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INTEGRAL TRANSFORMS AND AMERICAN OPTIONS: LAPLACE AND MELLIN GO GREEN

G. ALOBAIDI, R. MALLIER AND M. C. HASLAM

Abstract. We use Mellin and Laplace transforms to study the price of American options, and show that both transforms produce solutions and integral equations which are equivalent to the Green's function approach. Conventional rather than partial transforms are used. We also combine a boundary fixing transformation with the integral transforms.

1. Introduction

One of the classic problems of mathematical finance is the pricing of American options and the associated free boundary. For the uninitiated, financial derivatives are securities whose value is based on the value of some other underlying security, and options are an example of derivatives, carrying the right but not the obligation to enter into a specified transaction in the underlying security. A call option allows the holder to buy the underlying security at a specified strike price E, while a put option allows the holder to sell the underlying at the price E. Unlike Europeans, which can be exercised only at expiry. American options may be exercised at any time at or before expiry, For vanilla Americans, the pay-off from immediate exercise is the same as the pay-off at expiry, namely $\max(S-E,0)$ for a call and $\max(E-S,0)$ for a put. Naturally, a rational investor will choose to exercise early if that maximizes his return, and it follows that there will be regions where it is optimal to hold the option and others where exercise is optimal, with a free boundary known as the optimal exercise boundary separating these regions. For vanilla Americans, there have been numerous studies of this free boundary, as well a number of reviews such as [**6, 10, 11, 22**], but a closed form solution for its location remains elusive, as does a closed form expression for the value of an American option, although a number of series solutions have been presented such as [**18, 1, 41**] and an infinite sum of double integrals [**62**].

We would note that [**59**] proved that the free boundary was regular for vanilla Americans, and that the results of [**58**] on the analyticity of solutions to general

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Stefan problems are also applicable to American options, while [**13**] has shown existence and uniqueness for the free boundary for the put.

One popular approach over the years has been to reformulate the problem as an integral equation for the location of the free boundary, which can be solved using either asymptotics or numerics. Two forms of this integral equation approach are pertinent here. As our reference point, we will use the Green's function approach [**35, 44, 7, 30, 32, 29**]; by reference point, we mean that this is existing work to which we will compare our results obtained by other methods. The Green's function approach was originally developed for physical Stefan problems by Kolodner [**35**] and later applied to economics by McKean [**44**] who rederived the results in [**35**] using a partial Fourier transform, which was published as an appendix to a 1965 paper on the pricing of American warrants by Samuelson [**53**] which predated the publication of the Black-Scholes [**8**] and Merton [**45**] studies and involved hard-to-estimate discount rates rather than the risk-free rate which [**8, 45**] arrived at using a continuous-time arbitrage argument. These results were applied to vanilla Americans with great success by Kim [**32**] and Jacka [**29**], who independently derived the same results, Kim both by using McKean's formula and by taking the continuous limit of the Geske-Johnson formula [**25**] which is a discrete approximation for American options, and thereby demonstrating that those two approaches led to the same result, and Jacka by applying probability theory to the optimal stopping problem. [**9**] later used these results to show how to decompose the value of an American into intrinsic value and time value. The approach in [**44, 32, 29**] leads to an integral equation for the location of the free boundary, which was solved numerically by [**28**], by approximating the free boundary as a multipiece exponential function by [**31**], and, in a slightly different form, using asymptotics by [**12, 21, 26, 34, 36**]. Chiarella [**15**] has surveyed the Green's function approach to American options using the incomplete Fourier transform and demonstrated how the various representations are related, as well as considering their economic interpretations.

The other method we would mention is the integral transform approach, which is the subject of the present paper. The use of integral transform methods for Stefan-type free boundary problems dates back at least as far as the classic work of Evans *et al.* [**20**], who considered the recrystallization of an infinite metal slab. In that work, the governing partial differential equation (PDE) was the diffusion equation,

$$\frac{\partial V}{\partial \tau} = \sigma^2 \frac{\partial^2 V}{\partial x^2}, \tag{1}$$

and [**20**] were able to apply a partial Laplace transform in time to (1) to reduce the PDE to an ordinary differential equation in transform space. The solution of that ordinary differential equation together with the conditions of the free boundary enabled [**20**] to give an integral equation formulation of their problem, which they solved as a series. Overviews of partial transforms, sometimes called modified transforms or incomplete transforms, can be found in standard texts on diffusion and Stefan problems, such as [**19, 27, 47**]. Broadly speaking, the objective of applying an integral transform such as the Laplace transform, $\mathcal{L}\left[A\left(\tau\right)\right]=\int_0^\infty \mathrm{e}^{-p\tau}A\left(\tau\right)\mathrm{d}\tau$, or Mellin transform, $\mathcal{M}\left[A\left(S\right)\right]=\int_0^\infty A\left(S\right)S^{p-1}\mathrm{d}S$, to a PDE such as the diffusion equation (1) or the Black-Scholes-Merton equation,

$$\frac{\partial V}{\partial \tau}-\frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2}-\left(r-D\right)S\frac{\partial V}{\partial S}+rV=0, \tag{2}$$

is to reduce the dimension of the problem, by which we mean that if we apply a transform to an equation such as (1) or (2) with derivatives with respect to two (or more) variables, the transformed equation should have derivatives with respect to one fewer variables. In order to achieve this goal, it is obviously necessary to pick the correct integral transform, and for the Black-Scholes-Merton equation (2) the natural integral transforms are Laplace with respect to time τ and Mellin with respect to stock price S, where natural in this sense means that the transformed equation does indeed have derivatives with respect to fewer variables. For the diffusion equation (1), the natural integral transforms are again Laplace with respect to time τ but now either Fourier or two-sided Laplace with respect to x. Although it is straightforward to apply integral transforms to unbounded problems, where the governing equation applies everywhere, it is less straightforward to apply them to problems with boundaries, as typically the governing equation does not apply once the boundary is crossed, and it was for this reason that partial integral transforms were developed, in which the transform is only applied where the governing equation holds, and the function being transformed is essentially set to zero elsewhere. With a partial Laplace transform, for example, instead of taking an in integral from $\tau=0$ to $\tau=\infty$, there would typically be an integral from $\tau=\tau_f$ to $\tau=\infty$, where τ_f is the location of the boundary which often depends on the other variables in the problem. Although partial transforms address the issue that the governing equation does not apply once they boundary is crossed, they can be difficult to invert, because the inverse is required to be zero where the equation does not apply. In many problems, this is not an issue, especially in free boundary problems where it is often possible to obtain an integral equation for the location of the boundary in transform space without needing to invert the transform.

For securities which involve both equity and debt, such as equity-linked debt or convertible bonds, one approach has been to combine the Black-Scholes model with an interest rate model, which leads to a three-dimensional PDE involving derivatives with respect to the interest rate r as well as S and τ, and it is possible to take the Laplace transform with respect to τ and the Mellin transform with respect to S to reduce the dimension of the problem by two [**40, 42**].

Having given a very brief sketch of the use of integral transforms to solve diffusion-like PDEs in general, we now turn to American options. The price $V(S,\tau)$ of an American option is governed by the Black-Scholes-Merton PDE (2). American options can be exercised at any time at or before expiry, and this early exercise feature leads to a free boundary problem very similar to the Stefan problem, and indeed it is well-known that (2) can be transformed into the diffusion equation (1), this transformation having been crucial to the derivation of the original Black-Scholes formula [**8**]. As we mentioned above, the natural integral transforms for (2) are Laplace with respect to τ and Mellin with respect to S, and applying either of these transforms to (2) yields an ordinary differential equation which can be solved fairly easily. Both of these natural transforms have previously been applied to American options. In [**2, 3, 4, 5, 39**], we followed Evans and applied a partial Laplace transform in time to study American, and American-style exotic, options, while [**33**] used what was essentially a partial Mellin transform with respect to S. To be more precise, [**33**] used a change of variables which both fixed the boundary and turned (2) into a diffusion-like equation, and then used a Laplace transform with respect to the new spatial variable, but if the change of variables is reversed, it becomes apparent that this Laplace transform is equivalent to applying a partial Mellin transform with respect to stock price S to the original equation (2). More recently, [**23, 24, 48, 49, 50**] have used conventional Mellin transforms with respect to S. Partial Fourier transforms have been considered in [**15, 16, 17**], while [**54, 55, 57**] have combined a boundary fixing transformation with Fourier sine and cosine transformations on a half-space.

In the present study, we will again use integral transforms to study the Black-Scholes-Merton PDE, specifically Laplace and Mellin transforms, which are the natural transforms for this PDE, but rather than use partial transforms applied only to the region where it is optimal to hold the option, we will apply conventional transforms to the whole of space. To do so, we obviously need to have a PDE which covers the whole of space, not just the region where it is optimal to hold. There is actually a straightforward way to do this, which can be found in for example [**37**]. When an American option is exercised early, with exercise taking place on the free boundary, it is exchanged for another portfolio with value $P(S,\tau)$; for an American call, which carries the right to buy the underlying at strike price E, this new portfolio is $P(S,\tau) = S - E$, while for an American put, which carries the right to sell the underlying, $P(S,\tau) = E - S$. This new portfolio will obey the nonhomogeneous form of (2), meaning (2) together with a forcing term on the right-hand side, and so when we apply our integral transform, we apply it to (2) together with a forcing term, with this forcing term zero where it is optimal to hold the option. This explanation should become clearer in the next section when we present the analysis.

