{L})$ isomorphic to D_3. To obtain the symmetries that send N into S, it is enough to compose each



(a) Planar representation of $\mathcal{D}_b$

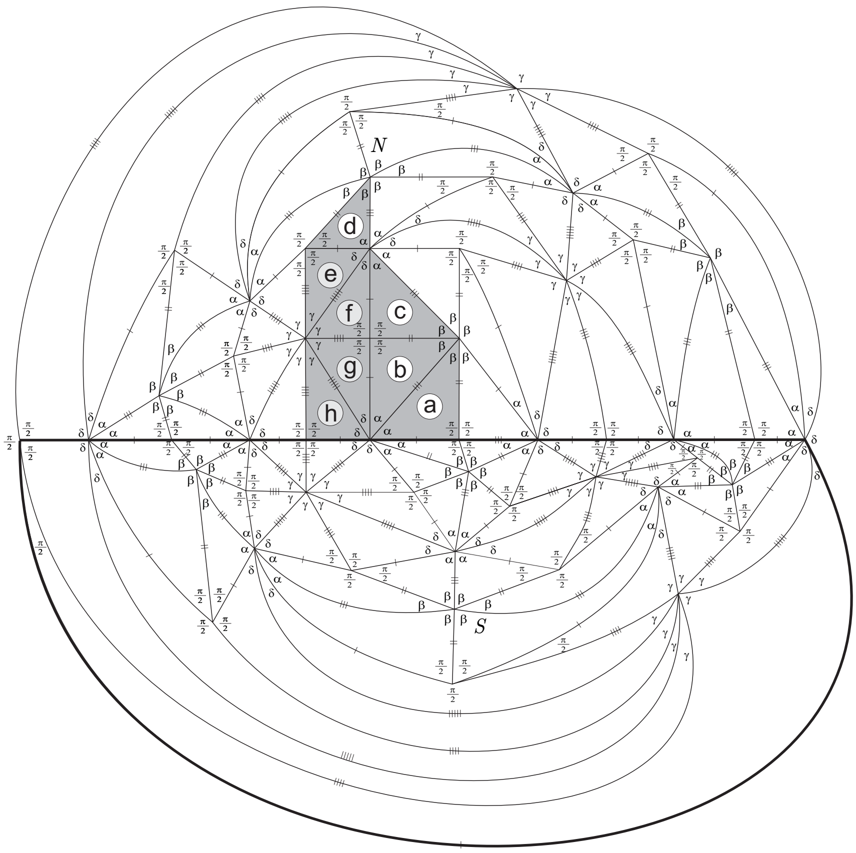


(b) 3D representation of $\mathcal{D}_b$

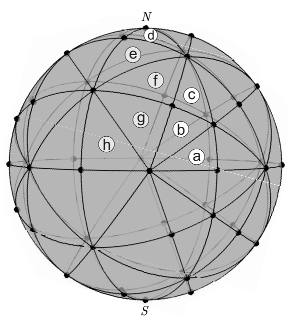
Figure 9. f-tiling $\mathcal{D}_b$.

element of G with $a = R^z_{\frac{\pi}{3}} \rho^{xy}$. Now, one has

$$a^5 \rho^{yz} = R^z_{\frac{5\pi}{3}} \rho^{xy} \rho^{yz} = R^z_{\frac{5\pi}{3}} R^y_{\pi} = R^y_{\pi} R^z_{\frac{\pi}{3}} = \rho^{yz} \rho^{xy} R^z_{\frac{\pi}{3}} = \rho^{yz} a.$$



