ON $b$-ORDER DUNFORD-PETTIS OPERATORS
AND THE $b$-AM-COMPACTNESS PROPERTY

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Abstract. In this paper, we introduce $b$-order Dunford-Pettis operators, that is, an operator $T$ from a normed Riesz space $E$ into a Banach space $X$ is called $b$-order Dunford-Pettis if $T$ carries each $b$-order bounded subset of $E$ into a Dunford-Pettis subset of $X$, and we investigate its relationship with order Dunford-Pettis operators. We also introduce the $b$-AM-compactness property for a Banach lattice and we study some of its topological properties and its relationships with the Dunford-Pettis property. We show that the identity operator of Banach lattice $E$ is $b$-order Dunford-Pettis if and only if $E$ has the $b$-AM-compactness property. We characterize Banach lattices $E$ and $F$ on which the adjoint of each operator from $E$ into $F$ which is $b$-order Dunford-Pettis and weak Dunford-Pettis, is Dunford-Pettis.

1. Introduction

Let us recall that a norm bounded subset $A$ of a Banach space $X$ is a Dunford-Pettis set whenever every weakly compact operator from $X$ to an arbitrary Banach space carries $A$ to a norm totally bounded set. An operator $T : X \to Y$ between two Banach spaces is called a Dunford-Pettis operator if $T$ carries weakly convergent sequences to norm convergent sequences. A Banach space $X$ is said to have the Dunford-Pettis property if every weakly compact operator $T$ defined on $X$ and taking values in a Banach space $Y$ is Dunford-Pettis. For example, the Banach space $\ell^\infty$ has the Dunford-Pettis property but the Banach space $\ell^2$ does not have the Dunford-Pettis property. In [6], Aqzzouz and Bouras introduced the AM-compactness property for Banach lattices. A Banach lattice $E$ is said to have the AM-compactness property if every weakly compact operator defined on $E$ and taking values in a Banach space $X$ is AM-compact. For example, the Banach lattice $\ell^1$ has the AM-compactness property, but $L^2[0,1]$ does not have the AM-compactness property. They used the AM-compactness property to characterize Banach lattices on which each positive weak Dunford-Pettis operator is almost Dunford-Pettis, and conversely. They proved that Banach lattice $E$ has the AM-compactness property if and only if for each $x \in E^+$, $[-x,x]$ is a Dunford-Pettis set. Also, they proved that a Banach lattice $E$ has the AM-compactness property if and only if for every weakly null sequence $\{f_n\} \subset E'$, we have $|f_n| \to 0$ for

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\( \sigma(E', E) \). They showed that Banach lattice \( E \) with the Dunford-Pettis property and order continuous norm has the \( AM \)-compactness property. In this paper, we introduce the \( b \)-\( AM \)-compactness property and investigate Banach lattices which under some conditions have the \( b \)-\( AM \)-compactness property.

The class of order Dunford-Pettis operators was introduced by Aqzzouz and Bouras in [5]. An operator \( T \) from a normed Riesz space \( E \) into a Banach space \( X \) is called order Dunford-Pettis if it carries each order bounded subset of \( E \) onto a Dunford-Pettis set of \( X \). For example, the identity operator of Banach lattice \( c_0 \) is order Dunford-Pettis. They studied the class of Dunford-Pettis sets in Banach lattices, and establish some sufficient conditions for which a Dunford-Pettis set is relatively weakly compact (resp., relatively compact). They proved that Banach lattice \( E \) has the \( AM \)-compactness property if and only if the identity operator of \( E \) is order Dunford-Pettis. In this paper, we introduce \( b \)-order Dunford-Pettis operators and prove some of their properties. Then we study relationship between order Dunford-Pettis operators and \( b \)-order Dunford-Pettis operators. We show that the identity operator of Banach lattice \( E \) is \( b \)-order Dunford-Pettis if and only if \( E \) has the \( b \)-\( AM \)-compactness property. Bouras, El Kaddouri, H'Michane, and Moussa characterized Banach lattices \( E \) and \( F \) on which the adjoint of each operator from \( E \) into \( F \) which is order Dunford-Pettis and weak Dunford-Pettis, is Dunford-Pettis, see [10]. In this paper, we characterize Banach lattices \( E \) and \( F \) on which the adjoint of each operator from \( E \) into \( F \) which is \( b \)-order Dunford-Pettis and weak Dunford-Pettis, is Dunford-Pettis.

