Acta Math. Univ. Comenianae
Vol. LXXXVII, 2 (2018), pp. 301–307

POLYNOMIALS AND MULTILINEAR MAPPINGS IN LOCALLY CONVEX ALGEBRAS

A. EL KINANI, R. CHOUKRI AND A. OUDADES

ABSTRACT. We introduce the pseudo-equicontinuity of a sequence of maps in topological algebras. This notion allows us to simplify the proofs of classical results such as theorems of Turpin for power maps in commutative locally convex algebras and that of B. Mitiagin, S. Rolewics and W. Żelazko for entire functions in commutative B_0-algebras. We also obtain that a commutative B_0-algebra A is m-convex if a power series with an appropriate control over its coefficients operates in some open subset of A.

1. INTRODUCTION

In [**6**], P. Turpin showed that a commutative locally convex algebra in which the sequence $(x \mapsto x^n)_n$ of power maps is equicontinuous at zero is necessarily m-convex. In the non-commutative case, W. Żelazko [**9**] gave an example of a complete non m-convex locally convex algebra in which the sequence $(x \mapsto x^n)_n$ of power maps is equicontinuous at zero. Concerning entire functions in locally convex algebra, it is known [**3**] that every entire function operates in any unital and M-complete locally m-convex algebra. In [**5**], B. Mitiagin, S. Rolewics and W. Żelazko showed that a unital commutative B_0-algebra in which every entire function operates is necessarily m-convex. Their proof is long and technical. A short and direct proof is given in [**2**] in the general context of Baire algebras. Analyzing the proofs of [**5**] and [**2**] leads us to consider the pseudo-equicontinuity of a sequence of maps in topological algebras. We examine the link between the equicontinuity of a family of homogeneous polynomials and the pseudo-equicontinuity of their generators (Proposition 2). We consider a sequence of homogeneous polynomials of degree n and we extricate sufficient conditions (Corollary 3), similar to that of the sequence $(x \mapsto x^n)_n$ of power maps, involving the m-convexity of the algebra. As a consequence, we obtain (Corollary 4) the main result of Turpin [**6**]. We show (Proposition 5) that the convergence of a continuous homogeneous polynomials series on an open subset of a B_0-algebra implies the equicontinuity at zero of its general term. As an application of the previous result, we obtain (Corollary 7) the main result of [**5**] wherein a commutative B_0-algebra in which every entire

---

Received March 17, 2017; revised April 7, 2018.
2010 *Mathematics Subject Classification.* Primary 46H05; Secondary 46H20.
Key words and phrases. Locally convex algebras; continuous multiplication; homogeneous polynomials; multilinear mapping; m-convexity; B_0-algebra; power maps; entire functions.

function operates is necessarily m-convex. In the direction of the cited result of B. Mitiagin, S. Rolewics and W. Żelazko [**5**], in a commutative B_0-algebra (A,τ), we consider a power series $\sum_n a_n z^n$ with $(a_n)_n \subset \mathbb{R}_+^*$, convergent in a certain open subset of (A,τ). We show (Theorem 9) that (A,τ) is m-convex provided that the sequence $\left(\frac{a_n a_m}{a_{m+n}}\right)_{n,m\in\mathbb{N}^*}$ is bounded.

2. Preliminaries

Let E, F be two vector spaces over a field $\mathbb{K}$ ($\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$) with a Hausdorff locally convex topology. A mapping $f\colon E\to F$ is called a *homogeneous polynomial of degree* n if there exists a n-linear symmetrical mapping $\overline{f}\colon E^n\to F$ such that $f(x)=\overline{f}(x,\dots,x)$ for every $x\in E$. We say that $\overline{f}$ is associated with f or that $\overline{f}$ generates f. $\overline{f}$ is symmetrical means that $\overline{f}(x_1,\dots,x_n)=\overline{f}(x_{j_1},\dots,x_{j_n})$ for every permutation $(j_1,\dots,j_n)$ of the sequence $(1,\dots,n)$. A sequence of maps $f_n\colon E\to F$, $n\in\mathbb{N}^*$, is said to be *equicontinuous at zero* if for every neighborhood U of zero, there exists a neighborhood V of zero such that $f_n(V)\subset U$ for every $n\in\mathbb{N}^*$. A locally convex algebra (A,τ), *l.c.a.* in brief, is an algebra over a field $\mathbb{K}$ ($\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$) with a Hausdorff locally convex topology for which the multiplication is separately continuous. If the multiplication is continuous in both variables, (A,τ) is said to be with jointly continuous multiplication. A locally convex algebra (A,τ) is said to be multiplicatively convex (m-convex, for short) if the topology τ can be given by a family of $(|\cdot|_i)_{i\in I}$ of seminorms such that