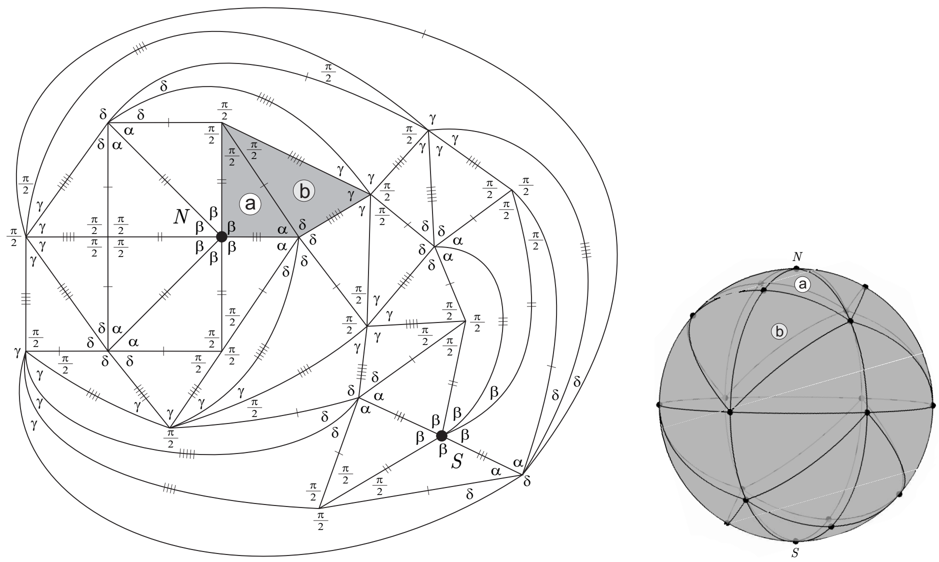
(a) Planar representation of $\mathcal{D}_c$



(b) 3D representation of $\mathcal{D}_c$

Figure 10. f-tiling $\mathcal{D}_c$.

Moreover, $| < a > | = 6$ and $\rho^{yz} \notin < a >$. Therefore, $< a, \rho^{yz} >= G(\mathcal{L}) \simeq D_6$ and $\mathcal{L}$ is 2-isohedral. Observe that $R^x_\pi \in G(\mathcal{L})$; in fact, $R^x_\pi = \rho^{yz} R^z_{\frac{2\pi}{3}} \circ R^z_{\frac{\pi}{3}} \rho^{xy}$.



(a) Planar representation of $\mathcal{L}$ (b) 3D representation of $\mathcal{L}$

Figure 11. f-tiling $\mathcal{L}$.

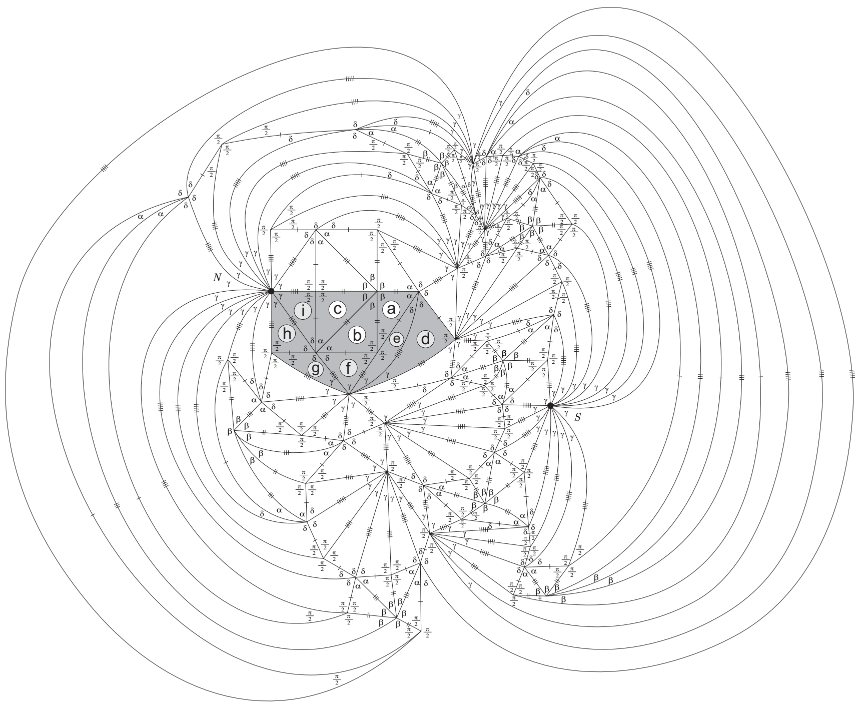
A planar representation of $\mathcal{M}$ is illustrated in Figure 12(a). One has $\beta = \frac{\pi}{3}$, $\gamma = \frac{\pi}{8}$, $\delta = \operatorname{arcsec}\left(4\cos\frac{\pi}{8}\right) \approx 74.3°$ and $\alpha = \pi - 2\delta \approx 31.4°$. Its 3D representation is given in Figure 12(b).

