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COHEN-MACAULAY FLAT DIMENSION AND LOCAL HOMOLOGY MODULES

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Abstract. Let $\mathfrak{a}$ be an ideal of a commutative Noetherian ring R, M a finitely generated R-module with finite flat dimension, and N an arbitrary R-module with finite Cohen-Macaulay flat dimension. We prove that the generalized local homology module $H_i^{\mathfrak{a}}(M,N) = 0$ for each i larger than the Cohen-Macaulay flat dimension of N. As an application, we present a characterization for regularity of local rings having dualizing modules.

1. Introduction

The study of local homology modules was initiated by Matlis [**Ma**], and it was continued by many authors, see, e.g., [**S**], [**GM**], [**LLT**]. Mohammadi and Divaani-Aazar in [**MD**] studied the connection between local homology modules and Gorenstein flat modules. Also, the author in [**Mo**] defined the generalized $\mathfrak{a}$-adic completion functor for any ideal $\mathfrak{a}$ of a Noetherian ring R and studied generalized local homology modules by using Gorenstein flat modules in order to find several ways for computing generalized local homology modules.

In this paper, we study vanishing of generalized local homology modules by using Cohen-Macaulay flat dimension which was defined by Holm and Jørgensen [**HJ1**]. Let $\mathfrak{a}$ be an ideal of a Noetherian ring R. We prove that if M is a finitely generated R-module with finite flat dimension, C is a semi-dualizing R-module and N is a C-flat R-module, then $H_i^{\mathfrak{a}}(M,N) = 0$ for every $i > 0$. By using this, we prove that for the same R-module M and any R-module U, $H_i^{\mathfrak{a}}(M,U) = 0$ for each i larger than the Cohen-Macaulay flat dimension of U, see Theorem 3.7. In the sequel, we give a characterization of the regularity of a local ring which possesses a dualizing module. More precisely, we show that if $(R,\mathfrak{m})$ is a local ring with a dualizing R-module ω, then R is regular if and only if for any ideal I and any ω-Gorenstein flat R-module N, $H_i^{\mathfrak{a}}(R/I,N) = 0$ for all $i > 0$, see Theorem 3.12. Finally, we prove that if a local ring $(R,\mathfrak{m})$ admits a non-zero Artinian R-module M with finite Cohen-Macaulay flat dimension such that $\sup \mathbf{L}\Lambda^{\mathfrak{m}}(M) = \dim R$, then R should be a Cohen-Macaulay ring.

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2. Prerequisites

Throughout this paper, R is a commutative Noetherian ring and $\mathcal{D}(R)$ denotes the derived category of R-modules. Modules are considered as complexes concentrated in degree zero. For a complex X, its supremum is defined by $\sup X := \sup\{i \in \mathbb{Z} | H_i(X) \neq 0\}$ with the usual convention that $\sup \emptyset = -\infty$. The full subcategory of complexes homologically bounded to the right (resp., left) is denoted by $\mathcal{D}_{\sqsupset}(R)$ (resp., $\mathcal{D}_{\sqsubset}(R)$). We use the symbol $\simeq$ for denoting isomorphisms in the category $\mathcal{D}(R)$. For any complex X in $\mathcal{D}_{\sqsupset}(R)$ (resp., $\mathcal{D}_{\sqsubset}(R)$), there is a bounded to the right (resp., left) complex U of flat (resp., injective) R-modules such that $U \simeq X$ (In fact, there is an actual chain map $U \to X$ (resp., $X \to U$) which is an isomorphism on homology). Such a complex U is called a flat (resp., injective) resolution of X. We say that a homologically bounded complex X has finite flat (resp., injective) dimension $\operatorname{fd}_R X$ (resp., $\operatorname{id}_R X$) if in $\mathcal{D}(R)$, it is isomorphic to a bounded complex of flat (resp., injective) R-modules. The left derived tensor product functor $-\otimes_R^{\mathbf{L}} \sim$ is computed by taking a flat resolution of the first argument or of the second one. Let $\mathfrak{a}$ be an ideal of R and $\mathcal{C}_0(R)$ denote the full subcategory of R-modules. The left derived functor of $\mathfrak{a}$-adic completion functor

$$\Lambda^{\mathfrak{a}}(-) := \varprojlim_n (R/\mathfrak{a}^n \otimes_R -) : \mathcal{C}_0(R) \to \mathcal{C}_0(R)$$