$$|xy|_i \le |x|_i|y|_i \qquad \text{for all } x,y\in A,\ i\in I.$$

A B_0*-algebra* is a *l.c.a.* whose underlying locally convex space is a complete metrisable space. An entire function $f(z)=\sum_{n=0}^{+\infty} a_n z^n$, $a_n\in\mathbb{K}$, operates in a *l.c.a.* (A,τ) if for every x in A, $f(x)=\sum_{n=0}^{+\infty} a_n x^n$ converges in (A,τ). For a detailed account of basic properties of general locally m-convex algebras and B_0-algebras, we refer the reader to [**8**] and [**4**].

3. Polynomials and multilinear mappings in locally convex algebras

Definition 1. Let (A,τ) be a topological algebra. A sequence of maps $f_n\colon A^n\to A$, $n\in\mathbb{N}^*$, is said to be *pseudo-equicontinuous at zero* if for every neighborhood U of zero, there exists a neighborhood V of zero such that $f_n(V^n)\subset U$ for every $n\in\mathbb{N}^*$, where $V^n=V\times V\times\cdots\times V$ is the product n times of V.

The following result gives the link between the equicontinuity at zero of a family of homogeneous polynomials and the pseudo-equicontinuity at zero of the family of their generators.

Proposition 2. *Let (A,τ) be a locally convex algebra and $(f_n)_n$ be a sequence of homogeneous polynomials of degree n. Then the following assertions are equivalent:*

1. *$(f_n)_n$ is equicontinuous at zero.*
2. *$(\overline{f_n})_n$ is pseudo-equicontinuous at zero.*

Proof. 2) $\Longrightarrow$ 1) It is obvious.

1) $\Longrightarrow$ 2) Let V be an absolutely convex and closed neighborhood of zero. By Mazur-Orlicz theorem (see [**1**]), we have

$$\overline{f_n}(x_1,\dots,x_n)=\frac{1}{n!}\sum_{\varepsilon_1,\dots,\varepsilon_n=0}^{1}(-1)^{n-(\varepsilon_1+\cdots+\varepsilon_n)}f_n(x_0+\varepsilon_1x_1+\cdots+\varepsilon_nx_n),$$

where $x_1,\dots,x_n\in A$, $n\in\mathbb{N}^*$, and x_0 is an arbitrary point of A. We take $x_0=0$. As $(f_n)_n$ is equicontinuous at zero and zero admits a fundamental system of convex, balanced and absorbing neighborhoods((A,τ) is locally convex), there exists an absolutely convex neighborhood U of zero such that

$$f_n(U)\subset V \qquad \text{for every } n\in\mathbb{N}^*.$$

Now let $x_1,\dots,x_n\in U$. Then $\frac{\varepsilon_1x_1+\cdots+\varepsilon_nx_n}{n}\in U$ and $f_n(\frac{\varepsilon_1x_1+\cdots+\varepsilon_nx_n}{n})=\frac{1}{n^n}f_n\times(\varepsilon_1x_1+\cdots+\varepsilon_nx_n)$. This implies that $f_n(\varepsilon_1x_1+\cdots+\varepsilon_nx_n)\in n^nV$. Hence, for every $n\in\mathbb{N}^*$,

$$\overline{f_n}(x_1,\dots,x_n)\in\frac{(2n)^n}{n!}V \qquad \text{for every } x_1,\dots,x_n\in U.$$

But there exists $c>0$ such that $\frac{(2n)^n}{n!}\le c^n$ for every integer n. Whence for every $n\in\mathbb{N}^*$,

$$\overline{f_n}(x_1,\dots,x_n)\in V \qquad \text{for every } x_1,\dots,x_n\in\frac{1}{c}U$$

and the implication is proved. □

As a consequence, we obtain the following results.

