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ON AN OPERATOR OF STANCU-TYPE WITH FIXED POINTS e_1 AND e_2

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Abstract. The objective of this paper is to introduce an operator of Stancu-type, with the properties that the test functions e_1 and e_2 are reproduced. Also, in our approach, a theorem of error approximation and a Voronovskaja-type theorem for this operator are obtained.

1. Introduction

By $e_j, j \in \{0,1,2\}$, we denote the monomial of j-degree.

Let $\mathbb{N}$ be a set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and α, β real numbers with $0 \le \alpha \le \beta$.

In 1969 D. D. Stancu [**14**], for $m \in \mathbb{N}$ and $0 \le \alpha \le \beta$ introduced the linear positive operators

$$\left(P_m^{(\alpha,\beta)} f\right)(x) = \sum_{k=0}^{m} \binom{m}{k} x^k (1-x)^{m-k} f\Big(\frac{k+\alpha}{m+\beta}\Big) \tag{1.1}$$

defined for any $f \in C([0,1])$ and $x \in [0,1]$. The author proved that the sequence $\left(P_m^{(\alpha,\beta)}\right)_{m\ge 1}$ converges uniformly to f on $[0,1]$. Note that the operator from (1.1) preserves the test function e_0.

A generalization of the operator (1.1) was defined by P. I. Braica, O. T. Pop, D. Bărbosu and L. I. Pişcoran [**4**] as follows: Let $m_0 \in \mathbb{N}$ be a fixed number. For $m \in \mathbb{N}, m \ge m_0$ and $0 \le \alpha \le \beta$, the authors consider the operators

$$\begin{aligned}&\left(Q_m^{(\alpha,\beta)} f\right)(x)\\ &= \frac{1}{m^m} \sum_{k=0}^{m} \binom{m}{k} ((m+\beta)x-\alpha)^k (m+\alpha-(m+\beta)x)^{m-k} f\Big(\frac{k+\alpha}{m+\beta}\Big)\end{aligned} \tag{1.2}$$

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defined for any $f \in C([0,1])$ and $x \in \left[\frac{\alpha}{m_0+\beta}; \frac{m_0+\alpha}{m_0+\beta}\right]$. It was also proved that the sequence $\left(Q_m^{(\alpha,\beta)}\right)_{m\geq 1}$ converges uniformly to f on $\left[\frac{\alpha}{m_0+\beta}; \frac{m_0+\alpha}{m_0+\beta}\right]$. Note that the operators (1.2) preserve the test functions e_0 and e_1.

Another generalization of the operator (1.1) of the following form

$$\left(V_m^{(\alpha,\beta)} f\right)(x) = \sum_{k=0}^{m} \binom{m}{k} r_m^k(x)(1-r_m(x))^{m-k} f\left(\frac{m+\alpha}{m+\beta}\right), \tag{1.3}$$

$m \in \mathbb{N}, m \geq m_0, m_0 \in \mathbb{N}$ a fixed number, and $0 \leq \alpha \leq \beta$ and

$$\begin{aligned} r_m(x) &= r_m^*(x) \\ &= \begin{cases} \frac{-m(1+2\alpha)+\sqrt{m^2(1+2\alpha)^2-4m(m-1)(\alpha^2-(m+\beta)^2)x^2}}{2m(m-1)} & m > 1, \\ x^2 & m = 1, \end{cases} \end{aligned} \tag{1.4}$$

appeared in [**11**].

This operator is defined for any $f \in C([0,1])$ and $x \in \left[\frac{\alpha}{m_0+\beta}; \frac{m_0+\alpha}{m_0+\beta}\right]$. It is proved that the sequence $\left(V_m^{(\alpha,\beta)}\right)_{m\geq 1}$ converges uniformly to f on $\left[\frac{\alpha}{m_0+\beta}; \frac{m_0+\alpha}{m_0+\beta}\right]$. Note that the operator (1.3) preserves the test functions e_0 and e_2.

In the last decade, next issue close to the linear approximation of operators was intensively investigated: to create classes of operators that reproduce two of the three test functions, see, for example, [**1**] and [**10**].

Following the ideas from [**4**], [**5**], [**9**]–[**11**] and [**13**], in this paper, we introduce a general class which preserves the test functions e_1 and e_2. For our operators approximation properties, a convergence theorem and a Voronovskaja-type theorem are obtained.