exists in $\mathcal{D}(R)$, and so for any complex $X \in \mathcal{D}_{\sqsupset}(R)$, the complex $\mathbf{L}\Lambda^{\mathfrak{a}}(X) \in \mathcal{D}_{\sqsupset}(R)$ is defined by $\mathbf{L}\Lambda^{\mathfrak{a}}(X) := \Lambda^{\mathfrak{a}}(F)$, where F is a flat resolution of X. Let $X \in \mathcal{D}_{\sqsupset}(R)$. For any integer i, the i-th local homology module of X with respect to $\mathfrak{a}$ is defined by $H_i^{\mathfrak{a}}(X) := H_i(\mathbf{L}\Lambda^{\mathfrak{a}}(X))$. The author in [**Mo**] defined the generalized $\mathfrak{a}$-adic completion functor for any two R-modules M and N by $\mathbf{L}\Lambda^{\mathfrak{a}}(M, N) := \mathbf{L}\Lambda^{\mathfrak{a}}(M \otimes_R^{\mathbf{L}} N)$ and the i-th generalized local homology module of M and N with respect to $\mathfrak{a}$ by $H_i^{\mathfrak{a}}(M, N) := H_i(\mathbf{L}\Lambda^{\mathfrak{a}}(M, N))$. Clearly if one of our modules M or N is R, then the generalized $\mathfrak{a}$-adic completion functor is the usual $\mathfrak{a}$-adic completion functor.

Our definition differs from the definition of generalized local homology modules by Nam [**N**]. More precisely, for any R-module N and any ideal of R like $\mathfrak{a}$, Nam [**N**] defined the i-th local homology module of N with respect to $\mathfrak{a}$ by $H_i^{\mathfrak{a}}(N) = \varprojlim_n \operatorname{Tor}_i^R(R/\mathfrak{a}^n, N)$ for all i. Also, for any R-modules M and N, he defined the i-th generalized local homology module $H_i^{\mathfrak{a}}(M, N)$ of M, N with respect to an ideal $\mathfrak{a}$ of R by $H_i^{\mathfrak{a}}(M, N) = \varprojlim_n \operatorname{Tor}_i^R(M/\mathfrak{a}^n M, N)$ for all i. By his definition, one can see that for any R-module N, $H_0^{\mathfrak{a}}(R, N) \cong H_0^{\mathfrak{a}}(N) = \varprojlim_n \operatorname{Tor}_0^R(R/\mathfrak{a}^n, N) \cong \Lambda^{\mathfrak{a}}(N)$. However, with my definition of generalized local homology, one can see that for any R-module N, we have $H_0^{\mathfrak{a}}(R, N) \cong H_0^{\mathfrak{a}}(N) = H_0(\mathbf{L}\Lambda^{\mathfrak{a}}(N))$ which is not necessarily isomorphism with $\Lambda^{\mathfrak{a}}(N)$, see [**S**]. So, my definition of generalized local homology is different with Nam's definition [**N**], and additionally is a right extension of the usual local homology module.

The notion of Gorenstein flat modules was introduced by Enochs, Jenda, and Torrecillas in [**EJT**]. An R-module M is said to be *Gorenstein flat* if there exists an exact complex F of flat R-modules such that $M \cong \mathrm{im}(F_0 \to F_{-1})$ and $F \otimes_R I$ is exact for all injective R-modules I. The Gorenstein flat dimension of M is defined by

$$\mathrm{Gfd}_R\, M := \inf\{\, \sup\{l \in \mathbb{Z} | Q_l \neq 0\} \mid Q \text{ is a bounded to the right complex of Gorenstein flat R-modules and } Q \simeq M\}.$$

A *dualizing complex* for R is a complex D which its homology complex is bounded, $H_i(D)$ is finitely generated for all i, the homothety morphism $R \to \mathbf{R}\,\mathrm{Hom}_R(D, D)$ is an isomorphism in $\mathcal{D}(R)$, and D has finite injective dimension. A *semi-dualizing* module for R is a finitely generated R-module C such that the natural homomorphism $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism and $\mathrm{Ext}^i_R(C, C) = 0$ for all $i \geq 1$. If in addition, the injective dimension of C is finite, then C is called a *dualizing* module. For example, R is a semi-dualizing R-module. Recall that for any R-module T, the direct sum $R \oplus T$ can be equipped with the product

$$(r_1, t_1).(r_2, t_2) = (r_1 r_2, r_1 t_2 + r_2 t_1).$$

This turns $R \oplus T$ into a ring which is called *trivial extension of R by T* and denoted by $R \ltimes T$. There are natural ring homomorphisms $R \to R \ltimes T \to R$, whose composition is the identity on R. These homomorphisms allow us to view any R-module as an $R \ltimes T$-module, and vice versa.

Following [**HJ1**, Definition 2.3], *Cohen-Macaulay flat dimension* of an R-module M is defined as

$$\mathrm{CMfd}_R\, M := \inf\{\mathrm{Gfd}_{R \ltimes C}\, M | C \text{ is a semi-dualizing R-module}\}.$$

Let C be a semi-dualizing R-module. An R-module M is called *C-flat* if $M = C \otimes_R F$ for a flat R-module F. Also, an R-module M is called *C-Gorenstein flat* if

i) $\mathrm{Tor}^R_i(\mathrm{Hom}_R(C, I), M) = 0$ for any injective R-module I and any integer $i \geq 1$.

ii) There exist C-flat R-modules $G_0, G_{-1}, \ldots$ together with an exact sequence

$$0 \to M \to G_0 \to G_{-1} \to \cdots,$$

such that this sequence stays exact when we apply the functor $\mathrm{Hom}_R(C,I) \otimes_R -$ to it for any injective R-module I.