2. Preliminary Information

We use the term operator \( T: E \to F \) between two Riesz spaces to mean a (maybe unbounded) linear mapping. Let \( E \) and \( F \) be two vector lattices (Riesz spaces), let \( x, y \in E \) with \( x \leq y \), and let the order interval \( [x, y] \) be the subset of \( E \) defined by \( [x, y] = \{ z \in E : x \leq z \leq y \} \). A subset of \( E \) is called order bounded if it is included in an order interval. Let \( T: E \to F \) be an operator between two Riesz spaces \( E \) and \( F \). \( T \) is order bounded if it maps order bounded subsets of \( E \) to order bounded subsets of \( F \).

By \( E' \) and \( E'' \), we denote the topological dual and topological bidual of \( E \), respectively. The vector space \( E^\sim \) of all order bounded linear functionals on \( E \) is called the order dual of \( E \). The vector space \( E^{\sim \sim} = (E^\sim)^\sim \) denotes the order bidual of \( E \). The algebraic adjoint of \( T \) denoted by \( T': F' \to E' \), and its order adjoint denoted by \( T^\sim: F^\sim \to E^\sim \).

The \( b \)-order bounded subsets of \( E \) are the order bounded in \( E^{\sim \sim} \). \( T \) is \( b \)-order bounded if it maps \( b \)-order bounded subsets of \( E \) to \( b \)-order bounded subsets of \( F \).

A vector lattice \( E \) is said to be discrete if it admits a complete disjoint system of discrete elements, where we say a nonzero element \( x \in E \) is discrete whenever the ideal generated by \( x \) coincides with the vector subspace generated by \( x \). A Banach lattice is a Banach space \((E, \|\cdot\|)\) such that \( E \) is a vector lattice and its norm satisfies the following property: for each \( x, y \in E \), if \( |x| \leq |y| \), then we have \( \|x\| \leq \|y\| \). A norm \( \|\cdot\| \) of a Banach lattice \( E \) is order continuous if for each net \( (x_\alpha)_{\alpha \in \Lambda} \) such that \( x_\alpha \downarrow 0 \), i.e., \( (x_\alpha) \) is decreasing and \( \inf\{x_\alpha : \alpha \in \Lambda\} = 0 \), we
have \( \|x_\alpha\| \to 0 \). A Banach lattice \( E \) is said to be a Kantorovich–Banach space (\( KB \)-space) whenever every increasing norm bounded sequence of \( E^+ \) is norm-convergent. If \( E \) is a Banach lattice and \( x \in E^+ \), then the principal ideal \( I_x \) generated by \( x \) is
\[
I_x = \{ y \in E : \text{there exists } \lambda > 0 \text{ with } |y| \leq \lambda x \},
\]
and thus \( I_x \) under the norm \( \|\cdot\|_\infty \), defined by
\[
\|y\|_\infty = \inf \{ \lambda > 0 : |y| \leq \lambda x \}, \quad y \in I_x,
\]
is an \( AM \)-space with the unit \( x \), whose closed unit ball is the order interval \([-x, x]\).

For a operator \( T : E \to F \) between two Riesz spaces, we say that its modulus \( |T| \) exists whenever \( |T| : = T \vee (-T) \) exists. By using [1, Theorem 1.18], for Riesz spaces \( E \) and \( F \) whenever \( F \) is Dedekind complete, each order bounded operator \( T : E \to F \) satisfies the following statement
\[
|T|(x) = \sup \{|Ty| : |y| \leq x\}
\]
for each \( x \in E^+ \). We refer to [1, 11] and [2] for any unexplained terms from vector lattice theory.

3. Main Results

In the following, we introduce the class of \( b \)-order Dunford-Pettis operators and the \( b \)-\( AM \)-compactness property, and we investigate some of their properties. We study application of these new concepts in the topological properties of Dunford-Pettis sets and operators.