The paper is organized as follows. In Section 2, we recall some results obtained in [**12**] which are essentially used for obtaining the main results of this paper. Section 3 is devoted to the construction of the general class of a new linear and positive operators. In Section 4, we obtain a Stancu type operator which is an operator that preserves the test functions e_1 and e_2.

2. Preliminaries

In this section, we recall some notions and results which we will use in what follows, see [**12**].

We consider real intervals I, J with the property $I \cap J \neq \emptyset$ and use the function sets $E(I)$, $F(J)$ that are subsets of the set of real functions defined on I, respectively J,

$$B(I) = \{f \mid f\colon I \to \mathbb{R}, f \text{ bounded on } I\},$$
$$C(I) = \{f \mid f\colon I \to \mathbb{R}, f \text{ continuous on } I\}$$

and

$$C_B(I) = B(I) \cap C(I).$$

For $x \in I$, we consider the function $\psi_x\colon I \to \mathbb{R}$, $\psi_x(t) = t - x$, $t \in I$.

For any $m\in\mathbb{N}_0$ and $k\in\{0,1,2,\ldots,m\}$, we consider the functions $\varphi_{m,k}\colon J\to\mathbb{R}$ with the property $\varphi_{m,k}(x)\geq 0$, $x\in J$, and the linear positive functionals $A_{m,k}\colon E(I)\to\mathbb{R}$. For $m\in\mathbb{N}_0$, we define the operator $L_m\colon E(I)\to F(J)$ by

$$(L_m f)(x)=\sum_{k=0}^{m}\varphi_{m,k}(x)A_{m,k}(f). \tag{2.1}$$

Remark 2.1. The operators $(L_m)_{m\in\mathbb{N}}$ are linear and positive on $E(I\cap J)$.

For any $f\in E(I), x\in I\cap J$, and for $i\in\mathbb{N}_0$, we define $T_{m,i}$ by

$$(T_{m,i}L_m)(x)=m^i(L_m\psi_x^i)(x)=m^i\sum_{k=0}^{m}\varphi_{m,k}(x)A_{m,k}(\psi_x^i) \tag{2.2}$$

In the following, let s be a fixed even natural number. We suppose that the operators $L_m, m\geq 0$ satisfy the following condition:
there exists the smallest $\alpha_s,\alpha_{s+1}\in[0,\infty)$ such that

$$\lim_{m\longrightarrow\infty}\frac{(T_{m,j}L_m)(x)}{m^{\alpha_j}}=B_j(x)\in\mathbb{R}, \tag{2.3}$$

for any $x\in I\cap J$, $j\in\{s,s+2\}$, and

$$\alpha_{s+2}<\alpha_s+2. \tag{2.4}$$

If $I\subset\mathbb{R}$ is a given interval and $f\in C_B(I)$, then the first order modulus of smoothness of f is the function $\omega(f;\cdot)\colon[0,+\infty)\to\mathbb{R}$ defined for any $\delta\geq 0$ by $\omega(f,\delta)=\sup\{|f(x')-f(x'')|\ :\ x',x''\in I,\ |x'-x''|\leq\delta\}$.

Theorem 2.1 ([**12**])**.** *Let $f\colon I\longrightarrow\mathbb{R}$ be a function. If $x\in I\cap J$, f is an s times derivable function in x, and the function $f^{(s)}$ is continuous in x, then*

$$\lim_{m\to\infty}m^{s-\alpha_s}\Big((L_m f)(x)-\sum_{i=0}^{s}\frac{f^{(i)}(x)}{m^i i!}(T_{m,i}L_m)(x)\Big)=0. \tag{2.5}$$

If f is an s times differentiable function on I, the function $f^{(s)}$ is continuous on I and there exist $m(s)\in\mathbb{N}$ and $k_j\in\mathbb{R}$ such that for any natural number $m\geq m(s)$ and for any $x\in I\cap J$, we have

$$\frac{(T_{m,j}L_m)(x)}{m^{\alpha_j}}\leq k_j, \tag{2.6}$$

where $j\in\{s,s+2\}$. Then the convergence given in (2.5) *is uniformly on $I\cap J$ and*

$$\begin{aligned}&m^{s-\alpha_s}\Big|(L_m f)(x)-\sum_{i=0}^{s}\frac{f^{(i)}(x)}{m^i i!}(T_{m,i}L_m)(x)\Big|\\&\leq\frac{1}{s!}(k_s+k_{s+2})\omega\Big(f^{(s)};\frac{1}{\sqrt{m^{2+\alpha_s-\alpha_{s+2}}}}\Big)\end{aligned} \tag{2.7}$$

for any $x\in I\cap J$ and $m\geq m(s)$.