It is easy to see that if $C := R$, then C-Gorenstein flat R-modules and Gorenstein flat R-modules are the same. By [**HJ2**, Example 2.8 c)], flat and C-flat R-modules are C-Gorenstein flat. By $GF_C(R)$, we denote the class of all C-Gorenstein flat R-modules. For an R-module M, we define C-Gorenstein flat dimension of M by

$$\mathrm{C\text{–}Gfd}_R\, M := \inf\{\, \sup\{l \in \mathbb{Z} | F_l \neq 0\} \mid F \text{ is a bounded to the right complex of C-Gorenstein flat R-modules and } F \simeq M\}.$$

By [**HJ2**, Theorem 2.16], $\mathrm{C\text{–}Gfd}_R M = \mathrm{Gfd}_{R \ltimes C}\, M$. Clearly, if $C = R$, then $\mathrm{C\text{–}Gfd}_R M = \mathrm{Gfd}_R\, M$.

3. The results

We start this section with the following lemma. Let $\mathfrak{a}$ be an ideal of the Noetherian ring R. We recall that an R-module M is $\Lambda^{\mathfrak{a}}$-acyclic if $H_i^{\mathfrak{a}}(M) = 0$ for any $i > 0$.

Lemma 3.1. *Let $\mathfrak{a}$ be an ideal of R and C a semi-dualizing R-module. Assume that M is an R-module. Then:*

i) *If $\mathfrak{a}^e$ is the extension of $\mathfrak{a}$ to $R \ltimes C$ under canonical homomorphism $R \to R \ltimes C$, then for any R-module M, there is an isomorphism $\mathbf{L}\Lambda^{\mathfrak{a}}(M) \simeq \mathbf{L}\Lambda^{\mathfrak{a}^e}(M)$ in $\mathcal{D}(R)$. In particular, $H_i^{\mathfrak{a}}(M) \cong H_i^{\mathfrak{a}^e}(M)$ for all integer i.*

ii) *Assume that G is a bounded to the right complex of C-Gorenstein flat R-modules such that $G \simeq M$, then $\mathbf{L}\Lambda^{\mathfrak{a}}(M) \simeq \Lambda^{\mathfrak{a}}(G)$, and so $H_i^{\mathfrak{a}}(M) = H_i(\Lambda^{\mathfrak{a}}(G))$ for all i. In particular, $\sup \mathbf{L}\Lambda^{\mathfrak{a}}(M) \leq \text{C–Gfd}_R(M)$.*

iii) *If M is C-Gorenstein flat, then M is $\Lambda^{\mathfrak{a}}$-acyclic and there is an R-isomorphism $H_0^{\mathfrak{a}}(M) \cong \Lambda^{\mathfrak{a}}(M)$. Also, if R admits a dualizing module, then $\Lambda^{\mathfrak{a}}(M)$ is C-Gorenstein flat.*

iv) *If M is C-flat, then $\Lambda^{\mathfrak{a}}(M)$ is C-flat.*

Proof. i) It is easy to see that the natural transformation of functors $\xi \colon \Lambda^{\mathfrak{a}}(\cdot) \to \Lambda^{\mathfrak{a}^e}(\cdot)$ is an equivalence. Let F be a flat resolution of M over R. By [**HJ2**, Example 2.8c)], any flat R-module is C-Gorenstein flat, and so by [**HJ2**, Proposition 2.15], it is Gorenstein flat over $R \ltimes C$. Hence, F is a Gorenstein flat resolution of M over $R \ltimes C$, and so by [**MD**, Theorem 2.5], we have

$$\mathbf{L}\Lambda^{\mathfrak{a}}(M) \simeq \Lambda^{\mathfrak{a}}(F) \simeq \Lambda^{\mathfrak{a}^e}(F) \simeq \mathbf{L}\Lambda^{\mathfrak{a}^e}(M).$$

ii) By [**HJ2**, Proposition 2.15], any C-Gorenstein flat R-module is a Gorenstein flat module over $R \ltimes C$. So, G is a bounded to the right complex of Gorenstein flat $R \ltimes C$-modules such that $G \simeq M$. So, part i) and its proof, and [**MD**, Theorem 2.5] imply that

$$\mathbf{L}\Lambda^{\mathfrak{a}}(M) \simeq \mathbf{L}\Lambda^{\mathfrak{a}^e}(M) \simeq \Lambda^{\mathfrak{a}^e}(G) \simeq \Lambda^{\mathfrak{a}}(G).$$

iii) The first assertion follows from part ii). By [**HJ2**, Proposition 2.15], M is a Gorenstein flat module over $R \ltimes C$. Assume that R has a dualizing R-module D. The natural homomorphism $R \ltimes C \to R$ turns that R is a finitely generated $R \ltimes C$-module, and so by the proof of [**Ha**, V.10.2], $R \ltimes C$ has a dualizing complex. Therefore, $\Lambda^{\mathfrak{a}^e}(M)$ is a Gorenstein flat module over $R \ltimes C$ by [**MD**, Lemma 2.7], and so by the proof of i), $\Lambda^{\mathfrak{a}}(M)$ is a Gorenstein flat module over $R \ltimes C$ as well. Thus by [**HJ2**, Proposition 2.15], $\Lambda^{\mathfrak{a}}(M)$ is C-Gorenstein flat.

