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ON b-ORDER DUNFORD-PETTIS OPERATORS AND THE b-AM-COMPACTNESS PROPERTY

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ABSTRACT. In this paper, we introduce b-order Dunford-Pettis operators, that is, an operator T from a normed Riesz space E into a Banach space X is called b-order Dunford-Pettis if T carries each b-order bounded subset of E into a Dunford-Pettis subset of X, and we investigate its relationship with order Dunford-Pettis operators. We also introduce the b-AM-compactness property for a Banach lattice and we study some of its topological properties and its relationships with the Dunford-Pettis property. We show that the identity operator of Banach lattice E is b-order Dunford-Pettis if and only if E has the b-AM-compactness property. We characterize Banach lattices E and F on which the adjoint of each operator from E into F which is b-order Dunford-Pettis and weak Dunford-Pettis, is Dunford-Pettis.

1. INTRODUCTION

Let us recall that a norm bounded subset A of a Banach space X is a Dunford-Pettis set whenever every weakly compact operator from X to an arbitrary Banach space carries A to a norm totally bounded set. An operator $T\colon X \to Y$ between two Banach spaces is called a Dunford-Pettis operator if T carries weakly convergent sequences to norm convergent sequences. A Banach space X is said to have the Dunford-Pettis property if every weakly compact operator T defined on X and taking values in a Banach space Y is Dunford-Pettis. For example, the Banach space ℓ^∞ has the Dunford-Pettis property but the Banach space ℓ^2 does not have the Dunford-Pettis property. In [**6**], Aqzzouz and Bouras introduced the AM-compactness property for Banach lattices. A Banach lattice E is said to have the AM-compactness property if every weakly compact operator defined on E and taking values in a Banach space X is AM-compact. For example, the Banach lattice ℓ^1 has the AM-compactness property, but $L^2[0,1]$ does not have the AM-compactness property. They used the AM-compactness property to characterize Banach lattices on which each positive weak Dunford-Pettis operator is almost Dunford-Pettis, and conversely. They proved that Banach lattice E has the AM-compactness property if and only if for each $x \in E^+$, $[-x,x]$ is a Dunford-Pettis set. Also, they proved that a Banach lattice E has the AM-compactness property if and only if for every weakly null sequence $\{f_n\} \subset E'$, we have $|f_n| \to 0$ for

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$\sigma(E', E)$. They showed that Banach lattice E with the Dunford-Pettis property and order continuous norm has the AM-compactness property. In this paper, we introduce the b-AM-compactness property and investigate Banach lattices which under some conditions have the b-AM-compactness property.

The class of order Dunford-Pettis operators was introduced by Aqzzouz and Bouras in [**5**]. An operator T from a normed Riesz space E into a Banach space X is called order Dunford-Pettis if it carries each order bounded subset of E onto a Dunford-Pettis set of X. For example, the identity operator of Banach lattice c_0 is order Dunford-Pettis. They studied the class of Dunford-Pettis sets in Banach lattices, and establish some sufficient conditions for which a Dunford-Pettis set is relatively weakly compact (resp,. relatively compact). They proved that Banach lattice E has the AM-compactness property if and only if the identity operator of E is order Dunford-Pettis. In this paper, we introduce b-order Dunford-Pettis operators and prove some of their properties. Then we study relationship between order Dunford-Pettis operators and b-order Dunford-Pettis operators. We show that the identity operator of Banach lattice E is b-order Dunford-Pettis if and only if E has the b-AM-compactness property. Bouras, El Kaddouri, H'Michane, and Moussa characterized Banach lattices E and F on which the adjoint of each operator from E into F which is order Dunford-Pettis and weak Dunford-Pettis, is Dunford-Pettis, see [**10**]. In this paper, we characterize Banach lattices E and F on which the adjoint of each operator from E into F which is b-order Dunford-Pettis and weak Dunford-Pettis, is Dunford-Pettis.

2. Preliminary Information

We use the term operator $T\colon E \to F$ between two Riesz spaces to mean a (maybe unbounded) linear mapping. Let E and F be two vector lattices (Riesz spaces), let $x, y \in E$ with $x \leq y$, and let the order interval $[x, y]$ be the subset of E defined by $[x, y] = \{z \in E : x \leq z \leq y\}$. A subset of E is called order bounded if it is included in an order interval. Let $T\colon E \to F$ be an operator between two Riesz spaces E and F. T is order bounded if it maps order bounded subsets of E to order bounded subsets of F.