$G(\mathcal{M})$ contains a subgroup S isomorphic to D_4 generated by $R^z_{\frac{\pi}{2}}$ and ρ^{yz}. On the other hand, $a = \rho^{xy} R^z_{\frac{\pi}{4}}$ is also a symmetry of $\mathcal{M}$ that maps N into S. Since a has order 4, then $G(\mathcal{M})$ is isomorphic to D_8 generated by a and ρ^{yz}. Finally, $\mathcal{M}$ is 9-isohedral.

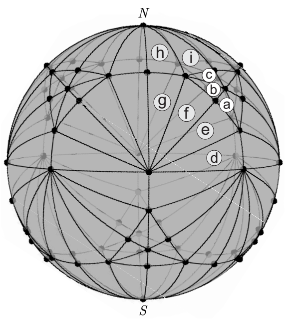
We illustrate planar and 3D representations of $\mathcal{Q}^2$ and $\mathcal{Q}^3$ in Figure 13 and Figure 14, respectively. One has $\beta = \frac{\pi}{3}$, $\gamma = \frac{\pi}{2k}$, $\delta = \operatorname{arcsec}\left(4\cos\frac{\pi}{2k}\right)$ and $\alpha = \pi - 2\delta$, $k = 2, 3$.

The great circle $x = 0$ depicted in the planar representation of $\mathcal{Q}^2$ has 6 segments of length $\frac{\pi}{3}$. Any symmetry of $\mathcal{Q}^2$ fixes C or maps C into L. As before, the symmetries that fix C are generated, for instance, by the rotation $R^x_{\frac{2\pi}{3}}$ and the reflection ρ^{xz}, and so $G(\mathcal{Q}^2)$ contains a subgroup G isomorphic to D_3. In order to obtain all the symmetries that send C into L, it is enough to compose each element of G with ρ^{yz} which commutes with all elements of G. And so $G(\mathcal{Q}^2) \cong C_2 \times D_3$. It follows immediately that $\mathcal{Q}^2$ is 3-isohedral.

The tiling $\mathcal{Q}^3$ has exactly four vertices surrounded by 12 angles γ and denoted by v_i, $i = 1, 2, 3, 4$. Any symmetry of $\mathcal{Q}^3$ that sends v_i into v_j, $i \neq j$, consists of a reflection on the great circle containing the remaining vertices. On the other hand, the symmetries of $\mathcal{Q}^3$ fixing one of these four vertices form a subgroup G