iv) By [**HJ2**, Example 2.8 c)], M is C-Gorenstein flat, and so $H_0^{\mathfrak{a}}(M) \cong \Lambda^{\mathfrak{a}}(M)$ by part iii). Since M is C-flat, one has $M = C \otimes_R F$ for a flat R-module F. We have

$$\begin{aligned} H_0^{\mathfrak{a}}(M) &= H_0^{\mathfrak{a}}(C \otimes_R F) = H_0(\mathbf{L}\Lambda^{\mathfrak{a}}(C \otimes_R F)) \cong H_0(\mathbf{L}\Lambda^{\mathfrak{a}}(C \otimes_R^{\mathbf{L}} F)) \\ &\overset{(a)}{\cong} H_0(C \otimes_R^{\mathbf{L}} \mathbf{L}\Lambda^{\mathfrak{a}}(F)) \cong H_0(C \otimes_R^{\mathbf{L}} \Lambda^{\mathfrak{a}}(F)) = C \otimes_R \Lambda^{\mathfrak{a}}(F), \end{aligned}$$

where (a) follows from [**MD**, Lemma 2.6]. Note that $\Lambda^{\mathfrak{a}}(F)$ is flat, see e.g., [**B**, 1.4.7]. □

The following gives a way for computing generalized local homology modules.

Lemma 3.2. *Let $\mathfrak{a}$ be an ideal of R and C be a semi-dualizing R-module. Assume that M is a finitely generated R-module and N is a C-Gorenstein flat R-module. Then $H_i^{\mathfrak{a}}(M,N) \cong \mathrm{Tor}_i^R(M,\Lambda^{\mathfrak{a}}(N))$ for all integer i.*

Proof. We have

$$\begin{aligned} H_i^{\mathfrak{a}}(M,N) &= H_i(\mathbf{L}\Lambda^{\mathfrak{a}}(M \otimes_R^{\mathbf{L}} N)) \overset{(a)}{\cong} H_i(M \otimes_R^{\mathbf{L}} \mathbf{L}\Lambda^{\mathfrak{a}}(N)) \\ &\overset{(b)}{\cong} H_i(M \otimes_R^{\mathbf{L}} \Lambda^{\mathfrak{a}}(N)) = \mathrm{Tor}_i^R(M,\Lambda^{\mathfrak{a}}(N)), \end{aligned}$$

where (a) follows from [**MD**, Lemma 2.6] and (b) follows from Lemma 3.1 ii). □

The following result mentions some conditions for the vanishing of generalized local homology modules.

Lemma 3.3. *Let $\mathfrak{a}$ be an ideal of R and C be a semi-dualizing R-module. Assume that M is a finitely generated R-module such that $\mathrm{Tor}_i^R(M,C)=0$ for all $i>0$. Then*

$$H_i^{\mathfrak{a}}(M,N)=0$$

for all C-flat R-modules N and all $i>0$.

Proof. Let N be a C-flat R-module and i a positive integer. By Lemma 3.1 iv), $\Lambda^{\mathfrak{a}}(N)$ is C-flat, and so $\Lambda^{\mathfrak{a}}(N) = C \otimes_R F$ for a flat R-module F. By Lemma 3.2, we have

$$H_i^{\mathfrak{a}}(M,N) \cong \mathrm{Tor}_i^R(M,\Lambda^{\mathfrak{a}}(N)) = \mathrm{Tor}_i^R(M, C\otimes_R F) \cong \mathrm{Tor}_i^R(M,C)\otimes_R F = 0. \quad \square$$

Next, we record the following corollary of Lemma 3.3.

Corollary 3.4. *Let $\mathfrak{a}$ be an ideal of R and C be a semi-dualizing R-module. Assume that M is a finitely generated R-module with finite flat dimension. Then*

$$H_i^{\mathfrak{a}}(M,N)=0$$

for all C-flat R-modules N and all $i>0$.

Proof. By [**HW**, Proposition 3.1 and Corollary 6.2], we have $\mathrm{Tor}_i^R(M,C)=0$ for all $i>0$, and so the assertion is followed by Lemma 3.3. □

Lemma 3.5. *Let $\mathfrak{a}$ be an ideal of R and M be an R-module. If*

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0,$$

is an exact sequence of R-modules, then there is an exact sequence

$$\begin{aligned} \cdots \longrightarrow H_{i+1}^{\mathfrak{a}}(M,N'') \longrightarrow H_i^{\mathfrak{a}}(M,N') \longrightarrow H_i^{\mathfrak{a}}(M,N) \longrightarrow \\ \cdots \longrightarrow H_0^{\mathfrak{a}}(M,N') \longrightarrow H_0^{\mathfrak{a}}(M,N) \longrightarrow H_0^{\mathfrak{a}}(M,N'') \longrightarrow 0. \end{aligned}$$