By E' and E'', we denote the topological dual and topological bidual of E, respectively. The vector space $E^{\sim}$ of all order bounded linear functionals on E is called the order dual of E. The vector space $E^{\sim\sim} = (E^{\sim})^{\sim}$ denotes the order bidual of E. The algebraic adjoint of T denoted by $T'\colon F' \to E'$, and its order adjoint denoted by $T^{\sim}\colon F^{\sim} \to E^{\sim}$.

The b-order bounded subsets of E are the order bounded in $E^{\sim\sim}$. T is b-order bounded if it maps b-order bounded subsets of E to b-order bounded subsets of F.

A vector lattice E is said to be discrete if it admits a complete disjoint system of discrete elements, where we say a nonzero element $x \in E$ is discrete whenever the ideal generated by x coincides with the vector subspace generated by x. A Banach lattice is a Banach space $(E, \|.\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$, if $|x| \leq |y|$, then we have $\|x\| \leq \|y\|$. A norm $\|\cdot\|$ of a Banach lattice E is order continuous if for each net $(x_\alpha)_{\alpha\in\Lambda}$ such that $x_\alpha \downarrow 0$, (i.e., (x_α) is decreasing and $\inf\{x_\alpha : \alpha \in \Lambda\} = 0$), we have $\|x_\alpha\| \to 0$. A Banach lattice E is said to be a Kantorovich–Banach space (KB-space) whenever every increasing norm bounded sequence of E^+ is norm-convergent. If E is a Banach lattice and $x \in E^+$, then the principal ideal I_x generated by x is

$$I_x = \{y \in E : \text{there exists } \lambda > 0 \text{ with } |y| \leq \lambda x\},$$

and thus I_x under the norm $\|\cdot\|_\infty$, defined by

$$\|y\|_\infty = \inf \{\lambda > 0 : |y| \leq \lambda x\}, \quad y \in I_x,$$

is an AM-space with the unit x, whose closed unit ball is the order interval $[-x, x]$. For an operator $T : E \to F$ between two Riesz spaces, we say that its modulus $|T|$ exists whenever

$$|T| := T \vee (-T)$$

exists. By using [**1**, Theorem 1.18], for Riesz spaces E and F whenever F is Dedekind complete, each order bounded operator $T : E \to F$ satisfies the following statement

$$|T|(x) = \sup \{|Ty| : |y| \leq x\}$$

for each $x \in E^+$. We refer to [**1, 11**] and [**2**] for any unexplained terms from vector lattice theory.

3. Main Results

In the following, we introduce the class of b-order Dunford-Pettis operators and the b-AM-compactness property, and we investigate some of their properties. We study application of these new concepts in the topological properties of Dunford-Pettis sets and operators.

Definition 3.1. An operator T from a normed Riesz space E into a Banach space X is called b-order Dunford-Pettis if T carries each b-order bounded subset of E into a Dunford-Pettis subset of X.

For example, we know that ℓ^1 has property (b) and its norm is order continuous. Therefore, if $A \subset \ell^1$ is a b-order bounded set, then it is weakly compact. On the other hand, ℓ^1 has the Schur property. Thus, A is norm compact, and hence Dunford-Pettis. Therefore, the identity operator of Banach lattice ℓ^1 is b-order Dunford-Pettis. Also, we can give Banach lattice c_0 as an example of a Banach lattice without property (b) that its identity operator is b-order Dunford-Pettis. Indeed, B_{c_0} (the closed unit ball of c_0) is a Dunford-Pettis set by Lemma 3.5. Therefore, each norm bounded subset of c_0, specifically each b-order bounded subset of c_0, is Dunford-Pettis. But the identity operator of ℓ^∞ is not a b-order Dunford-Pettis operator (since $[-1, 1]$, the closed unit ball of ℓ^∞, is not Dunford-Pettis). Let E be a normed Riesz space and let X be a Banach space. Recall that an operator T from E into X is AM-compact (resp., b-AM-compact) if it maps order bounded (resp., b-order bounded) subset of E to relatively compact subset of X. By $K(E, X)$, $AM(E, X)$, and $AM_b(E, X)$, we denote the collection of compact,