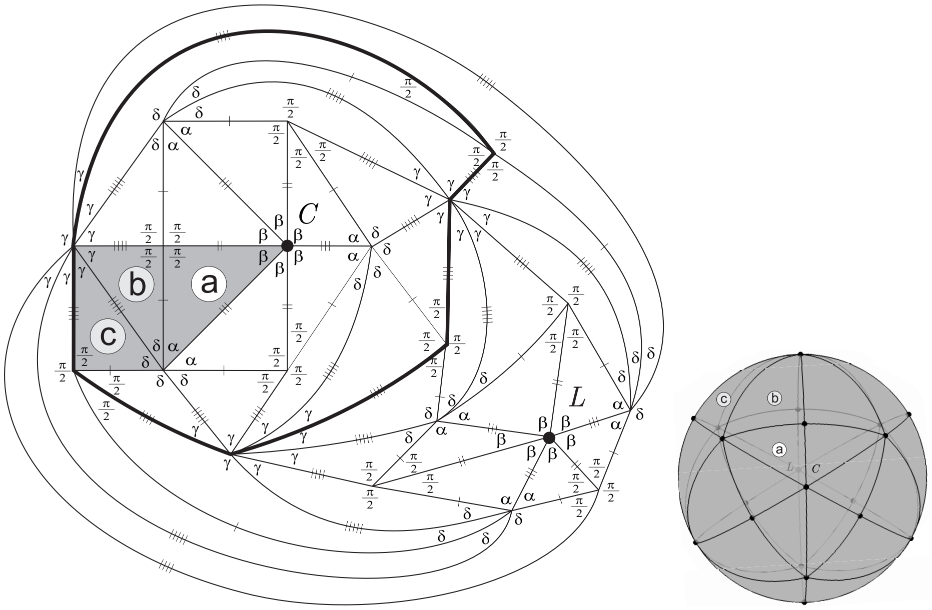
(a) Planar representation of $\mathcal{M}$



(b) 3D representation of $\mathcal{M}$

Figure 12. f-tiling $\mathcal{M}$.

isomorphic to D_3. Thus, $G(\mathcal{Q}^3)$ contains exactly 24 symmetries. Now, we easily conclude that $G(\mathcal{Q}^3)$ is the group of all symmetries of the regular tetrahedron or the group of all permutations of four objects, S_4. Finally, $\mathcal{Q}^3$ is 3-tile-transitive with respect to this group.



(a) Planar representation of $\mathcal{Q}^2$

(b) 3D representation of $\mathcal{Q}^2$

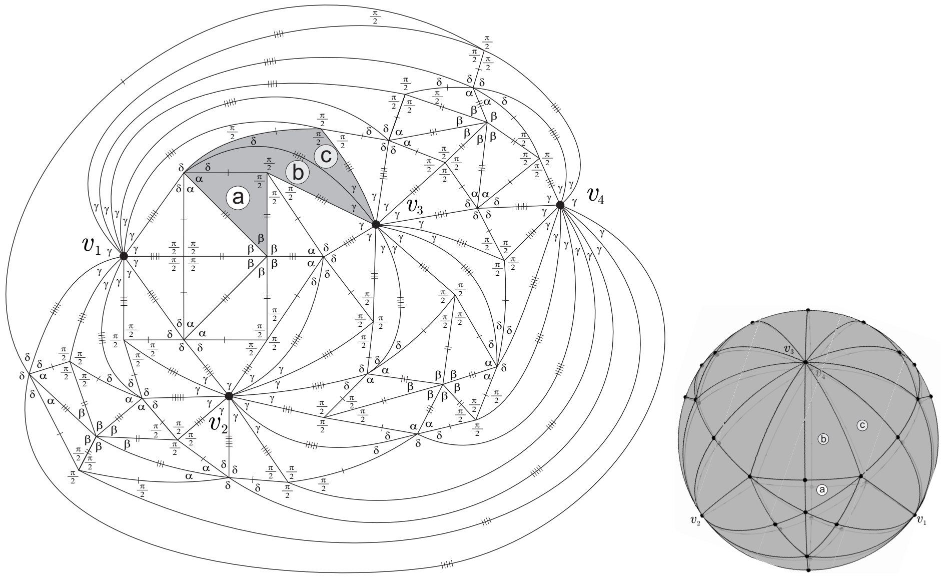
Figure 13. f-tiling $\mathcal{Q}^2$.

In the case (iii) we consider a family of f-tilings denoted by $\mathcal{G}_\alpha$, $\alpha \in \left(\frac{\pi}{6}, \frac{\pi}{3}\right)$. The corresponding planar and 3D representations are illustrated in Figure 15. Due to the condition (2) we have $\alpha = \gamma$, and so $\delta = \pi - 2\alpha$.