2. Analysis

Our starting point is the Black-Scholes-Merton partial differential equation (PDE) [**8, 45**] governing the price V of an equity derivative,

$$\mathcal{BS}\left(V\right) = \left[\frac{\partial}{\partial\tau} - \frac{\sigma^2 S^2}{2}\frac{\partial^2}{\partial S^2} - \left(r - D\right) S\frac{\partial}{\partial S} + r\right] V = 0\,, \tag{3}$$

where S is the price of the underlying stock and $\tau = T - t$ is the time remaining until expiry. In our analysis, the volatility σ, risk-free interest rate r, and dividend yield D are assumed constant. For European options, which can be exercised only at expiry, we must solve (3) together with the condition that the payoff at expiry $V(S,0)$ is specified. The value of a European option can be written in terms of a

Green's function as

$$
\begin{aligned}
V(S,\tau) &= \int_0^\infty V(Z,0) G\left(\frac{S}{Z},\tau\right)\frac{\mathrm{d}Z}{Z}, \\
G(S,\tau) &= \frac{\mathrm{e}^{-r\tau}}{\sigma\sqrt{2\pi\tau}}\exp\left(-\frac{[\ln(S)+\nu\tau]^2}{2\sigma^2\tau}\right),
\end{aligned} \tag{4}
$$

where $V(S,0)$ is the payoff at expiry, and we have introduced $\nu = r - D - \sigma^2/2$.

American options, by contrast, can be exercised at any time at or before expiry, with early exercise taking place on an optimal exercise boundary, which [**44, 45**] recognized was a free boundary. We will denote this free boundary by $S = S_f(\tau)$ or equivalently $\tau = \tau_f(S)$ and write the pay-off from early exercise as $P(S,\tau)$. For American options, we must solve (3) together with the condition that the payoff at expiry $V(S,0)$ is specified and the additional conditions that at the free boundary, we require $V(S_f(\tau),\tau) = P(S_f(\tau),\tau)$ and $(\partial V/\partial S)(S_f(\tau),\tau) = (\partial P/\partial S)(S_f(\tau),\tau)$. These two conditions at the free boundary mean that both the value of the option and its delta are continuous there. The condition on the delta is known as the high contact or smooth pasting condition [**53**]. It follows from [**35, 44, 7, 30, 32, 29**] that we can write the value of an American option as the combination of the European value together with another term coming from the right to exercise early,

$$
\begin{aligned}
V(S,\tau) = & \int_0^\infty V(Z,0) G\left(\frac{S}{Z},\tau\right)\frac{\mathrm{d}Z}{Z} \\
& + \int_0^\tau \int_0^\infty F(Z,\zeta) G\left(\frac{S}{Z},\tau-\zeta\right)\frac{\mathrm{d}Z\mathrm{d}\zeta}{Z},
\end{aligned} \tag{5}
$$

with $F(S,\tau) = 0$ where it is optimal to hold the option, while where exercise is optimal $F(S,\tau) = \mathcal{BS}(P)$ which is the result of substituting the early exercise payoff into (3). This solution (5) is a valid solution to $\mathcal{BS}(V) = F(S,\tau)$ for all $\tau \geq 0$ and all S, not just in the region where it is optimal to retain the option. Applying the conditions at the free boundary to (5) leads to a pair of integral equations for the location of the free boundary [**44, 32, 29**], which in the case of vanilla Americans were solved numerically by [**28**], by approximating the free boundary as a multipiece exponential function by [**31**], and using asymptotics by [**12, 21, 26, 34, 36**].

In what follows, we will discuss the relation of the expression (5) for the value of the option and the resulting integral equations with the corresponding expressions obtained using Mellin transforms and Laplace transforms. To facilitate this comparison, we will formulate the problem slightly differently and decompose the value of the option into the value from immediate exercise, $P(S,\tau)$ and the residual, $R(S,\tau)$,

$$
V(S,\tau) = P(S,\tau) + R(S,\tau). \tag{6}
$$

For vanilla Americans, $V(S,0) = P(S,0)$, so in this case we are decomposing the value of the option into its intrinsic value and time value [**9**].

With this formulation, $R(S,\tau)$ is zero where it is optimal to exercise the option, with $R\left(S_f(\tau),\tau\right) = \left(\partial R/\partial S\right)\left(S_f(\tau),\tau\right) = 0$ on the free boundary, while where it optimal to hold, R obeys

$$\mathcal{BS}\left(R\right) = -\mathcal{BS}\left(P\right), \tag{7}$$

where $\mathcal{BS}$ is the Black-Scholes-Merton operator defined in (3). At expiry, $R(S,0) = 0$. This has a solution using (5),

$$R(S,\tau) = \int_0^\tau \int_0^\infty F(Z,\zeta) G\left(\frac{S}{Z}, \tau - \zeta\right) \frac{\mathrm{d}Z\mathrm{d}\zeta}{Z}, \tag{8}$$

where now we have defined $F(S,\tau) = 0$ where it is optimal to exercise the option, while where holding is optimal $F(S,\tau) = -\mathcal{BS}\left(P\right)$. Once again, (8) is a valid solution to $\mathcal{BS}\left(R\right) = F(S,\tau)$ for all $\tau \geq 0$ and all S, not just in the region where it is optimal to retain the option. Applying the conditions on R and $\left(\partial R/\partial S\right)$ at the free boundary to (8) gives a pair of integral equations,

$$\begin{aligned} \int_0^\tau \int_0^\infty F(Z,\zeta) G\left(\frac{S_f(\tau)}{Z}, \tau - \zeta\right) \frac{\mathrm{d}Z\mathrm{d}\zeta}{Z} &= 0, \\ \int_0^\tau \int_0^\infty F(Z,\zeta) \frac{\partial G}{\partial S}\left(\frac{S_f(\tau)}{Z}, \tau - \zeta\right) \frac{\mathrm{d}Z\mathrm{d}\zeta}{Z} &= 0. \end{aligned} \tag{9}$$

For a call, we have $P(S,\tau) = (S-E)\,H\left(S-E\right)$, where E is the strike price of the option and H is the Heaviside step function, so that for a call,

$$\begin{aligned} F(S,\tau) = \Bigg(&\left[rE - DS\right] H\left(S-E\right) + \left[(r-D)\left(S-E\right) + \frac{\sigma^2 S}{2}\right] S\delta\left(S-E\right) \\ &+ \frac{\sigma^2 S^2}{2}\left[\left(S-E\right)\delta\left(S-E\right)\right]'\Bigg) H\left(S_f(\tau) - S\right), \end{aligned} \tag{10}$$

while for a put, we have $P(S,\tau) = (E-S)\,H\left(E-S\right)$ and

$$\begin{aligned} F(S,\tau) = \Bigg(&\left[DS - rE\right] H\left(E-S\right) + \left[(D-r)\left(E-S\right) + \frac{\sigma^2 S}{2}\right] S\delta\left(E-S\right) \\ &+ \frac{\sigma^2 S^2}{2}\left[\left(E-S\right)\delta\left(E-S\right)\right]'\Bigg) H\left(S - S_f(\tau)\right), \end{aligned} \tag{11}$$

with δ the Dirac delta function. The factors $H\left(S_f(\tau) - S\right)$ and $H\left(S - S_f(\tau)\right)$ in (10) and (11) ensure that F is zero where it is optimal to exercise. The use of the Heaviside step function to extend the governing PDE to region where exercise is optimal can be found in standard texts [**37, 61**] and studies such as [**16**].