AM-compact, and b-AM-compact operators from E into X, respectively. Clearly, we have

$$K(E,X) \subset AM_b(E,X) \subset AM(E,X).$$

By $DP_o(E,X)$ and $DP_b(E,X)$, we denote the collection of order Dunford-Pettis and b-order Dunford-Pettis operators, respectively. It is clear that

$$AM(E,X) \subset DP_o(E,X), \qquad AM_b(E,X) \subset DP_b(E,X).$$

Note that the inclusions may be proper. In fact, the identity operator $I\colon L^1[0,1] \to L^1[0,1]$ is both b-order Dunford-Pettis and order Dunford-Pettis but neither AM-compact nor b-AM-compact.

In the next example we show that the inclusion

$$DP_b(E,X) \subset DP_o(E,X)$$

also may be proper. Before giving an example which shows that the above inclusion is strict, we need a preliminary Lemma.

Lemma 3.2. *Let X be a Banach space and let A be a subset of X. Then, A is a Dunford-Pettis set as a subset of X if and only if A is a Dunford-Pettis set as a subset of X''.*

Proof. Let A be a Dunford-Pettis set as a subset of X and let $T\colon X'' \to Y$ be a weakly compact operator, where Y is an arbitrary Banach space. Put $S = T\big|_X$. Therefore, S is a weakly compact operator. Since $A \subset X$ is a Dunford-Pettis set, $S(A) = T(A)$ is relatively compact. Consequently, A is a Dunford-Pettis set as a subset of X''.

Conversely, let A be a Dunford-Pettis set as a subset of X'' and let T be a weakly compact operator from X into an arbitrary Banach space Y. We know that $T''\colon X'' \to Y''$ is weakly compact. Hence $T''(A)$ is relatively compact. Since $T = T''\big|_X$, $T(A)$ is relatively compact, the proof is complete. □

Example 3.3. Put

$$L = \{(a_n) \in c_0 \mid (na_n) \in c_0\}.$$

It is easy to see that L is a normed Riesz space and it is indeed an order ideal of c_0. First, we prove that ℓ^∞ is the topological bidual of L. It is sufficient to show that the linear operator $\psi\colon L' \to \ell^1$ defined by

$$\psi f = (f(e_n)),$$

is a well defined linear isometry which is also a lattice isomorphism, where $\{e_n\}$ is the standard basis of L. Let $\{e_n\}$ be the standard basis of L. Fix an arbitrary $f \in L'$. For each $n \in \mathbb{N}$, put

$$x_n(m) = \begin{cases} \operatorname{sgn}(f(e_m)), & m = 1,2,\ldots,n, \\ 0, & m > n \end{cases}$$

where $\operatorname{sgn}(x)$ is the sign of $x \in \mathbb{R}$ (i.e., $\operatorname{sgn}(0) = 0$ and $\operatorname{sgn}(x) = x/|x|$ for $x \neq 0$). Clearly, $(x_n) \in L$ and $\|x_n\| \leq 1$ for all $n \in \mathbb{N}$. We have

$$\|f\| \geq f(x_n) = f\left(\sum_{k=1}^{n} \operatorname{sgn}(f(e_k))e_k\right) = \sum_{k=1}^{n} \operatorname{sgn}(f(e_k))f(e_k) = \sum_{k=1}^{n} |f(e_k)|.$$

Hence $\sum_{k=1}^{\infty} |f(e_k)| \leq \|f\|$. That is, ψ is well defined. It is easy to see that ψ is a linear bijective map. The above argument also shows that

$$\|\psi f\| = \|(f(e_n))\|_1 = \sum_{k=1}^{\infty} |f(e_k)| \leq \|f\| .$$

On the other hand, let $x = (x_1, x_2, \cdots) \in L$ such that $\|x\| \leq 1$. Then,

$$\|f(x)\| = \left\| f\left(\sum_{k=1}^{\infty} x_k e_k \right) \right\| = \left\| \sum_{k=1}^{\infty} x_k f(e_k) \right\| \leq \sum_{k=1}^{\infty} |x_k| \, |f(e_k)| \leq \sum_{k=1}^{\infty} |f(e_k)| .$$

Thus, $\|f\| \leq \sum_{k=1}^{\infty} |f(e_k)|$. Therefore, $\|\psi f\| = \|f\|$, that is, ψ is an onto isometry. Since ψ and ψ^{-1} are positive, it follows from [**1**, Theorem 2.15] that ψ is an onto lattice isomorphism. We proved that ℓ^1 is the dual of L, and hence ℓ^∞ is the bidual of L.