$G(\mathcal{G}_\alpha)$ contains a subgroup G isomorphic to D_3 generated by $R^z_{\frac{2\pi}{3}}$ and ρ^{yz}. On the other hand, $a = \rho^{xy} R^z_{\frac{\pi}{3}}$ is a symmetry of $\mathcal{G}_\alpha$ that maps N into S. Similarly to some previous cases, we conclude that $G(\mathcal{G}_\alpha)$ is isomorphic to D_6 generated by a and ρ^{yz}. Finally, $\mathcal{G}_\alpha$ is 2-isohedral.

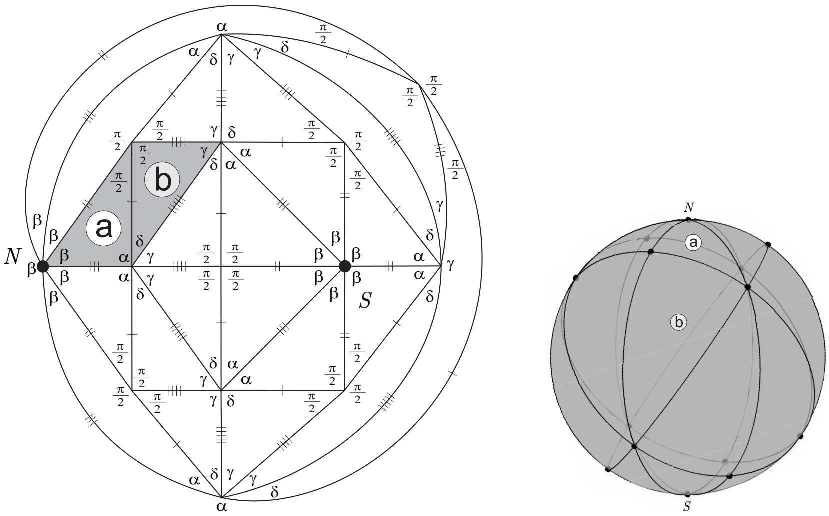
In the last case we consider an f-tiling denoted by $\mathcal{C}$ whose planar and 3D representations are presented in Figure 16. One has $\beta = \frac{\pi}{3}$, $\gamma = \frac{\pi}{4}$, $\delta = \frac{1}{2} \arccos \frac{\sqrt{2}-2}{4} \approx 49.2°$ and $\alpha = \pi - 3\delta \approx 32.4°$.

The dark line in the planar representation corresponds to the equator that is invariant under any symmetry of $\mathcal{C}$. With the labeling used in this figure, the symmetries that fix N are generated, for instance, by the rotation $R^z_{\frac{\pi}{2}}$ and the reflection ρ^{yz} giving rise to a subgroup G of $G(\mathcal{C})$ isomorphic to D_4. On the other hand, $a = \rho^{xy} R^z_{\frac{\pi}{4}}$ is also a symmetry of $\mathcal{C}$ that maps N into S. The symmetry group of $\mathcal{C}$ is then $G(\mathcal{C}) = < a, \rho^{yz} > \simeq D_8$, $\mathcal{C}$ is 12-isohedral.