\textbf{Definition 3.1.} An operator \( T \) from a normed Riesz space \( E \) into a Banach space \( X \) is called \( b \)-order Dunford-Pettis if \( T \) carries each \( b \)-order bounded subset of \( E \) into a Dunford-Pettis subset of \( X \).

For example, we know that \( \ell^1 \) has property (b) and its norm is order continuous. Therefore, if \( A \subset \ell^1 \) is a \( b \)-order bounded set, then it is weakly compact. On the other hand, \( \ell^1 \) has the Schur property. Thus, \( A \) is norm compact, and hence Dunford-Pettis. Therefore, the identity operator of Banach lattice \( \ell^1 \) is \( b \)-order Dunford-Pettis. Also, we can give Banach lattice \( c_0 \) as an example of a Banach lattice without property (b) that its identity operator is \( b \)-order Dunford-Pettis. Indeed, \( B_{c_0} \) (the closed unit ball of \( c_0 \)) is a Dunford-Pettis set by Lemma 3.5. Therefore, each norm bounded subset of \( c_0 \), specifically each \( b \)-order bounded subset of \( c_0 \), is Dunford-Pettis. But the identity operator of \( \ell^\infty \) is not a \( b \)-order Dunford-Pettis operator (since \([-1, 1]\), the closed unit ball of \( \ell^\infty \), is not Dunford-Pettis). Let \( E \) be a normed Riesz space and let \( X \) be a Banach space. Recall that an operator \( T \) from \( E \) into \( X \) is \( AM \)-compact (resp., \( b \)-\( AM \)-compact) if it maps order bounded (resp., \( b \)-order bounded) subset of \( E \) to relatively compact subset of \( X \). By \( K(E, X) \), \( AM(E, X) \), and \( AM_b(E, X) \), we denote the collection of compact,
AM-compact, and b-AM-compact operators from $E$ into $X$, respectively. Clearly, we have

$$K(E, X) \subset AM_b(E, X) \subset AM(E, X).$$

By $DP_o(E, X)$ and $DP_b(E, X)$, we denote the collection of order Dunford-Pettis and b-order Dunford-Pettis operators, respectively. It is clear that

$$AM(E, X) \subset DP_o(E, X), \quad AM_b(E, X) \subset DP_b(E, X).$$

Note that the inclusions may be proper. In fact, the identity operator $I: L^1[0, 1] \to L^1[0, 1]$ is both b-order Dunford-Pettis and order Dunford-Pettis but neither AM-compact nor b-AM-compact.

In the next example we show that the inclusion $DP_b(E, X) \subset DP_o(E, X)$ also may be proper. Before giving an example which shows that the above inclusion is strict, we need a preliminary Lemma.

**Lemma 3.2.** Let $X$ be a Banach space and let $A$ be a subset of $X$. Then, $A$ is a Dunford-Pettis set as a subset of $X$ if and only if $A$ is a Dunford-Pettis set as a subset of $X''$.

**Proof.** Let $A$ be a Dunford-Pettis set as a subset of $X$ and let $T: X'' \to Y$ be a weakly compact operator, where $Y$ is an arbitrary Banach space. Put $S = T|_X$. Therefore, $S$ is a weakly compact operator. Since $A \subset X$ is a Dunford-Pettis set, $S(A) = T(A)$ is relatively compact. Consequently, $A$ is a Dunford-Pettis set as a subset of $X''$.

Conversely, let $A$ be a Dunford-Pettis set as a subset of $X''$ and let $T$ be a weakly compact operator from $X$ into an arbitrary Banach space $Y$. We know that $T''': X'' \to Y''$ is weakly compact. Hence $T''(A)$ is relatively compact. Since $T = T''|_X$, $T(A)$ is relatively compact, the proof is complete. \qed