Corollary 3. *Let (A,τ) be a locally convex algebra and $(f_n)_n$ be a sequence of homogeneous polynomials of degree n. If*

(1) *$(f_n)_n$ is equicontinuous at zero,*

(2) *$\overline{f_n}(x_1,\dots,x_n)\overline{f_m}(y_1,\dots,y_m)=\overline{f_{n+m}}(x_1,\dots,x_n,y_1,\dots,y_m)$ for every $n,m\in\mathbb{N}^*$,*

(3) *the image by f_1 of any neighborhood of zero is also a neighborhood of zero, then (A,τ) is m-convex.*

Proof. Let V be an absolutely convex and closed neighborhood of zero. By (1) and Proposition 2, the associated sequence $(\overline{f_n})_n$ of $(f_n)_n$ is pseudo-equicontinuous at zero. Then there exists an absolutely convex and closed neighborhood U of zero such that

$$\overline{f_n}(U^n)\in V \qquad \text{for every } n\in\mathbb{N}^*.$$

So,

$$\Omega=\bigcup_{n\in\mathbb{N}^*}\overline{f_n}(U^n)\subset V.$$

Now let x and y be two elements of Ω. Then

$$x=\overline{f_n}(x_1,\dots,x_n),\quad y=\overline{f_m}(y_1,\dots,y_m),\quad (x_1,\dots,x_n)\in U^n,\quad (y_1,\dots,y_m)\in U^m.$$

Using (2), we have

$$\begin{aligned}&\overline{f_n}(x_1,\dots,x_n)\overline{f_m}(y_1,\dots,y_m)\\&=\overline{f_{n+m}}(x_1,\dots,x_n,y_1,\dots,y_m)\in f_{n+m}(U^{m+n})\subset\Omega.\end{aligned}$$

Thus $xy\in\Omega$, consequently Ω is an idempotent set. Moreover, we have $f_1(U)=\overline{f_1}(U)\subset\Omega$ and by using (3), $f_1(U)$ is a neighborhood of zero. Therefore, Ω is a neighborhood of zero. Whence the result. ☐

As an application of the previous result, we have the following.

Corollary 4 ([**6**]). *Let (A,τ) be a commutative locally convex algebra. If the sequence $(x\mapsto x^n)_n$ of power maps is equicontinuous at zero, then (A,τ) is necessarily m-convex.*

Proof. It suffices to observe that the sequence $(x\mapsto x^n)_n$ of power maps is a family of homogeneous polynomials whose sequence of generators, given by $[(x_1,\dots,x_n)\to x_1\dots x_n]_n$, satisfies (2) and (3) of Corollary 3. ☐

The following result shows that in the general case, the convergence of a continuous homogeneous polynomial series on an open subset of a B_0-algebra implies the equicontinuity at zero of its general term.

Proposition 5. *Let (A,τ) be a B_0-algebra and let $(f_n)_n$ be a sequence of continuous homogeneous polynomials of degree n such that the series $\sum_n f_n(x)$ converges on an open subset Ω of (A,τ). Then $(f_n)_n$ is equicontinuous at zero, and therefore, the sequence $(\overline{f_n})_{n\in\mathbb{N}^*}$ is pseudo-equicontinuous at zero.*

Proof. Let V be an absolutely convex and closed neighborhood of zero. Set

$$E_k=\{x\in\Omega: f_n(x)\in V,\ n\geq k\}=\bigcap_{n\geq k}\{x\in\Omega: f_n(x)\in V\}.$$

As f_n is continuous for every n, all sets E_k are closed. Moreover, $\bigcup_k E_k=\Omega$. By Baire's argument, there is an integer m such that E_m has non-void interior. Hence there is an $x_0\in E_m$ and an absolutely convex neighborhood of zero U such that, for every $n\geq m$,

$$f_n(x+x_0)\in V\qquad\text{for every } x\in U.$$

By the Mazur-Orlicz formula (see [**1**]), we have

$$f_n(x)=\frac{n^n}{n!}\sum_{j=0}^{n}(-1)^{n-j}C_n^j f_n\Big(x_0+\frac{j}{n}x\Big),$$

wehere $C_n^j=\frac{n!}{j!(n-j)!}$, $x\in A$, $n\in\mathbb{N}^*$.
In particular, if $x\in U$, one has $\frac{j}{n}x\in U$ because U is balanced. Thus $f_n(x_0+\frac{j}{n}x)\in V$, for and since V is also balanced we obtain

$$(-1)^{n-j}f_n\Big(x_0+\frac{j}{n}x\Big)\in V\qquad\text{for every } n\geq m.$$

Consequently, using the convexity of V, we have $\sum_{j=0}^{n}(-1)^{n-j}C_n^j f_n\left(x_0+\frac{j}{n}x\right)\in\left(\sum_{j=0}^{n}C_n^j\right)V=2^nV$. Then

$$f_n(x)\in\frac{(2n)^n}{n!}V,\qquad x\in U,\quad n\geq m.$$

But there exists $M>0$ such that $\frac{(2n)^n}{n!}\leq M^n$ for every $n\in\mathbb{N}^*$. Thus, for every $n\geq m$, one has

$$f_n(x)\in V\qquad\text{for every } x\in\frac{1}{M}U.$$

For $n<m$, using the continuity of the map $x\to x^n$, there exists a neighborhood W_n of zero such that

$$f_n(x)\in V\qquad\text{for every } x\in W_n.$$

Now, for $W=\left(\bigcap_{n<m}W_n\right)\cap\left(\frac{1}{M}U\right)$, we obtain

$$f_n(x)\in V\qquad\text{for every } n\in\mathbb{N}^*\text{ and } x\in W.$$

Whence the equicontinuity at zero of $(f_n)_n$. Finally the pseudo-equicontinuity at zero of $(\overline{f_n})_{n\in\mathbb{N}^*}$ is an immediate consequence of Proposition 2. □