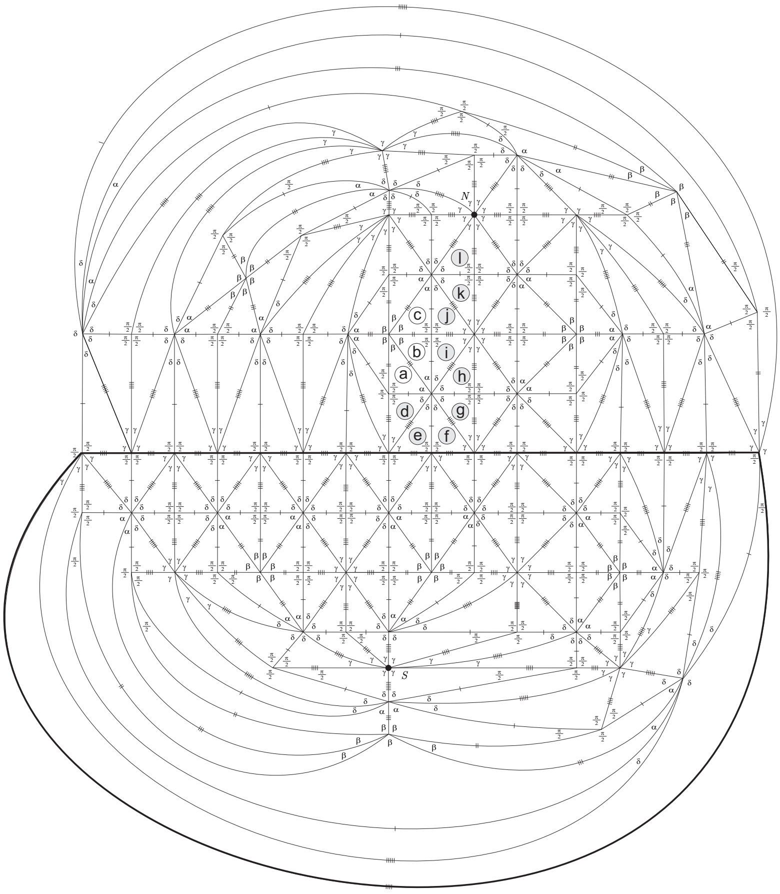
(a) Planar representation of $\mathcal{Q}^3$ (b) 3D representation of $\mathcal{Q}^3$

Figure 14. f-tiling $\mathcal{Q}^3$.

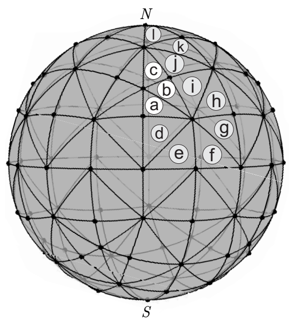


(a) Planar representation of $\mathcal{G}_\alpha$, $\alpha \in \left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ (b) 3D representation of $\mathcal{G}_\alpha$, $\alpha \in \left(\frac{\pi}{6}, \frac{\pi}{3}\right)$

Figure 15. f-tiling $\mathcal{G}_\alpha$, $\alpha \in \left(\frac{\pi}{6}, \frac{\pi}{3}\right)$.



(a) Planar representation of $\mathcal{C}$



(b) 3D representation of $\mathcal{C}$

Figure 16. f-tiling $\mathcal{C}$.

3. Summary

In Table 1 is shown a list of the presented spherical dihedral f-tilings whose prototiles are spherical right triangles, T_1 and T_2, of internal angles $\frac{\pi}{2}$, α, β, and $\frac{\pi}{2}$, γ, δ, respectively, in the case of adjacency illustrated in Figure 3. Our notation is as follows:

- $|V|$ is the number of distinct classes of congruent vertices,
- N_1 and N_2, respectively, are the number of triangles congruent to T_1 and T_2, respectively, used in each tiling.
- $G(\tau)$ is the symmetry group of each tiling $\tau \in \Omega(T_1, T_2)$; by C_n we mean the cyclic group of order n; D_n is the dihedral group of order $2n$; S_4 is the group of all permutations of four distinct objects; the octahedral group is $O_h \cong C_2 \times S_4$ (the symmetry group of the cube).
- $I(\tau)$ corresponds to the number of isohedrality classes of each tiling $\tau \in \Omega(T_1, T_2)$.