Now, we define $T\colon L \to \ell^\infty$ as follows:

$$T(a_1, a_2, \cdots) = (na_1, na_2, \cdots), \qquad (a_1, a_2, \cdots) \in L.$$

Let $0 \leq x = (x_n) \in c_0$. Thus, $T[0, x] \subset [0, (nx_n)]$. Since $I\colon c_0 \to c_0$ is order Dunford-Pettis, $[0, (nx_n)]$ is a Dunford-Pettis set. Therefore, $T[0, x]$ is Dunford-Pettis, that is, T is order Dunford-Pettis. Now, we claim that the operator T is not b-order Dunford-Pettis. Since the sequence $\{e_n\}$, the standard basis of L, is order bounded in ℓ^∞, it is order bounded in $L^{\sim\sim}$. Therefore, $\{e_n\}$ is a b-order bounded subset of L.

$$T(\{e_n \mid n \in \mathbb{N}\}) = \{ne_n \mid n \in \mathbb{N}\}.$$

Therefore, $T(\{e_n \mid n \in \mathbb{N}\})$ is a norm unbounded subset of ℓ^∞. Hence $T(\{e_n \mid n \in \mathbb{N}\})$ is not a Dunford-Pettis set. Consequently, T is not b-order Dunford-Pettis.

Definition 3.4. A Banach lattice E is said to have the b-AM-compactness property if every weakly compact operator from E into an arbitrary Banach space X is b-AM-compact.

For example, c_0 and ℓ^1 have the b-AM-compactness property. Clearly, each Banach lattice with the b-AM-compactness property has the AM-compactness property. We believe that the converse is false in general. However, right now we do not have an example. Nonetheless, there exists a Banach lattice which does not have the b-AM-compactness property. In fact, ℓ^∞ which does not have the AM-compactness property, does not have the b-AM-compactness property, either.

The next result characterizes the b-order Dunford-Pettis operators. For the proof of the next Theorem, we need the following Lemma.

Lemma 3.5. *Let T be an operator from a normed space X into a Banach space Y. Then, the adjoint $T'\colon Y' \to X'$ is Dunford-Pettis if and only if $T(B_X)$ is a Dunford-Pettis set, where B_X is the closed unit ball of X.*

Proof. One can repeat the argument of [**5**, Remark 4]. □

Theorem 3.6. *Let T be an operator from a Banach lattice E into a Banach space X. Then the following statements are equivalent:*

(a) *T is b-order Dunford-Pettis operator,*
(b) *for each weakly compact operator S from X into an arbitrary Banach space Z, the composed operator ST is b-AM-compact,*
(c) *for each $0 \leq x'' \in E''$, the operator $(T\big|_Y)'$ is a Dunford-Pettis operator, where $Y = I_{x''} \cap E$.*

Proof. (a)⇒(b) Obvious.

(b)⇒(a) Let A be a b-order bounded subset of E. It is sufficient to prove that an arbitrary weakly compact operator S from X into an arbitrary Banach space Z carries $T(A)$ into a norm totally bounded set. Since by our hypothesis, ST is b-AM-compact, $S(T(A)) = ST(A)$ is relatively compact. This proves that T is b-order Dunford-Pettis.

(a)⇒(c) Let $0 \leq x'' \in E''$ and let $Y = I_{x''} \cap E$. Since $B_Y = [-x'', x''] \cap E$ is b-order bounded, $T(B_Y)$ is a Dunford-Pettis set. Therefore, by using Lemma 3.5 the adjoint of the restriction of T to Y is a Dunford-Pettis operator.

(c)⇒(a) It sufficient to prove that for each $0 \leq x'' \in E''$, $T([-x'', x''] \cap E)$ is a Dunford-Pettis set.