The operator $(L_m)_{m\in\mathbb{N}}$ satisfies the condition (2.3), so it satisfies the condition (2.6) for any $x \in I \cap J$.

Now, let α, β be fixed real numbers with the property $0 \le \alpha \le \beta$.

We observe that if $m_1, m_2 \in \mathbb{N}$, $m_1 \le m_2$, then

$$\left[\frac{\alpha}{m_1+\beta}, \frac{m_1+\alpha}{m_1+\beta}\right] \subset \left[\frac{\alpha}{m_2+\beta}, \frac{m_2+\alpha}{m_2+\beta}\right]. \tag{2.8}$$

In the following, let $m_0 \in \mathbb{N}$ be fixed. One has

$$\left[\frac{\alpha}{m_0+\beta}, \frac{m_0+\alpha}{m_0+\beta}\right] \subset \left[\frac{\alpha}{m+\beta}, \frac{m+\alpha}{m+\beta}\right] \tag{2.9}$$

for any $m \in \mathbb{N}$ and $m \ge m_0$.

The relation (2.9) follows from (2.8).

Moreover, for $m \to \infty$, the interval $\left[\frac{\alpha}{m+\beta}; \frac{m+\alpha}{m+\beta}\right]$ becomes $[0,1]$.

3. The construction of a general linear and positive operators

For given $m_0 \in \mathbb{N}$, let $\mathbb{N}_1 = \{m \in \mathbb{N}_0 | m \ge m_0\}$, $0 \le \alpha \le \beta$, and the functions $a_m\colon J \to \mathbb{R}$, $b_m\colon J \to \mathbb{R}$ such that $a_m(x) \ge 0$, $b_m(x) \ge 0$ for any $x \in J$, $m \in \mathbb{N}_1$ and $I = [0,1]$.

We define the operator of the following form

$$(H_m^{(\alpha,\beta)} f)(x) = \sum_{k=0}^{m} \binom{m}{k} a_m^k(x) b_m^{m-k}(x) \cdot f\Big(\frac{k+\alpha}{m+\beta}\Big), \tag{3.1}$$

for any $m \in \mathbb{N}_1$, $x \in J$, and $f \in E([0,1])$, where $E([0,1])$ is a linear space of real valued functions defined on $[0,1]$.

In what follows, we impose some additional conditions to be fulfilled by our operators

$$(H_m^{(\alpha,\beta)} e_0)(x) = 1 + u_m(x), m \in \mathbb{N}_1, \qquad x \in J, \tag{3.2}$$

where $u_m\colon J \to \mathbb{R}$. We get

$$a_m(x) + b_m(x) = (1 + u_m(x))^{\frac{1}{m}} \tag{3.3}$$

for any $m \in \mathbb{N}_1$ and $x \in J$.

The second condition is read as follows

$$(H_m^{(\alpha,\beta)} e_1)(x) = x + v_m(x), \qquad m \in \mathbb{N}_1,\ x \in J, \tag{3.4}$$

where $v_m\colon J \to \mathbb{R}$. We get

$$\frac{m a_m(x)}{m+\beta}(a_m(x)+b_m(x))^{m-1} + \frac{\alpha}{m+\beta}(a_m(x)+b_m(x))^m = x + v_m(x) \tag{3.5}$$

for any $m \in \mathbb{N}_1$ and $x \in J$.

From (3.3) and (3.5), it follows

$$a_m(x) = (1+u_m(x))^{\frac{1}{m}} \Big(\frac{m+\beta}{m} \cdot \frac{x+v_m(x)}{1+u_m(x)} - \frac{\alpha}{m}\Big), \tag{3.6}$$

and

$$b_m(x) = (1+u_m(x))^{\frac{1}{m}} \Big(1 - \frac{m+\beta}{m} \cdot \frac{x+v_m(x)}{1+u_m(x)} + \frac{\alpha}{m}\Big) \tag{3.7}$$

for any $m \in \mathbb{N}_1$ and $x \in J$.