Using (10) and (11) in (8), for a call we have

$$\begin{aligned} R(S,\tau) = &\int_0^\tau \frac{\sigma^2}{2E} G\Big(\frac{S}{E}, \tau - \zeta\Big)\mathrm{d}\zeta \\ &+ \int_0^\tau \int_E^{S_f(\zeta)} \left(rE - DZ\right) G\Big(\frac{S}{Z}, \tau - \zeta\Big) \frac{\mathrm{d}Z\mathrm{d}\zeta}{Z}, \end{aligned} \tag{12}$$

while for a put, we have

$$
\begin{aligned}
R(S,\tau) = & \int_0^\tau \frac{\sigma^2}{2E} G\Big(\frac{S}{E}, \tau-\zeta\Big)\mathrm{d}\zeta \\
& + \int_0^\tau \int_{S_f(\zeta)}^E (DZ - rE)\, G\Big(\frac{S}{Z}, \tau-\zeta\Big)\frac{\mathrm{d}Z\mathrm{d}\zeta}{Z}.
\end{aligned}
\tag{13}
$$

The well-known put-call symmetry for American options [**14, 43**] is apparent in (12) and (13). The first equation in (9) then becomes

$$
\begin{aligned}
& \int_0^\tau \frac{\sigma^2}{2E} G\Big(\frac{S_f(\tau)}{E}, \tau-\zeta\Big)\mathrm{d}\zeta \\
& + \int_0^\tau \int_E^{S_f(\zeta)} (rE - DZ)\, G\Big(\frac{S_f(\tau)}{Z}, \tau-\zeta\Big)\frac{\mathrm{d}Z\mathrm{d}\zeta}{Z} = 0
\end{aligned}
\tag{14}
$$

for a call, while for a put we have

$$
\begin{aligned}
& \int_0^\tau \frac{\sigma^2}{2E} G\Big(\frac{S_f(\tau)}{E}, \tau-\zeta\Big) \\
& + \int_0^\tau \int_{S_f(\zeta)}^E (DZ - rE)\, G\Big(\frac{S_f(\tau)}{Z}, \tau-\zeta\Big)\frac{\mathrm{d}Z\mathrm{d}\zeta}{Z} = 0.
\end{aligned}
\tag{15}
$$

A similar pair of equations can be written using the second equation in (9).

2.1. Mellin Transforms

The main object of this study is to show that conventional Mellin and Laplace transforms can applied to American options. We will first consider a Mellin transform with respect to stock price S. To make the analysis as painless as possible, we will approach it from two directions simultaneously. We will apply a Mellin transform to the governing PDE and solve the resulting ordinary differential equation (ODE) in transform space to obtain an expression for the transform of the solution. Rather than invert this transform, we will apply a forward transform to the Green's function solution (8) presented above, which will be the same as the solution of the ODE, meaning that the inverse transform yields (8).

We recall that the Mellin transform of a function $A(S)$ and its inverse are [**52**]

$$
\begin{aligned}
\hat{A}(p) &= \mathcal{M}\left[A(S)\right] = \int_0^\infty A(S)\, S^{p-1}\mathrm{d}S, \\
A(S) &= \mathcal{M}^{-1}\left[\hat{A}(p)\right] = \frac{1}{2\pi\,\mathrm{i}}\int_{\sigma-\mathrm{i}\,\infty}^{\sigma+\mathrm{i}\,\infty} \hat{A}(p) S^{-p}\mathrm{d}p.
\end{aligned}
\tag{16}
$$

Convolution can be defined in several ways for Mellin transforms, including

$$
\begin{aligned}
A(S) * B(S) &= \int_0^\infty A\left(\frac{S}{Z}\right) B(Z)\frac{\mathrm{d}Z}{Z}, \\
\mathcal{M}\left[A(S) * B(S)\right] &= \hat{A}(p)\,\hat{B}(p),
\end{aligned}
\tag{17}
$$

so that (8) is a convolution integral,

$$
\begin{aligned}
R(S,\tau) &= \int_0^\tau F(S,\zeta) * G(S,\tau-\zeta)\mathrm{d}\zeta, \\
\hat{R}(p,\tau) &= \int_0^\tau \hat{F}(p,\zeta)\hat{G}(p,\tau-\zeta)\mathrm{d}\zeta,
\end{aligned} \tag{18}
$$

with $\hat{R}$ the transform of (8). We will need $\hat{G}$ and $\hat{F}$ shortly,

$$
\begin{aligned}
\hat{G}(p,\tau) &= \exp\left[\left(-r-\nu p+\frac{\sigma^2p^2}{2}\right)\tau\right], \\
\hat{F}(p,\tau) &= \frac{D\left(S_f^{2-p}(\tau)-E^{2-p}\right)}{p-2} - \frac{rE\left(S_f^{1-p}(\tau)-E^{1-p}\right)}{p-1} + \frac{\sigma^2E^{2-p}}{2},
\end{aligned} \tag{19}
$$

with $\hat{F}(p,\tau)$ the same for both the call and the put. We would stress that our Mellin transform (16) covers the whole of space, $0 \le S < \infty$, and therefore covers both the region where it is optimal to hold the option and that where exercise is optimal. In the second region (8) is a perfectly good solution, even though the option is no longer held. A consequence of this is that we are setting aside the conditions on R and $(\partial R/\partial S)$ at the free boundary. In effect, we are arguing that (8) is valid for all S, and that the location of the free boundary $S_f(\tau)$ can be determined by applying the conditions at the free boundary to (8), which leads to (9).

This is in contrast to the partial transform approach, where the transform is only applied where it is optimal to hold the option. At this point, we should briefly mention the work of [**33**], who used what was essentially a partial Mellin transform. If we return to our definition of the Mellin transform (16), and make the transformation $S = \mathrm{e}^{-x}$ and $A(S) = B(x)$, we arrive at the two-sided Laplace transform [**51**]

$$
\mathcal{L}^{(2)}\left[B\left(x\right)\right](p) = \int_{-\infty}^{\infty} B\left(x\right)\mathrm{e}^{-px}\,\mathrm{d}x. \tag{20}
$$

For the two-sided Laplace transform, convolution follows from (17),

$$
\begin{aligned}
A\left(x\right) * B\left(x\right) &= \int_{-\infty}^{\infty} A\left(x-z\right)B\left(z\right)\mathrm{d}z, \\
\mathcal{L}^{(2)}\left[A\left(x\right) * B\left(x\right)\right] &= \hat{A}\left(p\right)\hat{B}\left(p\right).
\end{aligned} \tag{21}
$$

By contrast, [**33**] used a one-sided Laplace transform,

$$
\begin{aligned}
\mathcal{L}^{(1)}\left[B\left(x\right)\right](p) &= \int_0^\infty B\left(x\right)\mathrm{e}^{-px}\,\mathrm{d}x. \\
A\left(x\right) * B\left(x\right) &= \int_0^x A\left(x-z\right)B\left(z\right)\mathrm{d}z, \\
\mathcal{L}^{(1)}\left[A\left(x\right) * B\left(x\right)\right] &= \hat{A}\left(p\right)\hat{B}\left(p\right),
\end{aligned} \tag{22}
$$

having first used the Landau boundary fixing transformation $X = S/S_f(\tau)$ [**38, 19, 57**] to fix the free boundary at $X = 1$. It is worth noting that the convolutions for one- and two-sided Laplace transforms (21) and (22) are different, with the limits of integration being $-\infty$ to $+\infty$ for the two-sided transform but 0 to x for the one-sided transform. This one-sided Laplace transform is essentially the same as taking a partial Mellin transform, which we define for the call as

$$\mathcal{M}_c\left[A(S,\tau)\right] = \int_0^{S_f(\tau)} A(S,\tau)S^{p-1}\mathrm{d}S, \tag{23}$$

while for the put, we define

$$\mathcal{M}_p\left[A(S,\tau)\right] = \int_{S_f(\tau)}^{\infty} A(S,\tau)S^{p-1}\mathrm{d}S. \tag{24}$$

These definitions are of course equivalent to setting the value of a function equal to zero where it is optimal to exercise, which is what the decomposition (6) implies. The definitions (23) and (24) differ from an earlier partial Mellin transform defined by Naylor [**46**] and discussed in [**56**]. Naylor defined finite Mellin transforms of the first and second kind to be

$$\begin{aligned} \mathcal{M}_1\left[V(S)\right] &= \int_0^{Z}\left(\frac{Z^{2p}}{S^{p+1}} - S^{p-1}\right)V(S)\mathrm{d}S, \\ \mathcal{M}_2\left[V(S)\right] &= \int_0^{Z}\left(\frac{Z^{2p}}{S^{p+1}} + S^{p-1}\right)V(S)\mathrm{d}S. \end{aligned} \tag{25}$$

In (18) and (19), we have the Mellin transform of our solution (8). We will now apply a (conventional) Mellin transform (16) to (7) and work forward trying to solve the problem in Mellin transform space and thereby recover (18,19) and thence (8). If we apply a conventional transform (16) to (7), we get an ODE for the transform of the residual R,

$$\left[\frac{d}{\mathrm{d}\tau} - \frac{\sigma^2p^2}{2} + p\nu + r\right]\hat{R} = \hat{F}(p,\tau). \tag{26}$$

Again, we would note that we have transformed (7) in both the region where it is optimal to hold and that where exercise is optimal. Interestingly, the conventional and partial Mellin transforms of F will be the same since F is zero where it is optimal to exercise. Since $R(S,0) = 0$ for vanilla Americans, it follows that $\hat{R}(p,0) = 0$, and (26) has a solution

$$\begin{aligned} \hat{R}(p,\tau) &= \int_0^{\tau}\hat{F}(p,\zeta)\exp\left[\left(\frac{\sigma^2p^2}{2} - p\nu - r\right)(\tau-\zeta)\right]\mathrm{d}z \\ &= \int_0^{\tau}\hat{F}(p,\zeta)\hat{G}(p,\tau-\zeta)\mathrm{d}\zeta, \end{aligned} \tag{27}$$

which of course recovers (18) and thence (8).