Proof. The exact sequence $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$ yields that

$$0 \longrightarrow M \otimes_R^{\mathbf{L}} N' \longrightarrow M \otimes_R^{\mathbf{L}} N \longrightarrow M \otimes_R^{\mathbf{L}} N'' \longrightarrow 0$$

is an exact sequence. As $\Lambda^{\mathfrak{a}}(.)$ is an exact functor on the category of flat R-modules by the proof of [**Ma**, Corollary 4.5], the following sequence

$$0 \longrightarrow \mathbf{L}\Lambda^{\mathfrak{a}}(M \otimes^{\mathbf{L}}_R N') \longrightarrow \mathbf{L}\Lambda^{\mathfrak{a}}(M \otimes^{\mathbf{L}}_R N) \longrightarrow \mathbf{L}\Lambda^{\mathfrak{a}}(M \otimes^{\mathbf{L}}_R N'') \longrightarrow 0$$

is exact, and so we have the following long exact sequence of generalized local homology modules

$$\cdots \longrightarrow H^{\mathfrak{a}}_{i+1}(M, N'') \longrightarrow H^{\mathfrak{a}}_i(M, N') \longrightarrow H^{\mathfrak{a}}_i(M, N) \longrightarrow$$
$$\cdots \longrightarrow H^{\mathfrak{a}}_0(M, N') \longrightarrow H^{\mathfrak{a}}_0(M, N) \longrightarrow H^{\mathfrak{a}}_0(M, N'') \longrightarrow 0.$$

□

Lemma 3.6. *Let $\mathfrak{a}$ be an ideal of R and C be a semi-dualizing R-module. If N is C-Gorenstein flat R-module, then there exists an exact sequence*

$$0 \longrightarrow N \longrightarrow Q_0 \longrightarrow Q_{-1} \longrightarrow Q_{-2} \longrightarrow \cdots \longrightarrow Q_{-j} \longrightarrow Q_{-(j+1)} \longrightarrow \cdots$$

such that each Q_{-j} is C-flat and $K_j := \operatorname{im}(Q_{-j} \longrightarrow Q_{-(j+1)})$ is C-Gorenstein flat for all $j \geq 0$.

Proof. Since N is C-Gorenstein flat, then by the definition, $\operatorname{Tor}^R_i(\operatorname{Hom}_R(C,I),N) = 0$ for any injective R-module I and any integer $i > 0$ and also there exists an exact sequence

$$\mathbf{X} = 0 \longrightarrow N \longrightarrow Q_0 \longrightarrow Q_{-1} \longrightarrow Q_{-2} \longrightarrow \cdots \longrightarrow Q_{-j} \longrightarrow Q_{-(j+1)} \longrightarrow \cdots,$$

such that each Q_{-j} is C-flat and $\operatorname{Hom}_R(C, I) \otimes_R X$ is an exact sequence for any injective R-module I. Let j be a non-negative integer and I be an injective R-module. We know that

$$\mathbf{Y} = 0 \longrightarrow K_j \longrightarrow Q_{-(j+1)} \longrightarrow Q_{-(j+2)} \longrightarrow Q_{-(j+3)} \longrightarrow \cdots$$

is an exact sequence. Since $\operatorname{Hom}_R(C, I) \otimes_R X$ is an exact sequence and by [**C**, Lemma 4.1.7 b)], we have $\operatorname{im}((Q_{-j} \otimes_R \operatorname{Hom}_R(C, I)) \longrightarrow (Q_{-(j+1)} \otimes_R \operatorname{Hom}_R(C, I))) \cong K_j \otimes_R \operatorname{Hom}_R(C, I)$ $\operatorname{Hom}_R(C, I) \otimes_R Y$, so is an exact sequence as well. Since any C-flat module is C-Gorenstein flat by [**HJ2**, Example 2.8 c)], then

$$\operatorname{Tor}^R_i(\operatorname{Hom}_R(C, I), Q_{-j}) = 0$$

for all $i > 0$. Thus [**C**, Lemma 4.1.7 c)] implies that

$$\operatorname{Tor}^R_i(\operatorname{Hom}_R(C, I), K_j) = 0$$

for any integer $i > 0$. Therefore, K_j is a C-Gorenstein flat R-module, as desired. □

Next, we present our main result.