The two following results constitute an application of the above to a particular class of homogeneous polynomials series.

Corollary 6. *Let (A,τ) be a B_0-algebra and $\sum_n a_nz^n$ be a power series which operates in a certain open subset of (A,τ). Then the sequence $(x\mapsto a_nx^n)_n$ is equicontinuous at zero. In particular, if A is commutative, then the sequence $((x_1,x_2,\ldots,x_n)\mapsto a_nx_1x_2\ldots x_n)_{n\in\mathbb{N}^*}$ is pseudo-equicontinuous at zero.*

Remark 7. In the non-commutative case, the sequence

$$\left((x_1,x_2,\ldots,x_n)\to\frac{a_n}{n!}\sum x_{j_1}\ldots x_{j_n}\right)_{n\in\mathbb{N}^*},$$

where the sum is taken over all permutations $(j_1,\ldots,j_n)$ of the sequence $(1,\ldots,n)$, is pseudo-equicontinuous at zero.

As a consequence of Corollary 6, we have the following result of B. Mitiagin, S. Rolewics and W. Żelazko (see [**5**]).

Corollary 8 ([**5**]). *Let (A,τ) be a commutative B_0-algebra. If entire functions operate, then (A,τ) is necessarily m-convex.*

Proof. If (A,τ) is not m-convex, one can find a neighborhood V of zero such that every neighborhood U of zero contains elements $y_1,\ldots,y_{k_n}$ satisfying $\frac{y_1\ldots y_{k_n}}{n^{k_n}}\notin V$. This contradicts the fact that the family

$$\left((x_1,x_2,\ldots,x_n)\mapsto\frac{1}{n^{k_n}}x_1x_2\ldots x_n\right)_{n\in\mathbb{N}^*}$$

is equicontinuous at zero. □

Theorem 9. *Let (A,τ) be a commutative B_0-algebra and $\sum_n a_n z^n$ be a power series with $(a_n)_n \subset \mathbb{R}_+^*$. If $\sum_n a_n z^n$ is convergent in some open subset of (A,τ), then the topology τ can be defined by a family $(|\cdot|_k)_k$ of seminorms such that*

$$|xy|_k \leq \frac{a_{n_x} a_{m_y}}{a_{n_x+m_y}} \; |x|_k \, |y|_k \,, \qquad x, y \in A, \; n_x, m_y \in \mathbb{N}^*.$$

In particular, if the sequence $\left(\frac{a_n a_m}{a_{m+n}}\right)_{n,m\in\mathbb{N}^*}$ is bounded, then (A,τ) is m-convex.

Proof. Let V_k be an absolutely convex and closed neighborhood of zero. By Corollary 6, the sequence $((x_1, x_2, \ldots, x_n) \mapsto a_n x_1 x_2 \ldots x_n)_{n\in\mathbb{N}^*}$ is pseudo-equicontinuous at zero. Then there exists an absolutely convex neighborhood U_k of zero such that

$$\Omega_k = \bigcup_n a_n U_k^n \subset V_k.$$

Let $\operatorname{conv}(\Omega_k)$ be the convex hull of Ω_k. It is an absolutely convex neighborhood of zero. Let $x, y \in \operatorname{conv}(\Omega_k)$. Then

$$x = \sum_{i\in I} \alpha_i x_i, \qquad y = \sum_{j\in J} \beta_j y_j, \qquad x_i, y_j \in \Omega_k, \qquad \sum_{i\in I} \alpha_i = \sum_{j\in J} \beta_j = 1,$$

where I and J are finite. Hence

$$xy = \Big(\sum_{i\in I} \alpha_i x_i\Big)\Big(\sum_{j\in J} \beta_j y_j\Big) = \sum_{j\in J; i\in I} \alpha_i \beta_j x_i y_j.$$