The next issue is whether we can arrive at an equation for the free boundary from (27). Obviously, if we invert the transform, we can again arrive at (9).

However, a rather different equation can be derived if we take the limit $\tau \to \infty$ of (8),

$$R_\infty(S) = \lim_{\tau\to\infty} \int_0^\tau \int_0^\infty F(Z,\zeta) G\left(\frac{S}{Z}, \tau-\zeta\right) \frac{\mathrm{d}Z\mathrm{d}\zeta}{Z}, \tag{28}$$

where R_∞ is the perpetual version of the residue. If we apply the same limit to (27), we arrive at the transform of (28), namely

$$\begin{aligned} \hat{R}_\infty(p) &= \lim_{\tau\to\infty} \int_0^\tau \hat{F}(p,\zeta)\hat{G}(p,\tau-\zeta)\mathrm{d}\zeta \\ &= \lim_{\tau\to\infty} \int_0^\tau \left[\frac{D\left(S_f^{2-p}(\zeta) - E^{2-p}\right)}{p-2} - \frac{rE\left(S_f^{1-p}(\zeta) - E^{1-p}\right)}{p-1} \right. \\ &\quad \left. + \frac{\sigma^2 E^{2-p}}{2}\right] \exp\left[\left(-r - \nu p + \frac{\sigma^2 p^2}{2}\right)(\tau-\zeta)\right] \mathrm{d}\zeta. \end{aligned} \tag{29}$$

For the call, (29) is the transform of

$$\begin{aligned} R_\infty(S) = \lim_{\tau\to\infty} \int_0^\tau \frac{\mathrm{e}^{-r(\tau-\zeta)}}{\sigma\sqrt{2\pi(\tau-\zeta)}} &\left[\frac{\sigma^2}{2E} \exp\left(-\frac{\left[\ln(S/E) + \nu(\tau-\zeta)\right]^2}{2\sigma^2(\tau-\zeta)} \right) \right. \\ &\left. + \int_E^{S_f(\zeta)} (rE - DZ) \exp\left(-\frac{\left[\ln(S/Z) + \nu(\tau-\zeta)\right]^2}{2\sigma^2(\tau-\zeta)} \right) \frac{\mathrm{d}Z}{Z} \right] \mathrm{d}\zeta, \end{aligned} \tag{30}$$

while for the put, it is the transform of

$$\begin{aligned} R_\infty(S) = \lim_{\tau\to\infty} \int_0^\tau \frac{\mathrm{e}^{-r(\tau-\zeta)}}{\sigma\sqrt{2\pi(\tau-\zeta)}} &\left[\frac{\sigma^2}{2E} \exp\left(-\frac{\left[\ln(S/E) + \nu(\tau-\zeta)\right]^2}{2\sigma^2(\tau-\zeta)} \right) \right. \\ &\left. + \int_{S_f(\zeta)}^{E} (DZ - rE) \exp\left(-\frac{\left[\ln(S/Z) + \nu(\tau-\zeta)\right]^2}{2\sigma^2(\tau-\zeta)} \right) \frac{\mathrm{d}Z}{Z} \right] \mathrm{d}\zeta. \end{aligned} \tag{31}$$

2.2. Laplace Transforms

We now turn our attention to Laplace transforms. Our analysis will follow a broadly similar path to that for Mellin transforms, and we will again tackle the problem from two directions. We recall that the Laplace transform of a function $A(S)$ and its inverse are [**52**]

$$\begin{aligned} \hat{A} = \mathcal{L}\left[A(\tau)\right] &= \int_0^\infty \mathrm{e}^{-p\tau} A(\tau)\,\mathrm{d}\tau, \\ A(\tau) = \mathcal{L}^{-1}\left[\hat{A}(p)\right] &= \frac{1}{2\pi\,\mathrm{i}} \int_{\sigma-\mathrm{i}\,\infty}^{\sigma+\mathrm{i}\,\infty} \hat{A}(p)\,\mathrm{e}^{p\tau}\,\mathrm{d}p. \end{aligned} \tag{32}$$

Convolution for Laplace transforms is defined by

$$\begin{aligned} A(\tau) * B(\tau) &= \int_0^\tau A(\zeta)\,B(\tau-\zeta)\,\mathrm{d}\zeta, \\ \mathcal{L}\left[A(\tau) * B(\tau)\right] &= \hat{A}(p)\hat{B}(p), \end{aligned} \tag{33}$$

so that (8) is a convolution integral

$$
\begin{aligned}
R(S,\tau) &= \int_0^\infty F(Z,\tau) * G\left(\frac{S}{Z},\tau\right)\frac{\mathrm{d}Z}{Z}, \\
\hat{R}(S,p) &= \int_0^\infty \hat{F}(Z,p)\hat{G}\left(\frac{S}{Z},p\right)\frac{\mathrm{d}Z}{Z},
\end{aligned}
\tag{34}
$$

with $\hat{R}$ the transform of (8). For the Laplace transform approach, we must write F in a slightly different, but equivalent, form to (10) and (11). For Mellin transforms, we wrote the free boundary as $S = S_f(\tau)$, while for Laplace transforms, we will invert this and write $\tau = \tau_f(S)$. For values of S for which an option is held until expiry, we define $\tau_f(S) = 0$. We can then replace the terms $H\left(S_f(\tau) - S\right)$ in (10) and $H\left(S - S_f(\tau)\right)$ in (11) by $H\left(\tau - \tau_f(S)\right)$, so that for the call,

$$
\begin{aligned}
F(S,\tau) = &\left([rE - DS]\, H\left(S - E\right) + \left[(r-D)\left(S-E\right) + \frac{\sigma^2 S}{2}\right] S\delta\left(S-E\right)\right. \\
&\left. + \frac{\sigma^2 S^2}{2}\left[\left(S-E\right)\delta\left(S-E\right)\right]'\right) H\left(\tau - \tau_f(S)\right) H\left(S_* - S\right),
\end{aligned}
\tag{35}
$$

while for the put,

$$
\begin{aligned}
F(S,\tau) = &\left([DS - rE]\, H\left(E - S\right) + \left[(D-r)\left(E-S\right) + \frac{\sigma^2 S}{2}\right] S\delta\left(E-S\right)\right. \\
&\left. + \frac{\sigma^2 S^2}{2}\left[\left(E-S\right)\delta\left(E-S\right)\right]'\right) H\left(\tau - \tau_f(S)\right) H\left(S - S_*\right).
\end{aligned}
\tag{36}
$$

S_* is the location of the free boundary in the limit $\tau \to \infty$ which can be deduced from the perpetual American option [**60, 61**]. We will need $\hat{G}$ and $\hat{F}$ shortly,

$$
\hat{G}(S,p) = \left[2\left(p+p_0\right)\sigma^2\right]^{-1/2} \times \begin{cases} S^{-\frac{\left(\nu+\sigma\sqrt{2(p+p_0)}\right)}{\sigma^2}} & S > 1 \\ S^{-\frac{\left(\nu-\sigma\sqrt{2(p+p_0)}\right)}{\sigma^2}} & S < 1 \end{cases},
\tag{37}
$$

with $p_0 = \left(4(D-r)^2 + 4\sigma^2(D+r) + \sigma^4\right)/(8\sigma^2)$. For the call

$$
\begin{aligned}
\hat{F}(S,p) = -&\left([rE - DS]\, H\left(S - E\right) + \left[(r-D)\left(S-E\right) + \frac{\sigma^2 S}{2}\right] S\delta\left(S-E\right)\right. \\
&\left. + \frac{\sigma^2 S^2}{2}\left[\left(S-E\right)\delta\left(S-E\right)\right]'\right) H\left(S_* - S\right)\frac{\mathrm{e}^{-p\tau_f(S)}}{p},
\end{aligned}
\tag{38}
$$

while for the put,

$$
\begin{aligned}
\hat{F}(S,p) = -&\left([DS - rE]\, H\left(E - S\right) + \left[(D-r)\left(E-S\right) + \frac{\sigma^2 S}{2}\right] S\delta\left(E-S\right)\right. \\
&\left. + \frac{\sigma^2 S^2}{2}\left[\left(E-S\right)\delta\left(E-S\right)\right]'\right) H\left(S - S_*\right)\frac{\mathrm{e}^{-p\tau_f(S)}}{p}.
\end{aligned}
\tag{39}
$$

Because of the split nature of $\hat{G}$, we note that we can write (34) as

$$\hat{R}(S,p) = \left[2\left(p+p_0\right)\sigma^2\right]^{-1/2} \int_0^S \hat{F}(Z,p) \left(\frac{S}{Z}\right)^{-\frac{\left(\nu-\sigma\sqrt{2(p+p_0)}\right)}{\sigma^2}} \frac{\mathrm{d}Z}{Z} + \left[2\left(p+p_0\right)\sigma^2\right]^{-1/2} \int_S^\infty \hat{F}(Z,p) \left(\frac{S}{Z}\right)^{-\frac{\left(\nu+\sigma\sqrt{2(p+p_0)}\right)}{\sigma^2}} \frac{\mathrm{d}Z}{Z}. \tag{40}$$