Put $Y = I_{x''} \cap E$. By our hypothesis, the adjoint of the restriction of T to Y is a Dunford-Pettis operator. So by using Lemma 3.5, the proof is complete. □

Some basic properties of b-order Dunford-Pettis operators are summarized in the next proposition. For an arbitrary pair of normed spaces X and Y, the symbol $B(X, Y)$ denote the vector space of all bounded operators from X into Y.

Proposition 3.7. *Let E and F be two normed Riesz spaces, and let X and Y be two Banach spaces. We have*

(a) *The vector space $DP_b(E, X) \cap B(E, X)$ is a closed vector subspace of vector space $B(E, X)$.*
(b) *The vector space $DP_b(E, X) \cap B(E, X)$ is a closed vector subspace of vector space $DP_o(E, X) \cap B(E, X)$.*
(c) *If $T \in DP_b(E, X)$, then for each bounded operator $S \colon X \to Y$, the composed operator ST is b-order Dunford-Pettis.*
(d) *If $T \colon E \to F$ is a b-order bounded operator, then for each operator $S \in DP_b(F, X)$, the composed operator ST is b-order Dunford-Pettis.*

Proof. (a) Let S be a bounded operator in norm closure $DP_b(E, X) \cap B(E, X)$. So there exist $\{T_n\} \subset DP_b(E, X) \cap B(E, X)$ such that $T_n \to S$ is norm. Let V be a weakly compact operator from X into an arbitrary Banach space Z. We have $\|VT_n - VS\| \to 0$. By using Theorem 3.6, VT_n is a b-AM-compact operator for each n. Therefore, by using [**9**, Theorem 2.9 (1)], VS is b-AM-compact. Now again by using Theorem 3.6, we have $S \in DP_b(E, X) \cap B(E, X)$.

(b) It follows from the fact that $DP_o(E, X) \cap B(E, X)$ is closed in $B(E, X)$, see [**10**, Proposition 3.1 (a)].

(c) Let $T \in DP_b(E, X)$ and let $S \colon X \to Y$ be a bounded operator. We prove that $ST \in DP_b(E, Y)$. By using Theorem 3.6, it is sufficient to prove that for a weakly compact operator V from Y into an arbitrary Banach space Z, $V(ST)$ is b-AM-compact. We know that VS is a weakly compact operator from X into Z.

Since $T \in DP_b(E, F)$, by using Theorem 3.6 we conclude that $V(ST) = (VS)T$ is b-AM-compact. That is, ST is a b-order Dunford-Pettis operator.

(d) Obvious. □

Recall from [**3**] that an operator $T: E \to F$ between two Riesz spaces is called strongly order bounded if it maps b-order bounded subsets of E into order bounded subsets of F. A Riesz space E is said to have the property (b) if every subset A of E is order bounded whenever it is order bounded in $E^{\sim\sim}$. In the following result, we give some sufficient conditions under which an operator is b-order Dunford-Pettis.

Proposition 3.8. (a) *Let E be a Banach lattice and let X be a Banach space. An operator $T: E \to X$ is b-order Dunford-Pettis whenever its second adjoint $T'': E'' \to X''$ is an order Dunford-Pettis operator.*

(b) *Let E and F be two normed Riesz spaces and let X be a Banach space. If $T: E \to F$ is a strongly order bounded operator, then for each operator $S \in DP_o(F, X)$, the composed operator ST is b-order Dunford-Pettis.*

Proof. (a) Let A be a b-order bounded subset of E. Since A is order bounded in E'', and T'' is order Dunford-Pettis, $T''(A)$ is a Dunford-Pettis subset of X''. We know that $T(A) = T''(A)$. It follows from Lemma 3.2, $T(A)$ is a Dunford-Pettis subset of X. Therefore, T is b-order Dunford-Pettis.