**Example 3.3.** Put

$$L = \{(a_n) \in c_0 \mid (na_n) \in c_0\}.$$  

It is easy to see that $L$ is a normed Riesz space and it is indeed an order ideal of $c_0$. First, we prove that $\ell^\infty$ is the topological bidual of $L$. It is sufficient to show that the linear operator $\psi: L' \to \ell^1$ defined by

$$\psi f = (f(e_n)),$$

is a well defined linear isometry which is also a lattice isomorphism, where $\{e_n\}$ is the standard basis of $L$. Let $\{e_n\}$ be the standard basis of $L$. Fix an arbitrary $f \in L'$. For each $n \in \mathbb{N}$, put

$$x_n(m) = \begin{cases} \text{sgn}(f(e_m)), & m = 1, 2, \ldots, n, \\ 0, & m > n \end{cases}$$

where sgn$(x)$ is the sign of $x \in \mathbb{R}$ (i.e., sgn$(0) = 0$ and sgn$(x) = x/|x|$ for $x \neq 0$). Clearly, $(x_n) \in L$ and $\|x_n\| \leq 1$ for all $n \in \mathbb{N}$. We have

$$\|f\| \geq f(x_n) = f\left(\sum_{k=1}^{n} \text{sgn}(f(e_k))e_k\right) = \sum_{k=1}^{n} \text{sgn}(f(e_k))f(e_k) = \sum_{k=1}^{n} |f(e_k)|.$$
Hence \( \sum_{k=1}^{\infty} |f(e_k)| \leq \|f\| \). That is, \( \psi \) is well defined. It is easy to see that \( \psi \) is a linear bijective map. The above argument also shows that

\[
\|\psi f\| = \|(f(e_n))\|_1 = \sum_{k=1}^{\infty} |f(e_k)| \leq \|f\|.
\]

On the other hand, let \( x = (x_1, x_2, \cdots) \in L \) such that \( \|x\| \leq 1 \). Then,

\[
\|f(x)\| = \left\| f \left( \sum_{k=1}^{\infty} x_k e_k \right) \right\| = \left\| \sum_{k=1}^{\infty} x_k f(e_k) \right\| \leq \sum_{k=1}^{\infty} |x_k| |f(e_k)| \leq \sum_{k=1}^{\infty} |f(e_k)|.
\]

Thus, \( \|f\| \leq \sum_{k=1}^{\infty} |f(e_k)| \). Therefore, \( \|\psi f\| = \|f\| \), that is, \( \psi \) is an onto isometry. Since \( \psi \) and \( \psi^{-1} \) are positive, it follows from [1, Theorem 2.15] that \( \psi \) is an onto lattice isomorphism. We proved that \( \ell^1 \) is the dual of \( L \), and hence \( \ell^\infty \) is the bidual of \( L \).

Now, we define \( T: L \to \ell^\infty \) as follows:

\[
T(a_1, a_2, \cdots) = (na_1, na_2, \cdots), \quad (a_1, a_2, \cdots) \in L.
\]

Let \( 0 \leq x = (x_n) \in c_0 \). Thus, \( T[0, x] \subset [0, (nx_n)] \). Since \( I: c_0 \to c_0 \) is order Dunford-Pettis, \( [0, (nx_n)] \) is a Dunford-Pettis set. Therefore, \( T[0, x] \) is Dunford-Pettis, that is, \( T \) is order Dunford-Pettis. Now, we claim that the operator \( T \) is not \( b \)-order Dunford-Pettis. Since the sequence \( \{e_n\} \), the standard basis of \( L \), is order bounded in \( \ell^\infty \), it is order bounded in \( L^{\infty} \). Therefore, \( \{e_n\} \) is a \( b \)-order bounded subset of \( L \).

\[
T(\{e_n | n \in \mathbb{N}\}) = \{ne_n | n \in \mathbb{N}\}.
\]

Therefore, \( T(\{e_n | n \in \mathbb{N}\}) \) is a norm unbounded subset of \( \ell^\infty \). Hence \( T(\{e_n | n \in \mathbb{N}\}) \) is not a Dunford-Pettis set. Consequently, \( T \) is not \( b \)-order Dunford-Pettis.

**Definition 3.4.** A Banach lattice \( E \) is said to have the \( b\)-AM-compactness property if every weakly compact operator from \( E \) into an arbitrary Banach space \( X \) is \( b\)-AM-compact.