Since $x_i, y_j \in \Omega_k$, one has $x_i = a_{n_i} x_{1_i} x_{2_i} \ldots x_{n_i}$ and $y_j = a_{m_j} y_{1_j} y_{2_j} \ldots y_{m_j}$, where $x_{1_i}, \ldots, x_{n_i}, y_{1_j}, \ldots, y_{m_j} \in U_k$. Whence

$$x_i y_j \in \frac{a_{n_i} a_{m_j}}{a_{m_j+n_i}} \Omega_k.$$

Let

$$\frac{a_{n_x} a_{m_y}}{a_{n_x+m_y}} = \sup_{i,j} \frac{a_{n_i} a_{m_j}}{a_{m_j+n_i}}.$$

It follows that $x_i y_j \in \frac{a_{n_i} a_{m_j}}{a_{m_j+n_i}} \Omega_k \subset \frac{a_{n_x} a_{m_y}}{a_{n_x+m_y}} \Omega_k$. Hence

$$\frac{a_{n_x+m_y}}{a_{n_x} a_{m_y}} \, x_i y_j \in \Omega_k \qquad \text{for every } i \in I \text{ and } j \in J.$$

And since

$$\frac{a_{n_x+m_y}}{a_{n_x} a_{m_y}} \; xy = \sum_{j\in J,\ i\in I} \alpha_i \beta_j \; \frac{a_{n_x+m_y}}{a_{n_x} a_{m_y}} x_i y_j$$

and

$$\sum_{j\in J,\ i\in I} \alpha_i \beta_j = \Big(\sum_{i\in I} \alpha_i\Big)\Big(\sum_{j\in J} \beta_j\Big) = 1,$$

one has $\frac{a_{n_x+m_y}}{a_{n_x} a_{m_y}} \; xy \in \operatorname{conv}(\Omega_k)$. Thus the family of semi-norms $(|\cdot|_k)_k$ associated to all $\operatorname{conv}(\Omega_k)$ defines a topology τ such that

$$|xy|_k \leq \frac{a_{n_x} a_{m_y}}{a_{n_x+m_y}} \, |x|_k \, |y|_k \,, \qquad x, y \in A, \; n_x, m_y \in \mathbb{N}^*.$$

Finally, it is clear that if the sequence $\left(\frac{a_n a_m}{a_{m+n}}\right)_{n,m\in\mathbb{N}^*}$ is bounded, then (A,τ) is m-convex. ☐

Remark 10. If $\sum\limits_n a_n z^n$ is an entire function, then the sequence $\left(\frac{a_n a_m}{a_{m+n}}\right)_{n,m\in\mathbb{N}^*}$ is not necessarily bounded. Indeed, in [**5**], it was shown that for every entire function φ, there is a non m-convex commutative algebra A_φ such that φ operates on A_φ.

REFERENCES

1. Bochnak J. and Siciak J., *Polynomials and multilinear mappings in topological vector space.* Studia. Math. **39** (1971), 59–76.

2. El Kinani A. and Oudadess M., *Entire functions and m-convex structure in commutative Baire algebra*, Bull. Belg. Math. Soc. **4** (1997), 685–687.

3. El Kinani A. and Oudadess M., *Entire functions in locally convex algebras*, Arab. J. Sci. Eng. Sect. A Sci. **27(1A)** (2002), 85–90.

4. Michael E. A., *Locally multiplicatively-convex topological algebras,* Mem. Amer. Math. Soc. **11** (1952), 79.

5. Mitiagin B. S., Rolewicz S. and Żelazko W., *Entire functions in B_0-algebras,* Studia Math. **21** (1962), 291–306.

6. Turpin P., *Une remarque sur les algèbres à inverse continu,* C. R. Acad. Sci. Paris, **A 270** Série A, 1686–1689 (1970).

7. Żelazko W., *Metric generalization of Banach algebras*, Rozprway Mat. (Dissertationes Math.) **47** (1965).

8. Żelazko W., *Selected topics in topological algebras,* Lecture Notes Series **31**, Aarhus, Univ. (1971).

9. Żelazko W., *Concerning entire functions in B_0-algebras,* Studia Math. **110(3)** (1994), 283–290.

A. El Kinani, Université Mohammed V, Ecole Normale Supérieure de Rabat, 10105 Rabat, Morocco, *e-mail*: `abdellah_elkinani@yahoo.fr`

R. Choukri, Université Mohammed V, Ecole Normale Supérieure de Rabat, 10105 Rabat, Morocco, *e-mail*: `rachoukri@yahoo.fr`

A. Oudades, Université Mohammed V, Ecole Normale Supérieure de Rabat, 10105 Rabat, Morocco, *e-mail*: `oudadess.majid@gmail.com`