(b) Obvious. □

Corollary 3.9. *Let E be a normed Riesz space with property (b) and let X be a Banach space. We have*

$$DP_o(E, X) = DP_b(E, X).$$

Proof. It follows from part 3.8 of Proposition 3.8 and the fact that $I: E \to E$ is strongly order bounded. □

Theorem 3.10. *Let E be a Banach lattice that its bidual has order continuous norm and let X be a Banach space with the Dunford-Pettis property. Then, each continuous operator $T: E \to X$ is b-order Dunford-Pettis.*

Proof. We use similar proof techniques as those which were developed in [**10**, Proposition 3.2]. Let T be an operator from E into X and let A be a b-order bounded subset of E. Thus, A is order bounded in E''. Therefore, by using [**1**, Theorem 4.9], A is $\sigma(E'', E''')$-compact. Since $A \subset E$, A is $\sigma(E, E')$-compact. Thus, $T(A)$ is relatively weakly compact. On the other hand, since X has the Dunford-Pettis property, the identity operator of X is weak Dunford-Pettis. Therefore, $T(A)$ is a Dunford-Pettis subset of X. □

In the next result we prove that a Banach lattice E has the b-AM-compactness property if and only if its b-order bounded subsets are Dunford-Pettis.

Theorem 3.11. *Let E be a Banach lattice. Then the following statements are equivalent:*

(a) *E has the b-AM-compactness property,*
(b) *$I: E \to E$ is b-order Dunford-Pettis,*

(c) *each positive operator from E into E is b-order Dunford-Pettis,*
(d) *for each $0 \leq x'' \in E''$, $[-x'', x''] \cap E$ is a Dunford-Pettis set.*

Proof. (a) $\Rightarrow$ (b) It follows from Theorem 3.6.

(b) $\Rightarrow$ (c) Let $T\colon E \to E$ be a positive operator and let A be a b-order bounded set in E. Since T is positive, it is b-order bounded. Therefore, $T(A)$ is a b-order bounded set in E. So by our hypothesis, $T(A) = I(T(A))$ is a Dunford-Pettis set. This proves that T is a b-order Dunford-Pettis operator.

(c) $\Rightarrow$ (b) Obvious.

(b) $\Rightarrow$ (a) Let T be a weakly compact operator from E into an arbitrary Banach space X. We prove that T is b-AM-compact. Let A be a b-order bounded set in E. Since I is b-order Dunford-Pettis, $A = I(A)$ is a Dunford-Pettis set. Therefore, $T(A)$ is a totally bounded subset of X. Consequently, T is a b-AM-compact operator.

(b) $\Rightarrow$ (d) Obvious.

(d) $\Rightarrow$ (b) Let A be a b-order bounded subset of E. So, there exists $0 \leq x'' \in E''$ such that $A \subset [-x'', x''] \cap E$. Therefore, by our hypothesis, A is a Dunford-Pettis set in E''. By using Lemma 3.2, A is a Dunford-Pettis set in E. Since $I(A) = A$, I is a b-order Dunford-Pettis operator. □

The following theorem gives some sufficient conditions under which a Banach lattice has the b-AM-compactness property.

Theorem 3.12. *A Banach lattice E has the b-AM-compactness property if one of the following assertions is valid:*

(a) *the bidual of E has order continuous norm and E has the Dunford-Pettis property,*
(b) *the lattice operations in E' are weakly sequentially continuous,*
(c) *the topological dual E' is discrete with order continuous norm,*
(d) *E has the AM-compactness property and each b-order bounded disjoint sequence in E^+ is Dunford-Pettis,*
(e) *E is a discrete KB-space,*
(f) *E has property* (b) *and the AM-compactness property,*
(g) *E'' has the AM-compactness property.*

Proof. (a) Follows from Theorem 3.10 and Theorem 3.11.

(b) It is sufficient to show that for each $0 \leq x'' \in E''$, $[-x'', x''] \cap E$ is Dunford-Pettis. Let $\{x_n\}$ be a sequence in $[-x'', x''] \cap E$ and let $\{f_n\}$ be an arbitrary weakly null sequence in E'. We have

$$0 \leq |f_n(x_n)| \leq |f_n|\,|x_n| \leq |f_n|\,(x'').$$

It follows from our hypothesis and $f_n(x'') \to 0$ that $|f_n|\,(x'') \to 0$. Therefore, $f_n(x_n) \to 0$. The result follows from [**4**, Theorem 1].

(c) It follows from [**8**, Proposition 2.6] that the lattice operations in E' are weakly sequentially continuous.