For example, \( c_0 \) and \( \ell^1 \) have the \( b\)-AM-compactness property. Clearly, each Banach lattice with the \( b\)-AM-compactness property has the AM-compactness property. We believe that the converse is false in general. However, right now we do not have an example. Nonetheless, there exists a Banach lattice which does not have the \( b\)-AM-compactness property. In fact, \( \ell^\infty \) which does not have the AM-compactness property, does not have the \( b\)-AM-compactness property, either.

The next result characterizes the \( b \)-order Dunford-Pettis operators. For the proof of the next Theorem, we need the following Lemma.

**Lemma 3.5.** Let \( T \) be an operator from a normed space \( X \) into a Banach space \( Y \). Then, the adjoint \( T^*: Y^* \to X^* \) is Dunford-Pettis if and only if \( T(B_X) \) is a Dunford-Pettis set, where \( B_X \) is the closed unit ball of \( X \).

**Proof.** One can repeat the argument of [5, Remark 4]. \( \square \)

**Theorem 3.6.** Let \( T \) be an operator from a Banach lattice \( E \) into a Banach space \( X \). Then the following statements are equivalent:

1. \( T \) is \( b \)-order Dunford-Pettis.
2. \( T \) is Dunford-Pettis.
(a) \( T \) is \( b \)-order Dunford-Pettis operator,
(b) for each weakly compact operator \( S \) from \( X \) into an arbitrary Banach space \( Z \), the composed operator \( ST \) is \( b \)-AM-compact,
(c) for each \( 0 \leq x'' \in E'' \), the operator \( (T|_{Y})' \) is a Dunford-Pettis operator, where \( Y = I_{x''} \cap E \).

*Proof.* (a)⇒(b) Obvious.

(b)⇒(a) Let \( A \) be a \( b \)-order bounded subset of \( E \). It is sufficient to prove that an arbitrary weakly compact operator \( S \) from \( X \) into an arbitrary Banach space \( Z \) carries \( T(A) \) into a norm totally bounded set. Since by our hypothesis, \( ST \) is \( b \)-AM-compact, \( S(T(A)) = ST(A) \) is relatively compact. This proves that \( T \) is \( b \)-order Dunford-Pettis.

(a)⇒(c) Let \( 0 \leq x'' \in E'' \) and let \( Y = I_{x''} \cap E \). Since \( B_Y = [-x'', x''] \cap E \) is \( b \)-order bounded, \( T(B_Y) \) is a Dunford-Pettis set. Therefore, by using Lemma 3.5 the adjoint of the restriction of \( T \) to \( Y \) is a Dunford-Pettis operator.

(c)⇒(a) It sufficient to prove that for each \( 0 \leq x'' \in E'' \), \( T([-x'', x''] \cap E) \) is a Dunford-Pettis set.

Put \( Y = I_{x''} \cap E \). By our hypothesis, the adjoint of the restriction of \( T \) to \( Y \) is a Dunford-Pettis operator. So by using Lemma 3.5, the proof is complete. □

Some basic properties of \( b \)-order Dunford-Pettis operators are summarized in the next proposition. For an arbitrary pair of normed spaces \( X \) and \( Y \), the symbol \( B(X, Y) \) denote the vector space of all bounded operators from \( X \) into \( Y \).

**Proposition 3.7.** Let \( E \) and \( F \) be two normed Riesz spaces, and let \( X \) and \( Y \) be two Banach spaces. We have

(a) The vector space \( DP_b(E, X) \cap B(E, X) \) is a closed vector subspace of vector space \( B(E, X) \).

(b) The vector space \( DP_b(E, X) \cap B(E, X) \) is a closed vector subspace of vector space \( DP_b(E, X) \cap B(E, X) \).

(c) If \( T \in DP_b(E, X) \), then for each bounded operator \( S: X \to Y \), the composed operator \( ST \) is \( b \)-order Dunford-Pettis.

(d) If \( T: E \to F \) is a \( b \)-order bounded operator, then for each operator \( S \in DP_b(F, X) \), the composed operator \( ST \) is \( b \)-order Dunford-Pettis.