As with the Mellin transforms, we would stress that our Laplace transform (32) covers the whole of space, $0 \le \tau < \infty$, and therefore covers both the region where it is optimal to hold the option and that where exercise is optimal. Once again, in the second region (8) is a perfectly good solution, even though the option is no longer held. This is in contrast to the partial transform approach, where the transform is only applied where it is optimal to hold the option. The partial Laplace transform is due to Evans [**20, 19**], and can be defined as

$$\mathcal{L}\left[A\left(\tau\right)\right] = \int_{\tau_f(S)}^\infty \mathrm{e}^{-p\tau}\, A\left(\tau\right) \mathrm{d}\tau. \tag{41}$$

We will now apply a (conventional) Laplace transform (32) to (7), and work forward to recover (8). We are transforming (7) both in the region where it is optimal to hold and where exercise is optimal. From (7), we get an ODE for the transform of the residual R,

$$\left[p - \frac{\sigma^2 S^2}{2}\frac{d^2}{dS^2} - (r-D)\, S\frac{d}{dS} + r\right]\hat{R} = \hat{F}(S,p), \tag{42}$$

which has homogeneous solutions,

$$\hat{R}_1 = S^{-\frac{\left(\nu-\sigma\sqrt{2(p+p_0)}\right)}{\sigma^2}}, \qquad \hat{R}_2 = S^{-\frac{\left(\nu+\sigma\sqrt{2(p+p_0)}\right)}{\sigma^2}}. \tag{43}$$

We note that since r, D and σ are all assumed to be positive, and we assume that p has a positive real part from the definition of the Laplace transform, then the real part of the exponent in $\hat{R}_1$ is assumed positive, while that in $\hat{R}_2$ is assumed negative. The Wronskian of $\hat{R}_1$ and $\hat{R}_2$ is

$$W = \hat{R}_1\hat{R}_2' - \hat{R}_1'\hat{R}_2 = -2\sigma^{-1}\left[2\left(p+p_0\right)\right]^{1/2} S^{-1-2\nu/\sigma^2}, \tag{44}$$

so that we can write the solution to (42) using variation of parameters as

$$\begin{aligned}\hat{R} =& -\hat{R}_1(S)\int_S^\infty \frac{2\hat{R}_2(Z)\hat{F}(Z,p)\mathrm{d}Z}{\sigma^2 Z^2 W} - \hat{R}_2(S)\int_0^S \frac{2\hat{R}_1(Z)\hat{F}(Z,p)\mathrm{d}Z}{\sigma^2 Z^2 W} \\ =& \left[2\left(p+p_0\right)\sigma^2\right]^{-1/2} \int_0^S \hat{F}(Z,p)\left(\frac{S}{Z}\right)^{-\frac{\left(\nu+\sigma\sqrt{2(p+p_0)}\right)}{\sigma^2}} \frac{\mathrm{d}Z}{Z} \\ &+ \left[2\left(p+p_0\right)\sigma^2\right]^{-1/2} \int_S^\infty \hat{F}(Z,p)\left(\frac{S}{Z}\right)^{-\frac{\left(\nu-\sigma\sqrt{2(p+p_0)}\right)}{\sigma^2}} \frac{\mathrm{d}Z}{Z},\end{aligned} \tag{45}$$

which of course recovers (40) and thence (8).

Once again, we wonder out loud whether we can arrive at an equation for the free boundary from (40). Again, if we invert the transform, we arrive at (9), but there does not appear to be any other obvious equation in Laplace transform space.

2.3. Fourier Transforms

The analysis using Fourier transforms is straightforward because of the well-known relation between Mellin transforms, two-sided Laplace transforms, and Fourier transforms. We should recall our definitions of the Mellin (16), and the two-sided Laplace (20) transforms discussed earlier,

$$\begin{aligned} \mathcal{M}\left[A\left(S\right)\right]\left(p\right) &= \int_0^\infty A\left(S\right)S^{p-1}\mathrm{d}S, \\ \mathcal{L}^{(2)}\left[B\left(x\right)\right]\left(p\right) &= \int_{-\infty}^\infty B\left(x\right)\mathrm{e}^{-px}\,\mathrm{d}x. \end{aligned} \tag{46}$$

which were linked via the transformation $S = \mathrm{e}^{-x}$ and $A(S) = B(x)$. If we write $p = \mathrm{i}\,s$ in (16), we arrive at one of the definitions of the Fourier transform,

$$\mathcal{F}\left[B\left(x\right)\right]\left(s\right) = \int_{-\infty}^\infty B\left(x\right)\mathrm{e}^{-\mathrm{i}\,sx}\,dx = \mathcal{L}^{(2)}\left[B\left(x\right)\right]\left(\mathrm{i}\,s\right). \tag{47}$$

Convolution for Fourier transforms is defined by

$$\begin{aligned} A\left(x\right)*B\left(x\right) &= \int_0^\infty A\left(x-z\right)B\left(z\right)\mathrm{d}z, \\ \mathcal{F}\left[A\left(x\right)*B\left(x\right)\right] &= \hat{A}\left(s\right)\hat{B}\left(s\right). \end{aligned} \tag{48}$$

If follows that combining the transformation $S = \mathrm{e}^{-x}$ with a Fourier transform is equivalent to taking a complex Mellin transform, so that many of the results for Mellin transforms carry over to Fourier transforms, with in particular

$$\hat{R}(s,\tau) \quad = \textstyle\int_0^\tau \hat{F}(s,\zeta)\hat{G}(s,\tau-\zeta)\mathrm{d}\zeta. \tag{49}$$

with

$$\begin{aligned} \hat{G}(s,\tau) &= \exp\left[\left(-r-\nu\,\mathrm{i}\,s-\frac{\sigma^2 s^2}{2}\right)\tau\right], \\ \hat{F}(s,\tau) &= \frac{D\left(S_f^{2-\mathrm{i}\,s}(\tau)-E^{2-\mathrm{i}\,s}\right)}{\mathrm{i}\,s-2} - \frac{rE\left(S_f^{1-\mathrm{i}\,s}(\tau)-E^{1-\mathrm{i}\,s}\right)}{\mathrm{i}\,s-1} \\ &\quad + \frac{\sigma^2 E^{2-\mathrm{i}\,s}}{2}. \end{aligned} \tag{50}$$

$\hat{F}(s,\tau)$ is the same for both the call and the put. From (26), the ODE for the transform of the residual R becomes

$$\left[\frac{d}{\mathrm{d}\tau}+\frac{\sigma^2 s^2}{2}+\mathrm{i}\,s\nu+r\right]\hat{R} \quad = \quad \hat{F}(s,\tau), \tag{51}$$

which has a solution

$$\begin{aligned}\hat{R}(s,\tau) &= \int_0^\tau \hat{F}(s,\zeta)\exp\left[\left(-\frac{\sigma^2 s^2}{2} - \mathrm{i}\, s\nu - r\right)(\tau-\zeta)\right]\mathrm{d}z \\ &= \int_0^\tau \hat{F}(s,\zeta)\hat{G}(s,\tau-\zeta)\mathrm{d}\zeta,\end{aligned} \tag{52}$$

which of course recovers (49) and thence (8).

3. Boundary Fixing Transformation

In our analysis so far, we have assumed that the boundary was a free boundary, located at $S = S_f(\tau)$ or equivalently at $\tau = \tau_f(S)$. However, it is also possible to use Landau's boundary fixing transformation and write $S = XS_f(\tau)$ so that the free boundary is fixed at $X = 1$. For completeness, we present the analysis for the Mellin transform combined with the Landau transform.

If we write $V(XS_f(\tau),\tau) = V_F(X,\tau)$, then (3) becomes

$$\begin{aligned}\mathcal{BS}_F(V| &= \left[\frac{\partial}{\partial\tau} - \frac{\sigma^2X^2}{2}\frac{\partial^2}{\partial X^2} - \left(r - D + \frac{S_f'(\tau)}{S_f(\tau)}\right)X\frac{\partial}{\partial X} + r\right]V_F \\ &= 0.\end{aligned} \tag{53}$$

The Landau transformation fixes the boundary, but introduces a new term given by $-\left(S_f'(\tau)/S_f(\tau)\right)X\,(\partial\,/\partial X)$ in (53), with the result that the conditions at the free boundary are easier to implement but the underlying PDE is more complicated. Both [**21, 36**] and [**33**] took this approach.