(d) Put $A = [-x'', x''] \cap E$ for some $0 \leq x'' \in E''$. It is sufficient to show that A is a Dunford-Pettis set. Since E has the AM-compactness property, for each $x \in A^+ = [0, x''] \cap E$, $[-x, x]$ is Dunford-Pettis. If $\{x_n\}$ is a disjoint sequence in A^+, then by our hypothesis, $\{x_n\}$ is Dunford-Pettis. It follows from [**5**, Corollary 2.13] that A is Dunford-Pettis.

(e) It follows from [**9**, Corollary 2.10].

(f) Obviuos.

(g) It follows from Lemma 3.2 and the fact that each subset of a Dunford-Pettis set is Dunford-Pettis. □

The next theorem characterizes Banach lattices E and F on which the adjoint of each operator from E into F which is b-order Dunford-Pettis and weak Dunford-Pettis, is Dunford-Pettis.

Theorem 3.13. *Let E and F be two Banach lattices. The following assertions are equivalent:*

(a) *each order Dunford-Pettis and weak Dunford-Pettis operator $T\colon E \to F$ has an adjoint $T'\colon F' \to E'$ that is Dunford-Pettis,*
(b) *each b-order Dunford-Pettis and weak Dunford-Pettis operator $T\colon E \to F$ has an adjoint $T'\colon F' \to E'$ that is Dunford-Pettis,*
(c) *one of the following is valid:*
    (1) *E' has order continuous norm,*
    (2) *F' has the Schur property.*

Proof. (a) ⇒ (b) Obvious.

(b) ⇒ (c) We use similar proof techniques as those which were developed in [**7**, Theorem 3.5]. Assume by way of contradiction that (c) is not correct, i.e., E' does not have order continuous norm and F' does not have the Schur property. We have to construct a b-order Dunford-Pettis and weak Dunford-Pettis operator $T\colon E \to F$ such that its adjoint is not Dunford-Pettis. Since the norm of E' is not order continuous, by [**1**, Theorem 4.69], ℓ^1 embeds complementably in E. That is, there exist, a positive projection $P\colon E \to \ell^1$. On the other hand, F' does not have the Schur property so there exists a weakly null sequence $\{f_n\}$ in F' such that $\|f_n\| = 1$ for all $n \in \mathbb{N}$. Since $\sup_{\|y\| \leq 1} \|f_n(y)\| = 1$, there exists a sequence $\{y_n\}$ of positive elements in the closed unit ball of F such that for some $\varepsilon > 0$, we have $|f_n(y_n)| \geq \varepsilon$ for all $n \in \mathbb{N}$. We define the operator $S\colon \ell^1 \to F$ as follow:

$$Sa = \sum_n a_n y_n, \qquad a = (a_1, a_2, \ldots) \in \ell^1.$$

Now we consider $T = S \circ P$. As ℓ^1 is a discrete KB-space and $P \geq 0$, P is b-AM-compact. Therefore, T is b-AM-compact. So clearly, T is a b-order Dunford-Pettis operator. Since ℓ^1 has the Dunford-Pettis property, T is also weak Dunford-Pettis. But as shown in [**10**, Theorem 3.1], its adjoint $T'\colon F' \to E'$ is not Dunford-Pettis.

(c) ⇒ (a) It follows from [**10**, Theorem 3.1]. □

In the next theorem we prove that b-order Dunford-Pettis operators satisfy the domination property.

Theorem 3.14. *Let $S, T\colon E \to F$ be two operators from a Banach lattice E into a Banach lattice F such that $0 \leq S \leq T$. If T is b-order Dunford-Pettis, then S is also b-order Dunford-Pettis.*

Proof. Let $S, T\colon E \to F$ be two operators from a Banach lattice E into a Banach lattice F such that $0 \leq S \leq T$, and let T be b-order Dunford-Pettis. Let $0 \leq x'' \in E''$ be arbitrary and fixed and $Y = I_{x''} \cap E$. Then by using Theorem 3.6, $(T\big|_Y)'\colon F' \to Y'$ is a Dunford-Pettis operator. Since $0 \leq (S\big|_Y)' \leq (T\big|_Y)'$, and Y' has order continuous norm (since Y is an M-space), by using [**1**, Theorem 5.90], $(S\big|_Y)'$ is a Dunford-Pettis operator. Therefore, by using Theorem 3.6, we conclude that S is a b-order Dunford-Pettis operator. $\square$

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