*Proof.* (a) Let \( S \) be a bounded operator in norm closure \( DP_b(E, X) \cap B(E, X) \). So there exist \( \{T_n\} \subset DP_b(E, X) \cap B(E, X) \) such that \( T_n \to S \) in norm. Let \( V \) be a weakly compact operator from \( X \) into an arbitrary Banach space \( Z \). We have \( \|VT_n - VS\| \to 0 \). By using Theorem 3.6, \( VT_n \) is a \( b \)-AM-compact operator for each \( n \). Therefore, by using [9, Theorem 2.9 (1)], \( VS \) is \( b \)-AM-compact. Now again by using Theorem 3.6, we have \( S \in DP_b(E, X) \cap B(E, X) \).

(b) It follows from the fact that \( DP_b(E, X) \cap B(E, X) \) is closed in \( B(E, X) \), see [10, Proposition 3.1 (a)].

(c) Let \( T \in DP_b(E, X) \) and let \( S: X \to Y \) be a bounded operator. We prove that \( ST \in DP_b(E, Y) \). By using Theorem 3.6, it is sufficient to prove that for a weakly compact operator \( V \) from \( Y \) into an arbitrary Banach space \( Z \), \( V(ST) \) is \( b \)-AM-compact. We know that \( VS \) is a weakly compact operator from \( X \) into \( Z \).
Since $T \in DP_b(E, F)$, by using Theorem 3.6 we conclude that $V(ST) = (VS)T$ is $b$-AM-compact. That is, $ST$ is a $b$-order Dunford-Pettis operator.

(d) Obvious. □

Recall from [3] that an operator $T: E \to F$ between two Riesz spaces is called strongly order bounded if it maps $b$-order bounded subsets of $E$ into order bounded subsets of $F$. A Riesz space $E$ is said to have the property (b) if every subset $A$ of $E$ is order bounded whenever it is order bounded in $E^\sim\sim$. In the following result, we give some sufficient conditions under which an operator is $b$-order Dunford-Pettis.

**Proposition 3.8.** (a) Let $E$ be a Banach lattice and let $X$ be a Banach space. An operator $T: E \to X$ is $b$-order Dunford-Pettis whenever its second adjoint $T'' : E'' \to X''$ is an order Dunford-Pettis operator.

(b) Let $E$ and $F$ be two normed Riesz spaces and let $X$ be a Banach space. If $T: E \to F$ is a strongly order bounded operator, then for each operator $S \in DP_o(F, X)$, the composed operator $ST$ is $b$-order Dunford-Pettis.

**Proof.** (a) Let $A$ be a $b$-order bounded subset of $E$. Since $A$ is order bounded in $E''$, and $T''$ is order Dunford-Pettis, $T''(A)$ is a Dunford-Pettis subset of $X''$. We know that $T(A) = T''(A)$. It follows from Lemma 3.2, $T(A)$ is a Dunford-Pettis subset of $X$. Therefore, $T$ is $b$-order Dunford-Pettis.

(b) Obvious. □

**Corollary 3.9.** Let $E$ be a normed Riesz space with property (b) and let $X$ be a Banach space. We have

$$DP_o(E, X) = DP_b(E, X).$$

**Proof.** It follows from part 3.8 of Proposition 3.8 and the fact that $I: E \to E$ is strongly order bounded. □

**Theorem 3.10.** Let $E$ be a Banach lattice that its bidual has order continuous norm and let $X$ be a Banach space with the Dunford-Pettis property. Then, each continuous operator $T: E \to X$ is $b$-order Dunford-Pettis.

**Proof.** We use similar proof techniques as those which were developed in [10, Proposition 3.2]. Let $T$ be an operator from $E$ into $X$ and let $A$ be a $b$-order bounded subset of $E$. Thus, $A$ is order bounded in $E''$. Therefore, by using [1, Theorem 4.9], $A$ is $\sigma(E'', E''')$-compact. Since $A \subset E$, $A$ is $\sigma(E, E')$-compact. Thus, $T(A)$ is relatively weakly compact. On the other hand, since $X$ has the Dunford-Pettis property, the identity operator of $X$ is weak Dunford-Pettis. Therefore, $T(A)$ is a Dunford-Pettis subset of $X$. □

In the next result we prove that a Banach lattice $E$ has the $b$-AM-compactness property if and only if its $b$-order bounded subsets are Dunford-Pettis.