At expiry $\tau = 0$, we have $V_F(X,0) = (XS_0 - E)\,H\,(XS_0 - E)$, where $S_0 = S_f(0)$ is the location of the free boundary at expiry. At the free boundary, $X = 1$, we exchange V_F for the portfolio $XS_f(\tau) - E$, and we require that the value of the option and its delta be continuous across the boundary, so $V_F(1,\tau) = S_f(\tau) - E$ and $(\partial V_F/\partial X)(1,\tau) = S_f(\tau)$. Once again, we will work in terms of the residual defined in (6), and to this end, we will introduce $R_F(X,\tau) = R(XS_f(\tau),\tau)$ and $F_F(X,\tau) = F(XS_f(\tau),\tau)$. We can then use (4,8) to write

$$\begin{aligned}R_F(X,\tau) &= \int_0^\tau\int_0^\infty F_F(Z,\zeta)G_F\left(\frac{X}{Z},\tau,\zeta\right)\frac{\mathrm{d}Z\mathrm{d}\zeta}{Z}, \\ G_F(X,\tau,\zeta) &= \frac{\mathrm{e}^{-r(\tau-\zeta)}}{\sigma\sqrt{2\pi(\tau-\zeta)}}\exp\left(-\frac{\left[\ln(X) + \ln\left(\frac{S_f(\tau)}{S_f(\zeta)}\right) + \nu(\tau-\zeta)\right]^2}{2\sigma^2(\tau-\zeta)}\right).\end{aligned} \tag{54}$$

From (54), we see that G_F is slightly more complicated than G. The integral equations (9) from continuity of the option and its delta at the free boundary become

$$\begin{aligned}\int_0^\tau\int_0^\infty F_F(Z,\zeta)G_F\left(\frac{1}{Z},\tau,\zeta\right)\frac{\mathrm{d}Z\mathrm{d}\zeta}{Z} &= 0, \\ \int_0^\tau\int_0^\infty F_F(Z,\zeta)\frac{\partial G_F}{\partial X}\left(\frac{1}{Z},\tau,\zeta\right)\frac{\mathrm{d}Z\mathrm{d}\zeta}{Z} &= 0.\end{aligned} \tag{55}$$

For a call, the counterpart of the forcing term (10) is

$$
\begin{aligned}
F_F(X,\tau) = {} & \Big([rE - DXS_f(\tau)]\, H\,(XS_f(\tau) - E) \\
& + \left[(r-D)\,(XS_f(\tau) - E) + \frac{\sigma^2 XS_f(\tau)}{2} \right] XS_f(\tau)\delta\,(XS_f(\tau) - E) \\
& + \frac{\sigma^2 X^2 S_f^2(\tau)}{2} \left[(XS_f(\tau) - E)\,\delta\,(XS_f(\tau) - E) \right]' \Big)\, H\,(1 - X)\,,
\end{aligned}
\tag{56}
$$

while for a put, (11) is replaced by

$$
\begin{aligned}
F_F(X,\tau) = {} & \Big([DXS_f(\tau) - rE]\, H\,(E - XS_f(\tau)) \\
& + \left[(D-r)\,(E - XS_f(\tau)) + \frac{\sigma^2 XS_f(\tau)}{2} \right] XS_f(\tau)\delta\,(E - XS_f(\tau)) \\
& + \frac{\sigma^2 X^2 S_f^2(\tau)}{2} \left[(E - XS_f(\tau))\,\delta\,(E - XS_f(\tau)) \right]' \Big)\, H\,(X - 1)\,.
\end{aligned}
\tag{57}
$$

Using (56) and (57) in (54), for a call (12) is replaced by

$$
\begin{aligned}
R_F(X,\tau) = {} & \int_0^\tau \frac{\sigma^2 S_f(\zeta)}{2E} G_F\left(\frac{XS_f(\zeta)}{E}, \tau, \zeta \right) \mathrm{d}\zeta \\
& + \int_0^\tau \int_{E/S_f(\zeta)}^1 (rE - DZS_f(\zeta))\, G_F\left(\frac{X}{Z}, \tau, \zeta \right) \frac{\mathrm{d}Z\mathrm{d}\zeta}{Z},
\end{aligned}
\tag{58}
$$

and for a put (13) is replaced by

$$
\begin{aligned}
R_f(X,\tau) = {} & \int_0^\tau \frac{\sigma^2 ES_f(\zeta)}{2} G_F\left(\frac{XS_f(\zeta)}{E}, \tau, \zeta \right) \mathrm{d}\zeta \\
& + \int_0^\tau \int_1^{E/S_f(\zeta)} (DZS_f(\zeta) - rE)\, G_F\left(\frac{X}{Z}, \tau, \zeta \right) \frac{\mathrm{d}Z\mathrm{d}\zeta}{Z}.
\end{aligned}
\tag{59}
$$

The first equation in (55) then becomes

$$
\begin{aligned}
& \int_0^\tau \frac{\sigma^2 S_f(\zeta)}{2E} G_F\left(\frac{S_f(\zeta)}{E}, \tau, \zeta \right) \mathrm{d}\zeta \\
& + \int_0^\tau \int_{E/S_f(\zeta)}^1 (rE - DZS_f(\zeta))\, G_F\left(\frac{1}{Z}, \tau, \zeta \right) \frac{\mathrm{d}Z\mathrm{d}\zeta}{Z} = 0,
\end{aligned}
\tag{60}
$$

for a call, while for a put we have

$$
\begin{aligned}
& \int_0^\tau \frac{\sigma^2 ES_f(\zeta)}{2} G_F\left(\frac{S_f(\zeta)}{E}, \tau, \zeta \right) \mathrm{d}\zeta \\
& + \int_0^\tau \int_1^{E/S_f(\zeta)} (DZS_f(\zeta) - rE)\, G_F\left(\frac{1}{Z}, \tau, \zeta \right) \frac{\mathrm{d}Z\mathrm{d}\zeta}{Z} = 0.
\end{aligned}
\tag{61}
$$

A similar pair of equations can be written using the second equation in (55).

3.1. Mellin Transforms

The analysis in this section is very similar to that presented without the use of the boundary fixing transformation, and once again, we will approach the problem from two directions simultaneously. We will apply a Mellin transform to the governing PDE after the Landau transformation and solve the resulting ordinary differential equation (ODE) in transform space to obtain an expression for the transform of the solution. Rather than invert this transform, we will apply a forward transform to the Green's function solution (54) presented above, which will be the same as the solution of the ODE, meaning that the inverse transform yields (54).

The Mellin transform used here is (16), but taking a transform with respect to X rather than S. As with (8), (54) is a convolution integral,

$$\begin{aligned} R_F(X,\tau) &= \int_0^\tau F_F(X,\zeta) * G_F\left(X,\tau,\zeta\right) \mathrm{d}\zeta, \\ \hat{R}_F(p,\tau) &= \int_0^\tau \hat{F}_F(p,\zeta) \hat{G}_F\left(p,\tau,\zeta\right) \mathrm{d}\zeta, \end{aligned} \tag{62}$$

with $\hat{R}_F$ the transform of R_F. We can write $\hat{G}_F$ and $\hat{F}_F$ in terms of $\hat{G}$ and $\hat{F}$ (19), the transforms obtained without fixing the boundary,

$$\begin{aligned} \hat{G}_F(p,\tau,\zeta) &= \hat{G}(p,\tau)\left(\frac{S_f(\tau)}{S_f(\zeta)}\right)^{-p} \\ &= \exp\left[\left(-r-\nu p+\frac{\sigma^2 p^2}{2}\right)(\tau-\zeta)\right]\left(\frac{S_f(\tau)}{S_f(\zeta)}\right)^{-p}, \\ \hat{F}_F(p,\tau) &= S_f^{-p}(\tau)\hat{F}\left(p,\tau\right) \\ &= \left[\frac{D\left(S_f^{2-p}(\tau)-E^{2-p}\right)}{p-2}-\frac{rE\left(S_f^{1-p}(\tau)-E^{1-p}\right)}{p-1}\right. \\ &\qquad \left. + \frac{\sigma^2 E^{2-p}}{2}\right] S_f^{-p}(\tau). \end{aligned} \tag{63}$$

$\hat{F}_F(p,\tau)$ is the same for both the call and the put. Once again, our Mellin transform covers the whole of space, $0 \le X < \infty$, rather than just the region where it is optimal to hold the option.

We will now apply a Mellin transform to the equation $\mathcal{BS}_F\left(R_F\right) = F_F$ and work forward to recover (54). This yields an ODE for the transform of R_F,

$$\left[\frac{d}{\mathrm{d}\tau} - \frac{\sigma^2 p^2}{2} + p\left(\nu + \frac{S_f'(\tau)}{S_f(\tau)}\right) + r\right]\hat{R}_F = \hat{F}_F(p,\tau). \tag{64}$$

Thus has an additional term $p\left(S_f'(\tau)/S_f(\tau)\right)\hat{R}_F$ compared to (26). Again, we have transformed the equation for $0 \le X < \infty$, not just where it is optimal to hold. Since $R_F(S,0) = 0$ for vanilla Americans, it follows that $\hat{R}_F(p,0) = 0$, and

(64) has a solution

$$
\begin{aligned}
\hat{R}_F(p,\tau) &= \int_0^\tau \hat{F}_F(p,\zeta)\exp\left[\left(\frac{\sigma^2p^2}{2} - p\nu - r\right)(\tau-\zeta)\right]\left(\frac{S_f(\tau)}{S_f(\zeta)}\right)^{-p}\mathrm{d}z \\
&= \int_0^\tau \hat{F}_F(p,\zeta)\hat{G}_F(p,\tau,\zeta)\mathrm{d}\zeta,
\end{aligned}
\tag{65}
$$

which of course recovers (62) and thence (54).