Theorem 3.7. *Let $\mathfrak{a}$ be an ideal of R. Assume that M is a finitely generated R-module with finite flat dimension and N is an R-module with finite Cohen-Macaulay flat dimension. Then $H^{\mathfrak{a}}_i(M, N) = 0$ for all $i > \operatorname{CMfd}_R(N)$.*

Proof. Set $n := \operatorname{CMfd}_R(N)$. So, there exists a semi-dualizing R-module C such that $\operatorname{Gfd}_{R \ltimes C} N = n$. We prove the assertion by induction on n. Assume that $n = 0$. Then N is a Gorenstein flat module over $R \ltimes C$. Hence N is a C-Gorenstein flat R-module by [**HJ2**, Proposition 2.15], and so by Lemma 3.6, there exists an exact sequence

$$\mathbf{X} = 0 \longrightarrow N \longrightarrow Q_0 \longrightarrow Q_{-1} \longrightarrow Q_{-2} \longrightarrow \cdots \longrightarrow Q_{-j} \longrightarrow Q_{-(j+1)} \longrightarrow \cdots$$

such that each Q_{-j} is C-flat and $K_j := \operatorname{im}(Q_{-j} \longrightarrow Q_{-(j+1)})$ is C-Gorenstein flat for all $j \geq 0$. Set $K_{-1} = N$. For each $j \geq 0$, the exact sequence

$$0 \longrightarrow K_{j-1} \longrightarrow Q_{-j} \longrightarrow K_j \longrightarrow 0$$

yields that the following sequence of generalized local homology modules

$$\cdots \longrightarrow H^{\mathfrak{a}}_{i+1}(M, Q_{-j}) \longrightarrow H^{\mathfrak{a}}_{i+1}(M, K_j) \longrightarrow H^{\mathfrak{a}}_{i}(M, K_{j-1}) \longrightarrow H^{\mathfrak{a}}_{i}(M, Q_{-j}) \longrightarrow \cdots$$

is exact by Lemma 3.5. From this sequence and Corollary 3.4, we conclude that $H^{\mathfrak{a}}_{i}(M, K_{j-1}) \cong H^{\mathfrak{a}}_{i+1}(M, K_j)$ for all $j \geq 0$ and all $i \geq 1$. Let $m := \operatorname{fd}_R M$ and $s := \operatorname{cd}_{\mathfrak{a}}(R)$, where $\operatorname{cd}_{\mathfrak{a}}(R)$ is the supremum of i's such that ith local cohomology module of R with respect to $\mathfrak{a}$ is nonzero. Then by [**MD**, Lemma 2.1], one has

$$H^{\mathfrak{a}}_{i}(M, N) \cong H^{\mathfrak{a}}_{i+1}(M, K_0) \cong \ldots \cong H^{\mathfrak{a}}_{i+m+s}(M, K_{m+s-1}) = 0$$

for all $i > 0$. Now, we assume that $n > 0$. By [**Ho**, Lemma 3.17], there exists a short exact sequence

$$(*) \qquad 0 \longrightarrow K \longrightarrow Q \longrightarrow N \longrightarrow 0,$$

such that Q is a Gorenstein flat module over $R \ltimes C$ and $\operatorname{fd}_{R\ltimes C} K = n - 1$. So Q is C-Gorenstein flat by [**HJ2**, Proposition 2.15] and $\operatorname{Gfd}_{R\ltimes C} K = n - 1$ by [**Ho**, Theorem 3.19]. Thus $\operatorname{CMfd}_R Q = 0$ and $\operatorname{CMfd}_R K \leq n - 1$, and so $H^{\mathfrak{a}}_{i}(M, Q) = 0$ for all $i > 0$ and $H^{\mathfrak{a}}_{i}(M, K) = 0$ for all $i > n - 1$. Also, from $(*)$ and Lemma 3.5, we have the following long exact sequence of generalized local homology modules

$$\cdots \longrightarrow H^{\mathfrak{a}}_{i+1}(M, Q) \longrightarrow H^{\mathfrak{a}}_{i+1}(M, N) \longrightarrow H^{\mathfrak{a}}_{i}(M, K) \longrightarrow H^{\mathfrak{a}}_{i}(M, Q) \longrightarrow \cdots$$

which implies that $H^{\mathfrak{a}}_{i}(M, N) = 0$ for all $i > n$. □

The following immediate corollary gives an upper bound for the vanishing of local homology modules.

Corollary 3.8. *Let $\mathfrak{a}$ be an ideal of R. Then for any R-module N,* $\sup \mathbf{L}\Lambda^{\mathfrak{a}}(N) \leq \operatorname{CMfd}_R(N)$.

Corollary 3.9. *Let $\mathfrak{a}$ be an ideal of R. Assume that M is a finitely generated R-module with finite flat dimension. If N is an R-module and C is a semi-dualizing R-module, then $H^{\mathfrak{a}}_{i}(M, N)$ can be computed by applying the functor $H^{\mathfrak{a}}_{0}(M, -)$ to any $GF_C(R)$-resolution of N.*

Proof. Set $T := H^{\mathfrak{a}}_{0}(M, -)$. Then T is a right exact covariant functor and $\mathbf{L}_i T(N) \cong H^{\mathfrak{a}}_{i}(M, N)$ for any R-module N and any integer i. By Theorem 3.7, every C-Gorenstein flat R-module is T-acyclic. This finishes the proof. Recall that if T is a right exact additive functor from the category of R-modules and R-homomorphisms to itself, then for any R-module N, the left derived functors $\mathbf{L}_i T$ of T at N can be computed by using any left resolution of N consisting of T-acyclic modules. □

For proving Lemma 3.11 and Theorem 3.12, we need the following lemma.