Taking into account (3.6) and (3.7), the operator (3.1) becomes

$$\begin{aligned}(H_m^{(\alpha,\beta)} f)(x) = (1+u_m(x)) \sum_{k=0}^{m} \binom{m}{k} \Big(\frac{m+\beta}{m} \cdot \frac{x+v_m(x)}{1+u_m(x)} - \frac{\alpha}{m}\Big)^k \\ \cdot \Big(1 - \frac{m+\beta}{m} \cdot \frac{x+v_m(x)}{1+u_m(x)} + \frac{\alpha}{m}\Big)^{m-k} \cdot f\Big(\frac{k+\alpha}{m+\beta}\Big)\end{aligned} \tag{3.8}$$

for any $m \in \mathbb{N}_1, x \in J$ and $f \in E([0,1])$.

From (3.1), we have

$$\begin{aligned}(H_m^{(\alpha,\beta)} e_2)(x) = {} & \frac{m-1}{m} \cdot \frac{x^2}{1+u_m(x)} \\ & + \Big(\frac{2(m-1)}{m(1+u_m(x))} \cdot v_m(x) + \frac{m+2\alpha}{m(m+\beta)}\Big) \cdot x \\ & + \frac{m-1}{m(1+u_m(x))} \cdot v_m^2(x) + \frac{m+2\alpha}{m(m+\beta)} v_m(x) \\ & + \Big(\frac{-\alpha^2 - \alpha m}{m(m+\beta)^2}\Big) \cdot (1+u_m(x)),\end{aligned} \tag{3.9}$$

for any $m \in \mathbb{N}_1$ and $x \in J$.

Coming back to Theorem 2.1, for our operator (3.1), we have $I = [0,1]$, $E([0,1]) = C([0,1])$,

$$\begin{aligned}\varphi_{m,k} = {} & (1+u_m(x)) \cdot \binom{m}{k} \Big(\frac{m+\beta}{m} \cdot \frac{x+v_m(x)}{1+u_m(x)} - \frac{\alpha}{m}\Big)^k \\ & \cdot \Big(1 - \frac{m+\beta}{m} \cdot \frac{x+v_m(x)}{1+u_m(x)} + \frac{\alpha}{m}\Big)^{m-k}\end{aligned} \tag{3.10}$$

and

$$A_{m,k}(f) = f\Big(\frac{k+\alpha}{m+\beta}\Big) \tag{3.11}$$

for any $m \in \mathbb{N}_1$, $x \in J$ and $f \in C([0,1])$.

4. $H_m^{(\alpha,\beta)}$ OPERATORS PRESERVING TEST FUNCTIONS e_1 AND e_2

Let $m_0 \in \mathbb{N}$ be fixed. If $H_m^{(\alpha,\beta)} e_1 = e_1$ for any $m \in \mathbb{N}_1$, then $v_m(x) = 0$. From (3.6) and (3.7), we have

$$a_m(x) = (1+u_m(x))^{\frac{1}{m}} \Big(\frac{m+\beta}{m} \cdot \frac{x}{1+u_m(x)} - \frac{\alpha}{m}\Big) \tag{4.1}$$

and

$$b_m(x) = (1+u_m(x))^{\frac{1}{m}}\Big(1 - \frac{m+\beta}{m}\cdot\frac{x}{1+u_m x} + \frac{\alpha}{m}\Big) \tag{4.2}$$

for any $m \in \mathbb{N}_1$ and $x \in J$.

After some calculus, the following result is immediately seen.

Lemma 4.1. *Let $m \in \mathbb{N}_1$. The following relations are equivalent*

(i) $a_m(x) \geq 0$ *and* $b_m(x) \geq 0$,

(ii) $x \in J$ *and* $J = \Big[\dfrac{\alpha}{m_0+\beta}, \dfrac{m_0+\alpha}{m_0+\beta}\Big]$.