Again, we must consider if we can arrive at an equation for the free boundary from (65), other than the obvious transform of (55). Once again, we will apply the limit $\tau \to \infty$ to (54),

$$
R_{F\infty}(X) = \lim_{\tau\to\infty}\int_0^\tau\int_0^\infty F_F(Y,\zeta)G_F\left(\frac{X}{Y},\tau,\zeta\right)\frac{dY\mathrm{d}\zeta}{Y}, \tag{66}
$$

where $R_{F\infty}$ is the perpetual version of the residue. If we apply the same limit to (65), we arrive at the transform of (66), namely

$$
\begin{aligned}
\hat{R}_{F\infty}(p) &= \lim_{\tau\to\infty}\int_0^\tau \hat{F}_F(p,\zeta)\hat{G}_F(p,\tau,\zeta)\mathrm{d}\zeta \\
&= \lim_{\tau\to\infty}\int_0^\tau \left[\frac{D\left(S_f^{2-p}(\zeta)-E^{2-p}\right)}{p-2} - \frac{rE\left(S_f^{1-p}(\zeta)-E^{1-p}\right)}{p-1}\right. \\
&\quad \left. + \frac{\sigma^2E^{2-p}}{2}\right]\exp\left[\left(-r-\nu p+\frac{\sigma^2p^2}{2}\right)(\tau-\zeta)\right]S_f^{-p}(\tau)\mathrm{d}\zeta.
\end{aligned}
\tag{67}
$$

For the call, (67) is the transform of

$$
\begin{aligned}
R_{F\infty}(X) = &\int_0^\tau \frac{\mathrm{e}^{-r(\tau-\zeta)}}{\sigma\sqrt{2\pi(\tau-\zeta)}}\left[\frac{\sigma^2S_f(\zeta)}{2E}\exp\left(-\frac{\left[\ln\left(\frac{XS_f(\tau)}{E}\right)+\nu(\tau-\zeta)\right]^2}{2\sigma^2(\tau-\zeta)}\right)\right. \\
&+ \int_{E/S_f(\zeta)}^1 (rE - DZS_f(\zeta)) \\
&\left.\times\exp\left(-\frac{\left[\ln\left(\frac{XS_f(\tau)}{ZS_f(\zeta)}\right)+\nu(\tau-\zeta)\right]^2}{2\sigma^2(\tau-\zeta)}\right)\frac{\mathrm{d}Z}{Z}\right]\mathrm{d}\zeta,
\end{aligned}
\tag{68}
$$

while for the put, it is the transform of

$$
\begin{aligned}
R_{F\infty}(X) = \int_0^\tau \frac{\mathrm{e}^{-r(\tau-\zeta)}}{\sigma\sqrt{2\pi(\tau-\zeta)}} \Bigg[& \frac{\sigma^2 S_f(\zeta)}{2E} \exp\left(- \frac{\left[\ln\left(\frac{XS_f(\tau)}{E}\right) + \nu(\tau-\zeta)\right]^2}{2\sigma^2(\tau-\zeta)} \right) \\
& + \int_1^{E/S_f(\zeta)} \left(DZS_f(\zeta) - rE\right) \\
& \times \exp\left(- \frac{\left[\ln\left(\frac{XS_f(\tau)}{ZS_f(\zeta)}\right) + \nu(\tau-\zeta)\right]^2}{2\sigma^2(\tau-\zeta)} \right) \frac{\mathrm{d}Z}{Z} \Bigg] \mathrm{d}\zeta.
\end{aligned} \tag{69}
$$

4. Discussion

In the previous two sections, we revisited the use of integral transforms to tackle American options, using conventional Mellin, Laplace and Fourier transforms (16,32,47) rather than their partial counterparts. We were able to use conventional transforms rather than partial transforms because we were able to extend the governing PDE to the region where exercise is optimal by using the PDE obeyed by the early exercise pay-off, which is in effect also the approach taken by the Green's function approach. In each case, we transformed the governing PDE into an ODE in transform space and solved the ODE, finding that in each case, the solution of the ODE was the same as the transform of the Green's function solution, which we used as our reference solution, and the same was true when we combined a Mellin transform with Landau's boundary fixing transformation. Each of these approaches leads to leads to a pair of integral equations for the location of the free boundary, and again we found that the integral equations from the transform approaches were the same as those from the Green's function approach. This is the principal result of this paper, and the motivation for our study, that all roads (or at least the roads considered here) lead to Rome, by which we mean that the differing approaches produce equivalent solutions and equivalent integral equations. In one sense, our analysis is not a surprise, because the Green's function solution (8) is both a Mellin and a Laplace convolution. In another sense, however, it is a surprise as our analysis required us to use a PDE in the region where exercise is optimal, and historically, there have been concerns over whether it is possible to apply full-space transforms in such situations, which led to the development of partial transforms [**19, 20, 46, 56**] and their application to free boundary problems, although recent studies such as [**16**] have done much to allay those concerns.

The pair of integral equations for the location of the free boundary (9) has been written in a number of different forms in the extensive literature on the Green's function approach. Chiarella [**15**] has discussed how the various representations are related. The integral equations produced by our analysis do not have any particular advantage over the existing literature, and that was not the goal of our study. It is probably true that some of the integral equations in the existing literature are easier to solve either as a series close to expiry or numerically than those produced by our analysis, and it is probably also true that some of the integral equations in the existing literature are in a form that makes some property or other of the solution more readily apparent than those produced by our analysis, but this is an aspect of the problem which we have not explored as that was not the object of our study. We have not concerned ourselves with whether one approach or another is in some sense better, or whether one approach is efficacious or another meretricious, although if other researchers choose to pursue that path we would encourage them. For much of our analysis, we applied integral transforms directly to the Black-Scholes-Merton PDE (2) rather than following many earlier authors and transforming this PDE into a diffusion-like equation. There is of course some value in making such a transformation, not least because the diffusion equation has been studied for longer, and in more depth, than the Black-Scholes-Merton PDE, and in this respect at least, it could be argued that studies such as [**15, 16, 17, 54, 55, 57**], which apply various flavors of Fourier transform to a diffusion-like equation, and [**33**], which applies a Laplace transform, have an advantage over studies working directly with the Black-Scholes-Merton PDE. With such a transformation, the pair of integral equations (9) for the location of the free boundary would simply be replaced by the same equations with S and Z replaced by the transformed variables, so that the equations would involve the Green's function for the diffusion-like equation rather than the Green's function for the Black-Scholes-Merton PDE.

In our study, we wrote the pair of integral equations (9) for the location of the free boundary in a very simple form which is both a Mellin (in S) and a Laplace (in τ) convolution of a nonhomogeneous forcing term $F(S,\tau)$ with the Green's function $G(S,\tau)$. This was deliberate, as in this study we have not been concerned with solving these integral equations or discussing their various properties. Rather, we chose this particular form to ensure that the integral transforms were straightforward and to allow us to square the circle and show that the roads considered here all lead to Rome, with the different approaches producing equivalent integral equations.

As we mentioned above, for each case studied here, the solution of the ODE in transform space is simply the transform of the Green's function solution, so that the equations for the free boundary stemming from these transforms are simply those stemming from the Green's function approach (55). For the Mellin transform, if we take the limit $\tau \to \infty$ we arrive at an alternative, and more complicated, set of equations (31), (30), (68), (69) in addition to (55).

In the Introduction, we said that for the Black-Scholes-Merton PDE (2), the natural transforms are Laplace with respect to τ and Mellin with respect to S, where natural in this sense means that the transformed equation does indeed have derivatives with respect to fewer variables. Equivalently, if we transform the Black-Scholes-Merton PDE into a diffusion-like equation, the natural transforms are again Laplace with respect to τ but now either two-sided Laplace or Fourier with respect to the transformed variable x. From our analysis, it would seem that Mellin with respect to S (or equivalently two-sided Laplace or Fourier with respect to x) is the more natural and direct approach because it leads to a first order ODE in transform space (26), (51), (64) while Laplace with respect to τ leads to a more complicated second order ODE in transform space (42).

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References

1. Alobaidi G. and Mallier R., *Asymptotic analysis of American call options*, International J. Mathematics and Mathematical Sciences **27** (2001), 177–188.

2. ______, *Laplace transforms and the American straddle*, J. Applied Math. **2** (2002), 121–129.

3. ______, *Laplace Transforms and the American Call Option*, in Mathematical Control Theory and Finance, (ed.s A. Sarychev, et al.), Springer (2008), 15–27.