**Theorem 3.11.** Let $E$ be a Banach lattice. Then the following statements are equivalent:

(a) $E$ has the $b$-AM-compactness property,

(b) $I: E \to E$ is $b$-order Dunford-Pettis,
(c) each positive operator from $E$ into $E$ is $b$-order Dunford-Pettis,
(d) for each $0 \leq x'' \in E''$, $[-x'', x''] \cap E$ is a Dunford-Pettis set.

Proof. (a) $\Rightarrow$ (b) It follows from Theorem 3.6.

(b) $\Rightarrow$ (c) Let $T : E \to E$ be a positive operator and let $A$ be a $b$-order bounded set in $E$. Since $T$ is positive, it is $b$-order bounded. Therefore, $T(A)$ is a $b$-order bounded set in $E$. So by our hypothesis, $T(A) = I(T(A))$ is a Dunford-Pettis set. This proves that $T$ is a $b$-order Dunford-Pettis operator.

(c) $\Rightarrow$ (b) Obvious.

(b) $\Rightarrow$ (a) Let $T$ be a weakly compact operator from $E$ into an arbitrary Banach space $X$. We prove that $T$ is $b$-AM-compact. Let $A$ be a $b$-order bounded set in $E$. Since $I$ is $b$-order Dunford-Pettis, $A = I(A)$ is a Dunford-Pettis set. Therefore, $T(A)$ is a totally bounded subset of $X$. Consequently, $T$ is a $b$-AM-compact operator.

(d) $\Rightarrow$ (b) Obvious.

(d) $\Rightarrow$ (a) Let $A$ be a $b$-order bounded subset of $E$. So, there exists $0 \leq x'' \in E''$ such that $A \subset [-x'', x''] \cap E$. Therefore, by our hypothesis, $A$ is a Dunford-Pettis set in $E''$.

The following theorem gives some sufficient conditions under which a Banach lattice has the $b$-AM-compactness property.

**Theorem 3.12.** A Banach lattice $E$ has the $b$-AM-compactness property if one of the following assertions is valid:

(a) the bidual of $E$ has order continuous norm and $E$ has the Dunford-Pettis property,
(b) the lattice operations in $E'$ are weakly sequentially continuous,
(c) the topological dual $E'$ is discrete with order continuous norm,
(d) $E$ has the AM-compactness property and each $b$-order bounded disjoint sequence in $E^+$ is Dunford-Pettis,
(e) $E$ is a discrete KB-space,
(f) $E$ has property (b) and the AM-compactness property,
(g) $E''$ has the AM-compactness property.

Proof. (a) Follows from Theorem 3.10 and Theorem 3.11.

(b) It is sufficient to show that for each $0 \leq x'' \in E''$, $[-x'', x''] \cap E$ is Dunford-Pettis. Let $\{x_n\}$ be a sequence in $[-x'', x''] \cap E$ and let $\{f_n\}$ be an arbitrary weakly null sequence in $E'$. We have

$$0 \leq |f_n(x_n)| \leq |f_n| |x_n| \leq |f_n| (x'').$$

It follows from our hypothesis and $f_n(x'') \to 0$ that $|f_n| (x'') \to 0$. Therefore, $f_n(x_n) \to 0$. The result follows from [4, Theorem 1].

(c) It follows from [8, Proposition 2.6] that the lattice operations in $E'$ are weakly sequentially continuous.
(d) Put \( A = [-x'', x''] \cap E \) for some \( 0 \leq x'' \in E'' \). It is sufficient to show that \( A \) is a Dunford-Pettis set. Since \( E \) has the AM-compactness property, for each \( x \in A^+ = [0, x''] \cap E, \{ x_n \} \) is Dunford-Pettis. If \( \{ x_n \} \) is a disjoint sequence in \( A^+ \), then by our hypothesis, \( \{ x_n \} \) is Dunford-Pettis. It follows from \([5, Corollary 2.13]\) that \( A \) is Dunford-Pettis.

(e) It follows from \([9, Corollary 2.10]\).

(f) Obvious.