Next, taking into account (3.9), from $H_m^{(\alpha,\beta)} e_2 = e_2$ and $v_m(x) = 0$, $m \in \mathbb{N}_1$, we have

$$\frac{m-1}{m(1+u_m(x))}x^2 + \frac{m+2\alpha}{m(m+\beta)}x + \Big(\frac{-\alpha^2-\alpha m}{m(m+\beta)^2}\Big)(1+u_m(x)) = x^2. \tag{4.3}$$

The relation above can be written as

$$\begin{aligned}&(\alpha^2+\alpha m)u_m^2(x)\\ &+\Big(m(m+\beta)^2x^2 - (m+\beta)(m+2\alpha)x + 2(\alpha^2+\alpha m)\Big)u_m(x)\\ &+(m+\beta)^2x^2 - (m+\beta)(m+2\alpha)x + (\alpha^2+\alpha m) = 0.\end{aligned} \tag{4.4}$$

The relation (4.4) is a second degree equation in $u_m(x)$ and we denote

$$\begin{aligned}a &= \alpha^2+\alpha m,\\ b &= \Big(m(m+\beta)^2x^2 - (m+\beta)(m+2\alpha)x + 2(\alpha^2+\alpha m)\Big),\\ c &= (m+\beta)^2x^2 - (m+\beta)(m+2\alpha)x + (\alpha^2+\alpha m).\end{aligned} \tag{4.5}$$

Remark 4.1. For $m \in \mathbb{N}_1$, $x \in J$, $J = \Big[\frac{\alpha}{m_0+\beta}, \frac{m_0+\alpha}{m_0+\beta}\Big]$, we have $a > 0$, $c \leq 0$, consequently $\Delta \geq 0$ and the positive solution of equation (4.4) is given as follows:

$$u_m(x) = \frac{(m+\beta)(m+2\alpha)x - m(m+\beta)^2x^2 - 2(\alpha^2+\alpha m) + \sqrt{\Delta}}{2(\alpha^2+\alpha m)}, \tag{4.6}$$

Proof. If $(m+\beta)^2x^2-(m+\beta)(m+2\alpha)x+(\alpha^2+\alpha m) = 0$, then this equation has the solutions $\frac{\alpha}{m+\beta}, \frac{m+\alpha}{m+\beta}$. Because $(m+\beta)^2 > 0$, then $c \leq 0$ for $x \in \Big[\frac{\alpha}{m+\beta}, \frac{m+\alpha}{m+\beta}\Big]$ and from (2.9), we have $c \leq 0$ for $x \in J$, $J = \Big[\frac{\alpha}{m_0+\beta}, \frac{m_0+\alpha}{m_0+\beta}\Big]$. □

Lemma 4.2. *If $x \in \Big[\frac{\alpha}{m_0+\beta}, \frac{m_0+\alpha}{m_0+\beta}\Big]$ and $m \in \mathbb{N}_1$, then the operators $H_m^{(\alpha,\beta)}$ are linear and positive on $C([0,1])$.*

Proof. Taking into account (2.9) and Lemma 4.1, the results follow. □

Lemma 4.3. *If* $x \in \left[\frac{\alpha}{m_0+\beta}, \frac{m_0+\alpha}{m_0+\beta}\right]$, *then*

$$\lim_{m\to\infty} u_m(x) = 0 \tag{4.7}$$

and

$$\lim_{m\to\infty} m u_m(x) = \frac{1-x}{x}. \tag{4.8}$$

Proof. From (4.6), we have

$$\begin{aligned}
&\lim_{m\to\infty} u_m(x)\\
&= \lim_{m\to\infty} \frac{\Delta - \Big(m(m+\beta)^2x^2 + 2(\alpha^2+\alpha m) - (m+\beta)(m+2\alpha)x\Big)^2}{2(\alpha^2+\alpha m)\Big(\sqrt{\Delta} + m(m+\beta)^2x^2 + 2(\alpha^2+\alpha m) - (m+\beta)(m+2\alpha)x\Big)}\\
&= \lim_{m\to\infty} \frac{-4(\alpha^2+\alpha m)\Big((m+\beta)^2x^2 - (m+\beta)(m+2\alpha)x + (\alpha^2+\alpha m)\Big)}{2(\alpha^2+\alpha m)\Big(\sqrt{\Delta} + m(m+\beta)^2x^2 + 2(\alpha^2+\alpha m) - (m+\beta)(m+2\alpha)x\Big)}\\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{m\to\infty} m u_m(x)\\
&= \lim_{m\to\infty} \frac{-4m(\alpha^2+\alpha m)\Big((m+\beta)^2x^2 - (m+\beta)(m+2\alpha)x + (\alpha^2+\alpha m)\Big)}{2(\alpha^2+\alpha m)\Big(\sqrt{\Delta} + m(m+\beta)^2x^2 + 2(\alpha^2+\alpha m) - (m+\beta)(m+2\alpha)x\Big)}\\
&= \frac{1-x}{x}.
\end{aligned}$$