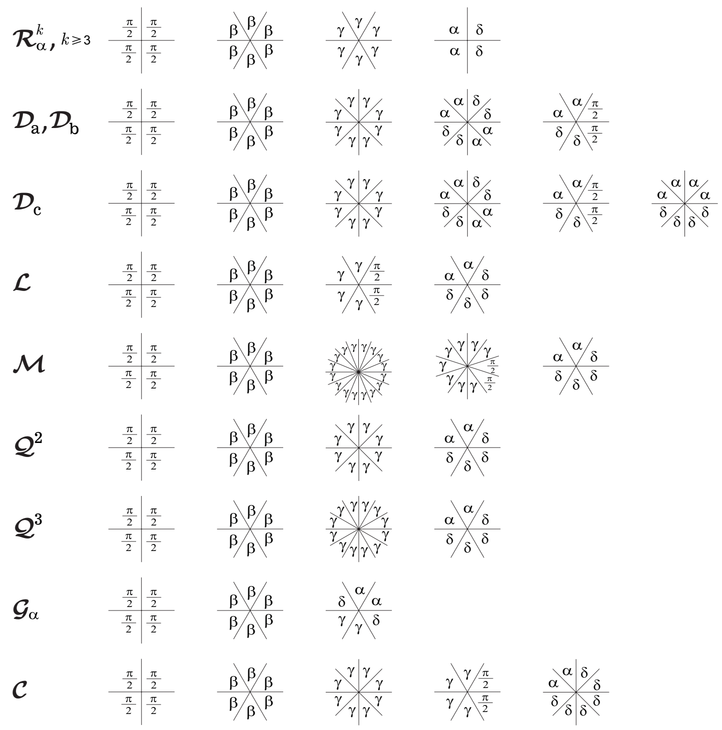


Figure 17. Distinct classes of congruent vertices.

The distinct classes of congruent vertices of each f-tiling are illustrated in Figure 17.

f-tiling	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{\gamma}$	$\boldsymbol{\delta}$	$\|V\|$	N_1	N_2	$G(\tau)$	$I(\tau)$
$\mathcal{R}^k_\alpha$	$\left(d\frac{(k-2)\pi}{2k}, \frac{(k+2)\pi}{2k}\right)$	$\frac{\pi}{k}$	$\frac{\pi}{k}$	$\pi-\alpha$	4	2k	2k	D_k	2
$\mathcal{D}_a$	$\frac{\pi}{2}-\delta$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\arctan\sqrt{2}$	5	48	48	D_3	16
$\mathcal{D}_b$	$\frac{\pi}{2}-\delta$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\arctan\sqrt{2}$	5	48	48	$C_2\times S_4$	2
$\mathcal{D}_c$	$\frac{\pi}{2}-\delta$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\arctan\sqrt{2}$	6	48	48	$C_2\times D_3$	8
$\mathcal{L}$	$\pi-2\delta$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\arccos\frac{\sqrt{2}}{4}$	4	12	24	D_6	2
$\mathcal{M}$	$\pi-2\delta$	$\frac{\pi}{3}$	$\frac{\pi}{8}$	$\operatorname{arcsec}\left(4\cos\frac{\pi}{8}\right)$	5	48	96	D_8	9
$\mathcal{Q}^2$	$\pi-2\delta$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\operatorname{arcsec}\left(4\cos\frac{\pi}{4}\right)$	4	12	24	$C_2\times D_3$	3
$\mathcal{Q}^3$	$\pi-2\delta$	$\frac{\pi}{3}$	$\frac{\pi}{6}$	$\operatorname{arcsec}\left(4\cos\frac{\pi}{6}\right)$	4	24	48	S_4	3
$\mathcal{G}_\alpha$	$\left(\frac{\pi}{6},\frac{\pi}{3}\right)$	$\frac{\pi}{3}$	α	$\pi-2\alpha$	3	12	12	D_6	2
$\mathcal{C}$	$\pi-3\delta$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{1}{2}\arccos\frac{\sqrt{2}-2}{4}$	5	48	144	D_8	12

Table 1. Combinatorial structure of some dihedral f–tilings of S^2 by right triangles on the case of adjacency of Figure 3.

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