(g) It follows from Lemma 3.2 and the fact that each subset of a Dunford-Pettis set is Dunford-Pettis. □

The next theorem characterizes Banach lattices \( E \) and \( F \) on which the adjoint of each operator from \( E \) into \( F \) which is \( b \)-order Dunford-Pettis and weak Dunford-Pettis, is Dunford-Pettis.

**Theorem 3.13.** Let \( E \) and \( F \) be two Banach lattices. The following assertions are equivalent:

(a) each order Dunford-Pettis and weak Dunford-Pettis operator \( T : E \rightarrow F \) has an adjoint \( T' : F' \rightarrow E' \) that is Dunford-Pettis,

(b) each \( b \)-order Dunford-Pettis and weak Dunford-Pettis operator \( T : E \rightarrow F \) has an adjoint \( T' : F' \rightarrow E' \) that is Dunford-Pettis,

(c) one of the following is valid:

(1) \( E' \) has order continuous norm,

(2) \( F' \) has the Schur property.

**Proof.** (a) \( \Rightarrow \) (b) Obvious.

(b) \( \Rightarrow \) (c) We use similar proof techniques as those which were developed in \([7, Theorem 3.5]\). Assume by way of contradiction that (c) is not correct, i.e., \( E' \) does not have order continuous norm and \( F' \) does not have the Schur property. We have to construct a \( b \)-order Dunford-Pettis and weak Dunford-Pettis operator \( T : E \rightarrow F \) such that its adjoint is not Dunford-Pettis. Since the norm of \( E' \) is not order continuous, by \([1, Theorem 4.69]\), \( \ell^1 \) embeds complementably in \( E \). That is, there exist, a positive projection \( P : E \rightarrow \ell^1 \). On the other hand, \( F' \) does not have the Schur property so there exists a weakly null sequence \( \{ f_n \} \) in \( F' \) such that \( \| f_n \| = 1 \) for all \( n \in \mathbb{N} \). Since \( \sup_{\| y \| \leq 1} \| f_n (y) \| = 1 \), there exists a sequence \( \{ y_n \} \) of positive elements in the closed unit ball of \( F \) such that for some \( \varepsilon > 0 \), we have \( \| f_n (y_n) \| \geq \varepsilon \) for all \( n \in \mathbb{N} \). We define the operator \( S : \ell^1 \rightarrow F \) as follow:

\[
Sa = \sum_n a_n y_n, \quad a = (a_1, a_2, \ldots) \in \ell^1.
\]

Now we consider \( T = S \circ P \). As \( \ell^1 \) is a discrete KB-space and \( P \geq 0, P \) is \( b \)-AM-compact. Therefore, \( T \) is \( b \)-AM-compact. So clearly, \( T \) is a \( b \)-order Dunford-Pettis operator. Since \( \ell^1 \) has the Dunford-Pettis property, \( T \) is also weak Dunford-Pettis. But as shown in \([10, Theorem 3.1]\), its adjoint \( T' : F' \rightarrow E' \) is not Dunford-Pettis.

(c) \( \Rightarrow \) (a) It follows from \([10, Theorem 3.1]\). □
In the next theorem we prove that b-order Dunford-Pettis operators satisfy the domination property.

Theorem 3.14. Let $S, T : E \to F$ be two operators from a Banach lattice $E$ into a Banach lattice $F$ such that $0 \leq S \leq T$. If $T$ is b-order Dunford-Pettis, then $S$ is also b-order Dunford-Pettis.

Proof. Let $S, T : E \to F$ be two operators from a Banach lattice $E$ into a Banach lattice $F$ such that $0 \leq S \leq T$, and let $T$ be b-order Dunford-Pettis. Let $0 \leq x'' \in E''$ be arbitrary and fixed and $Y = I_{x''} \cap E$. Then by using Theorem 3.6, $(T|_Y)': Y' \to Y'$ is a Dunford-Pettis operator. Since $0 \leq (S|_Y)' \leq (T|_Y)'$, and $Y'$ has order continuous norm (since $Y$ is an $M$-space), by using [1, Theorem 5.90], $(S|_Y)'$ is a Dunford-Pettis operator. Therefore, by using Theorem 3.6, we conclude that $S$ is a b-order Dunford-Pettis operator. □

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