Lemma 3.10. *Let $(R, \mathfrak{m})$ be a local ring with a dualizing R-module ω. Then for any R-module M,* $\mathrm{Gfd}_{R\ltimes\omega} M \leq \dim R$.

Proof. As a special case of [**J**, Theorem 2.2], one can see that $R \ltimes \omega$ is a Gorenstein ring. Hence, $\mathrm{Gfd}_{R\ltimes\omega} M$ is finite and

$$\mathrm{Gfd}_{R\ltimes\omega} M \leq \dim R \ltimes \omega = \dim R$$

by [**C**, Theorem 5.2.10 and Corollary 5.2.15]. □

Lemma 3.11. *Let $(R, \mathfrak{m})$ be a local ring with a dualizing R-module ω. Assume that $\mathfrak{a}$ is an ideal of R. If N is a ω-Gorenstein flat R-module, then $H_0^{\mathfrak{a}}(M, N)$ is a ω-Gorenstein flat R-module for any finitely generated flat R-module M.*

Proof. Since N is a ω-Gorenstein flat R-module, by Lemma 3.6, there exists an exact sequence

$$0 \longrightarrow N \longrightarrow Q_0 \longrightarrow Q_{-1} \longrightarrow Q_{-2} \longrightarrow \cdots \longrightarrow Q_{-i} \longrightarrow Q_{-(i+1)} \longrightarrow \cdots$$

such that each Q_{-i} is ω-flat and $K_i := \mathrm{im}(Q_{-i} \longrightarrow Q_{-(i+1)})$ is ω-Gorenstein flat for all $i \geq 0$. Let i be a non-negative integer. Then Q_{-i} is ω-Gorenstein flat by [**HJ2**, Example 2.8 c)] and K_i is a Gorenstein flat module over $R \ltimes \omega$ by [**HJ2**, Proposition 2.15], and so $\mathrm{CMfd}_R K_i = 0$. Also, there exists a flat R-module F_i such that $\Lambda^{\mathfrak{a}}(Q_{-i}) \cong \omega \otimes_R F_i$ by Lemma 3.1 iv). Hence, Lemma 3.2 yields that

$$H_0^{\mathfrak{a}}(M, Q_{-i}) \cong M \otimes_R \Lambda^{\mathfrak{a}}(Q_{-i}) \cong M \otimes_R (\omega \otimes_R F_i) \cong \omega \otimes_R (M \otimes_R F_i).$$

This implies that $H_0^{\mathfrak{a}}(M, Q_{-i})$ is ω-flat, and so it is Gorenstein flat module over $R \ltimes \omega$ by [**HJ2**, Example 2.8 c) and Proposition 2.15]. Set $K_{-1} := N$. For each $i \geq 0$, the exact sequence

$$0 \longrightarrow K_{i-1} \longrightarrow Q_{-i} \longrightarrow K_i \longrightarrow 0$$

yields the following sequence of generalized local homology modules

$$\begin{aligned} \cdots \longrightarrow H_1^{\mathfrak{a}}(M, Q_{-i}) \longrightarrow H_1^{\mathfrak{a}}(M, K_i) \longrightarrow H_0^{\mathfrak{a}}(M, K_{i-1}) \\ \longrightarrow H_0^{\mathfrak{a}}(M, Q_{-i}) \longrightarrow H_0^{\mathfrak{a}}(M, K_i) \longrightarrow 0, \end{aligned}$$

which is exact by Lemma 3.5. Also, $H_1^{\mathfrak{a}}(M, Q_{-i}) = 0$ by Corollary 3.4 and $H_1^{\mathfrak{a}}(M, K_i) = 0$ by Theorem 3.7. So, using the above exact sequences successively yields that the following sequence

$$\mathbf{X} = 0 \longrightarrow H_0^{\mathfrak{a}}(M, N) \longrightarrow H_0^{\mathfrak{a}}(M, Q_0) \longrightarrow H_0^{\mathfrak{a}}(M, Q_{-1}) \longrightarrow H_0^{\mathfrak{a}}(M, Q_{-2}) \longrightarrow \cdots$$

is exact.

Let $n := \dim R$ and L be the $(-n)$-th cokernel of X. Then $\mathrm{Gfd}_{R\ltimes\omega} L \leq n$ by Lemma 3.10, and so $H_0^{\mathfrak{a}}(M, N)$ should be a Gorenstein flat module over $R \ltimes \omega$ by [**Ho**, Theorem 3.14]. So, $H_0^{\mathfrak{a}}(M, N)$ is ω-Gorenstein flat by [**HJ2**, Proposition 2.15], as desired. □

Next, we present a characterization for regularity of local rings having dualizing modules.