□

Remark 4.2. If $x \in \left[\frac{\alpha}{m_0+\beta}, \frac{m_0+\alpha}{m_0+\beta}\right]$, then there exists $m_1 \in \mathbb{N}$ such that for any natural number $m \geq m_1$, we have

$$\frac{\beta - m_0 - 2\alpha}{m(m_0+\alpha)} \leq \frac{1-2x}{mx} \leq u_m(x) \leq \frac{1}{mx} \leq \frac{m_0+\beta}{m\alpha}. \tag{4.9}$$

Proof. We take (4.8) into account. For $x \in \left[\frac{\alpha}{m_0+\beta}, \frac{m_0+\alpha}{m_0+\beta}\right]$, since the functions $\frac{1-2x}{mx}, \frac{1}{mx}$ are decreasing for $x \in [0,1]$, we have $\frac{\beta-m_0-2\alpha}{m(m_0+\alpha)} \leq \frac{1-2x}{mx}$ and $\frac{1}{mx} \leq \frac{m_0+\beta}{m\alpha}$, respectively. □

Considering (2.2) and (3.2), we can state the following lemma.

Lemma 4.4. *For* $m \in \mathbb{N}_1$ *and* $x \in \left[\frac{\alpha}{m_0+\beta}, \frac{m_0+\alpha}{m_0+\beta}\right]$, *the following identities*

$$(T_{m,0}H_m^{(\alpha,\beta 0)})(x) = 1 + u_m(x), \tag{4.10}$$

$$(T_{m,1}H_m^{(\alpha,\beta)})(x) = mxu_m(x) \tag{4.11}$$

and

$$(T_{m,2}H_m^{(\alpha,\beta)})(x) = m^2x^2u_m(x) \tag{4.12}$$

hold.

In accordance with Theorem 2.1, from (4.10)–(4.12), we obtain $k_0 = 2, k_2 = \frac{5}{4}$, $\alpha_0 = 0$ and $\alpha_2 = 1$.

Theorem 4.1. *Let $f\colon [0,1] \to \mathbb{R}$ be continuous function s times differentiable on $[0,1]$, having the s-order derivative continuous on $[0,1]$.*

For $s = 0$, we have

$$\lim_{m\to\infty} H_m^{(\alpha,\beta)} f = f \tag{4.13}$$

uniformly on $J = \left[\frac{\alpha}{m_0+\beta}, \frac{m_0+\alpha}{m_0+\beta}\right]$. There exists $m^* = max(m_0, m(0), m_1)$ such that

$$|(H_m^{(\alpha,\beta)} f)(x) - f(x)| \le M \cdot \frac{m_0+\beta}{m\alpha} + \frac{13}{4}\omega\Big(f; \frac{1}{\sqrt{m}}\Big) \tag{4.14}$$

for any $x \in \left[\frac{\alpha}{m_0+\beta}, \frac{m_0+\alpha}{m_0+\beta}\right], m \in \mathbb{N}, m \ge m^*$. In $\sup_{x\in J} |f(x)| = M$ above, for $s = 2$, we have

$$\begin{aligned}\lim_{m\to\infty} m\Big((H_m^{(\alpha,\beta)} f(x)) - f(x)\Big) &= \frac{1-x}{x} f(x) + (1-x)f^{(1)}(x) \\ &\quad + \frac{x(1-x)}{2} f^{(2)}(x).\end{aligned} \tag{4.15}$$

Proof. We use Theorem 2.1, Remark 4.2 with $\left|\frac{\beta-m_0-2\alpha}{m(m_0+\alpha)}\right| \le \frac{m_0+\beta}{m\alpha}$, Lemma 4.4 and the relations (4.7)–(4.13). Applying in the classical inequality (4.14)

$$|d| - |e| \le |d - e|, \qquad d,\ e \in \mathbb{R},$$

the conclusion follows. □

The relation (4.15) is a Voronovskaja-type theorem.

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