4. Alobaidi G., Mallier R. and Deakin A. S., *Laplace transforms and installment options*, Math. Models Methods Appl. Sci. **14** (2004), 1167–1189.

5. Alobaidi G., Mallier R. and Mansi S., *Laplace transforms and shout options*, Acta Math. Univ. Comenian. **80** (2011), 79–102.

6. Barone-Adesi G., *The saga of the American put*, J. Banking & Fin. **29** (2005), 2909–2918.

7. Barone-Adesi G. and Elliot R.J., *Free boundary problems in the valuation of securities*, Working Paper, University of Alberta 1989.

8. Black F. and Scholes M., *The pricing of options and corporate liabilities*, J. Political Economy **81** (1973), 637–659.

9. Carr P., Jarrow R. and Myneni R., *Alternative characterizations of the American put option*, Math. Fin. **2** (1992), 87–106.

10. Chadam J., *Free boundary problems in mathematical finance*, in Progress in Industrial Mathematics at ECMI 2006, (ed.s L.L. Bonilla, et al.), Mathematics in Industry Volume 12, Springer (2008), 655–665.

11. ______, *Integral equation methods for free boundary problems*, in Encyclopedia of Quantitative Finance Volume 2, (ed. R. Cont), Wiley (2010), 956– 962.

12. Chen X. and Chadam J., *Analytical and numerical approximations for the early exercise boundary for American options*, Dynamics of Continuous, Discrete and Impulsive Systems **10** (2003), 649–660.

13. ______, *A mathematical analysis of the optimal exercise boundary for american put options*, SIAM J. Math. Anal. **38** (2006) 1613–1641.

14. Chesney M. and Gibson R., *State space symmetry and two-factor option pricing models*, Advances in Futures and Options Research **8** (1993), 85–112.

15. Chiarella C., Kucera A. and Ziogas A., *A survey of the integral representation of American option prices*, Quantitative Finance Research Centre Research Paper 118, University of Technology, Sydney, Australia 2004.

16. Chiarella C. and Ziogas A., *A Fourier transform analysis of the American call option on assets driven by jump-diffusion processes*, Quantitative Finance Research Centre Research Paper 174, University of Technology, Sydney, Australia 2006.

17. ______, *American call options under jump-diffusion processes – a Fourier transform approach*, Appl. Math. Fin. **16** (2009), 37–79.

18. Cook J., *Asymptotic Analysis of American-Style Options*, PhD thesis, University of Bath 2009.

19. Crank J., *Free and Moving Boundary Problems*, Clarendon 1984.

20. Evans G.W., Isaacson E. and MacDonald J. K. L., *Stefan-like problems*, Q. Appl. Math. **8** (1950), 312–319.

21. Evans J.D., Kuske R.E. and Keller J.B., *American options with dividends near expiry*, Math. Fin. **12** (2002), 219–237.

22. Firth N., *High Dimensional American Options*, PhD thesis, University of Oxford 2005.

23. Frontczak R., *On the Application of Mellin Transforms in the Theory of Option Pricing*, Dissertation, University of Tübingen 2010.

24. Frontczak R. and Schöbel R., *Pricing American options with Mellin transforms*, Tübinger Diskussionsbeitrag No. 319, University of Tbingen 2008.

25. Geske R. and Johnson H., *The American put option valued analytically*, J. Fin. **39** (1984), 1511–1524.

26. Goodman J. and Ostrov D. N., *On the early exercise boundary of the American put option*, SIAM J. Appl. Math. **62** (2002), 1823–1835.

27. Hill J. M., *One-dimensional Stefan Problems: an Introduction*, Pitman Monographs and Surveys in Pure and Applied Mathematics, Longman 1987.

28. Huang J.-Z., Subrahamanyan M. G., and Yu G. G., *Pricing and hedging American options: a recursive integration method*, Rev. Financial Studies **9** (1998), 277–300.

29. Jacka S. D., *Optimal stopping and the American put*, Math. Fin. **1** (1991), 1–14.

30. Jamshidian F., *An analysis of American options*, Review of Futures Markets **11** (1992), 72–80.

31. Ju N., *Pricing an American option by approximating its early exercise boundary as a multipiece exponential function*, Rev. Financial Studies **11** (1998), 627–646.

32. Kim I. J., *The analytic valuation of American options*, Rev. Financial Studies **3** (1990), 547–552.

33. Knessl C., *A note on a moving boundary problem arising in the American put option,* Stud. Appl. Math. **107** (2001), 157–183.

34. ______, *Asymptotic analysis of the American call option with dividends,* Euro. J. Appl. Math. **13** (2002), 587–616.

35. Kolodner I. I., *Free boundary problems for the heat conduction equation with applications to problems of change of phase*, Communications in Pure and Applied Math. **9** (1956), 1-31.

36. Kuske R. E. and Keller J. B., *Optimal exercise boundary for an American put option*, Appl. Math. Fin. **5** (1998), 107–116.

37. Kwok Y., *Mathematical Models of Financial Derivatives*, 1st edn, Springer-Verlag, Singapore 1998.

38. Landau H. G., *Heat conduction in a melting solid*, Quart. J. Appl. Math. **8**(1950), 81–94.

39. Mallier R. and Alobaidi G., *Laplace transforms and American options*, Appl. Math. Fin. **7** (2000), 241–256.

40. ______, *Pricing equity-linked debt using the Vasicek model*, Acta Math. Univ. Comenian. **71** (2002), 209–218.

41. ______, *The American put option close to expiry*, Acta Math. Univ. Comenian. **73** (2004), 161–174.

42. Mallier R. and Deakin A. S., *A Green's function for a convertible bond using the Vasicek model*, J. Appl. Math. **2** (2002), 219–232.

43. McDonald R. and Schroder M., *A parity result for American options*, J. Comp. Fin. **1** (1998), 5–13.

44. McKean H. P. Jr, *Appendix: A free boundary problem for the heat equation arising from a problem in mathematical economics*, Industrial Management Review **6** (1965), 32–29.

45. Merton R. C., *The theory of rational option pricing*, J. Econ. and Management Science **4** (1973), 141–183.

46. Naylor D., *On a Mellin-type integral transform*, J. Math. Mech. **12** (1963), 265–274.

47. Ockendon J.R., *Techniques of analysis*, in Moving Boundary Problems in Heat Flow and Diffusion, (eds. J.R. Ockendon and W.R. Hodgkins), Clarendon (1975), 138–149.

48. Panini R., *Option Pricing with Mellin Transforms*, PhD thesis, Stony Brook University 2004.

49. Panini R. and Srivastav R., *Option pricing with Mellin transforms*, Math. Comput. Modelling **40** (2004), 43–56.
50. ———, *Pricing perpetual options using Mellin transforms*, Appl. Math. Letters **18** (2005), 471–4.
51. Pipkin A. C., *A Course on Integral Equations*, Springer-Verlag 1991.
52. Polyanin A. D. and Manzhirov V., *Handbook of Integral Equations*, CRC Press 1998.
53. Samuelson P. A., *Rational theory of warrant pricing*, Industrial Management Review **6** (1965), 13–31.
54. Ševčovič D., *Analysis of the free boundary for the pricing of an American call option*, Euro. J. Appl. Math. **12** (2001), 25–37.
55. ———, *Transformation methods for evaluating approximations to the optimal exercise boundary for linear and nonlinear Black-Scholes equations*, In Nonlinear Models in Mathematical Finance: New Research Trends in Option Pricing, (ed. M. Ehrhardt), Nova Sci. Publ., New York, (2008), 153–198.
56. Sneddon I. N., *The Use of Integral Transforms*, McGraw-Hill 1972.
57. Stamicar R., Ševčovič D. and Chadam J., *The early exercise boundary for the American put near expiry: numerical approximation*, Can. Appl. Math. Quart **7** (1999), 427–444.
58. Tao L. N., *The analyticity of solutions of the Stefan problem*, Arch. Rat. Mech. Anal. **25** (1980), 285–301.
59. Van Moerbeke P., *On optimal stopping and free boundary problems*, Archives of Rational Mechanical Analysis **60** (1976), 101–148.
60. Wilmott P., *Paul Wilmott on Quantitative Finance*, Wiley 2000.
61. Wilmott P., Dewynne J. and Howison S., *Option Pricing.* Oxford Financial Press, Oxford 1993.
62. Zhu S.-P., *An exact and explicit solution for the valuation of American put options*, Quant. Fin. **6** (2006), 229–242.

G. Alobaidi, Department of Mathematics and Statistics American University of Sharjah Sharjah, United Arab Emirates, *e-mail*: `galobaidi@aus.edu`

R. Mallier, Department of Mathematics and Statistics York University Toronto ON M3J 1P3 Canada, *e-mail*: `mallier@mathstat.yorku.ca`

M. C. Haslam, Department of Mathematics and Statistics York University Toronto ON M3J 1P3 Canada, *e-mail*: `mchaslam@mathstat.yorku.ca`