Theorem 3.12. *Let $(R, \mathfrak{m}, k)$ be a local ring with a dualizing R-module ω and $\mathfrak{a}$ be an ideal of R. The following statements are equivalent:*

i) *For any finitely generated R-module M and any R-module N with finite Cohen-Macaulay flat dimension,* $H_i^{\mathfrak{a}}(M,N)=0$ *for all* $i>\operatorname{CMfd}_R N$.

ii) *For any ideal I and any R-module N with finite Cohen-Macaulay flat dimension,* $H_i^{\mathfrak{a}}(R/I,N)=0$ *for all* $i>\operatorname{CMfd}_R N$.

iii) *For any ideal I and any ω-Gorenstein flat R-module N,* $H_i^{\mathfrak{a}}(R/I,N)=0$ *for all* $i>0$.

iv) *Any* $\mathfrak{a}$*-adic complete R-module has finite flat dimension.*

v) *R is regular.*

Proof. $i\Rightarrow ii)$ is trivial.

ii ⇒ iii) is trivial. Note that any ω-Gorenstein flat R-module N is Gorenstein flat over $R\ltimes\omega$ by [**HJ2**, Proposition 2.15], and so $\operatorname{CMfd}_R N=0$.

iii ⇒ iv) Let N be an $\mathfrak{a}$-adic complete R-module. By [**S**, Proposition 2.5] and its proof, there exists a flat resolution

$$\cdots F_i\longrightarrow F_{i-1}\longrightarrow\cdots\longrightarrow F_2\longrightarrow F_1\longrightarrow F_0\longrightarrow N\longrightarrow 0$$

such that all F_i's and the kernels of all of its maps are $\mathfrak{a}$-adic complete. Set $K_1:=\ker(F_0\longrightarrow N)$ and $K_{i+1}:=\ker(F_i\longrightarrow F_{i-1})$ for all $i\geq 1$. Note that any flat R-module is ω-Gorenstein flat, and so it is Gorenstein flat module over $R\ltimes\omega$ by [**HJ2**, Example 2.8 c) and Proposition 2.15]. By Lemma 3.10, $\operatorname{Gfd}_{R\ltimes\omega}N\leq\dim R$, and so for $n:=\dim R$, K_n is a Gorenstein flat module over $R\ltimes\omega$ by [**Ho**, Theorem 3.14], and then K_n is ω-Gorenstein flat by [**HJ2**, Proposition 2.15]. Hence, Lemma 3.2 and our assumption imply that

$$\operatorname{Tor}_i^R(R/I,K_n)\cong\operatorname{Tor}_i^R(R/I,\Lambda^{\mathfrak{a}}(K_n))\cong H_i^{\mathfrak{a}}(R/I,K_n)=0$$

for any ideal I and any $i>0$. So K_n is flat which implies that flat dimension of N is finite.

iv ⇒ v) Since k is $\mathfrak{m}$-adic complete, it follows that flat dimension of k is finite, and so R is regular.

v ⇒ i) Since R is regular, any R-module has finite flat dimension, and so the assertion follows from Theorem 3.7. □

We end the paper with the following results.

Proposition 3.13. *Let* $(R,\mathfrak{m})$ *be a local ring and M be an Artinian R-module such that* $\operatorname{Gfd}_R M$ *is finite. Then* $\operatorname{Gfd}_R M=\operatorname{depth}R$.

Proof. Since M is Artinian, [**ET**, Theorem 3.5] yields that $\operatorname{Gfd}_R M=\operatorname{Gfd}_{\widehat{R}} M$. So, without loss of generality, we may assume that R is $\mathfrak{m}$-adic complete. Hence, R has a dualizing complex by [**Ha**, V.10.4]. Also $\operatorname{Hom}_R(M,E_R(R/\mathfrak{m}))$ is a finitely generated R-module and

$$M\cong\operatorname{Hom}_R(\operatorname{Hom}_R(M,E_R(R/\mathfrak{m})),E_R(R/\mathfrak{m})).$$

So,

$$\operatorname{Gfd}_R M=\operatorname{Gfd}_R\operatorname{Hom}_R(\operatorname{Hom}_R(M,E_R(R/\mathfrak{m})),E_R(R/\mathfrak{m}))=\operatorname{depth}R$$

by [**CFH**, Proposition 5.1 and Theorem 6.3]. □

Proposition 3.14. *Let* $(R, \mathfrak{m})$ *be a local ring. If* R *admits a non-zero Artinian* R*-module* M *with finite Cohen-Macaulay flat dimension and* $\sup \mathbf{L}\Lambda^{\mathfrak{m}}(M) = \dim R$*, then* R *is Cohen-Macaulay.*

Proof. Since $\mathrm{CMfd}_R M$ is finite, $\mathrm{CMfd}_R M = \mathrm{Gfd}_{R\ltimes C} M$ for a semi-dualizing R-module C. So, $\mathrm{Gfd}_{R\ltimes C} M = \operatorname{depth} R \ltimes C$ by Proposition 3.13. Then the assumption, Corollary 3.8 and the fact that $\operatorname{depth} R \ltimes C = \operatorname{depth} R$ imply that

$$\dim R = \sup \mathbf{L}\Lambda^{\mathfrak{m}}(M) \leq \mathrm{CMfd}_R M = \operatorname{depth} R,$$

and so R is Cohen-Macaulay. □

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