SUBTANGENT-LIKE STATISTICAL MANIFOLDS

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ABSTRACT. Subtangent-like statistical manifolds are introduced and characterization theorems for them are given. The special case when the conjugate connections are projectively (or dual-projectively) equivalent is considered.

1. Introduction

A statistical manifold is a Riemannian manifold, whose each point is a probability space. It connects information geometry, affine differential geometry and Hessian geometry. Information geometry is a branch of mathematics that applies the techniques of differential geometry to the field of probability theory. This is done by taking probability distributions for a statistical model as the points of a Riemannian manifold, forming a statistical manifold. The Fisher information metric provides the Riemannian metric. Information geometry can be applied in various areas, where parametrized distributions play a role such as in statistical inference, time series and linear systems, quantum systems, neuronal networks, machine learning, statistical mechanics, biology, mathematical finance, etc. In [1], [2], [3], [8], [9] the statistical manifolds are studied from the point of view of information geometry and it is given a new description of the statistical distributions by using the obtained geometric structures. The statistical manifold was introduced by S. Amari [1] as being a triple \((M, \nabla, g)\) consisting of a smooth manifold \(M\), a non-degenerate metric \(g\) on it and a torsion-free affine connection \(\nabla\) with the property that \(\nabla g\) is symmetric. With it we can associate another torsion-free affine connection \(\nabla^\ast\) defined by the relation

\[
X (g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)
\]

for any \(X, Y, Z \in \mathfrak{X}(M)\), called the conjugate (or the dual) connection of \(\nabla\) with respect to \(g\). In this case, \(\nabla^\ast g\) is also symmetric \((\nabla^\ast g = -\nabla g)\), therefore, \((M, \nabla^\ast, g)\) is a statistical manifold, too. It’s easy to see that the conjugate of the conjugate connection of \(\nabla\) coincides with \(\nabla\), i.e., \((\nabla^\ast)^\ast = \nabla\).

Starting from this, different notions of generalized connections were also defined. These were contained in the following general formulation, namely, two connections

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are conjugate in a larger sense if there exists a $(0, 3)$-tensor field $C$ on $M$ satisfying
\begin{equation}
X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z) + C(X, Y, Z)
\end{equation}
for any $X, Y, Z \in \mathfrak{X}(M)$. A simple condition on $C$ is implied by the symmetry of
$\nabla g$, that is, $g((\nabla^*)_X Y - \nabla_X Y, Z) = C(X, Y, Z) - C(X, Z, Y)$ [4]. For a certain
tensor field $C$, some particular cases were stated in [4], [11], [12], [13] defining
that $\nabla$ and $\nabla^*$ are said to be
\begin{enumerate}
\item \textit{generalized conjugate} [11], [12] with respect to $g$ by a $1$-form $\eta$ if
\begin{equation}
X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z) - \eta(X)g(Y, Z);
\end{equation}
\item \textit{semi-conjugate} [11], [13] with respect to $g$ by a $1$-form $\eta$ if
\begin{equation}
X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z) + \eta(Z)g(X, Y);
\end{equation}
\item \textit{dual semi-conjugate} [4] with respect to $g$ by a $1$-form $\eta$ if
\begin{equation}
X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z) - \eta(X)g(Y, Z) - \eta(Y)g(X, Z).
\end{equation}
\end{enumerate}
The motivation of studying the first types of conjugate connections comes for
the first ones from Weyl geometry [10]. The second ones naturally appear in affine
hypersurface theory [13] and the last ones establish the connection between them
[4]. One important feature of generalized connections is their invariance under
gauge transformations [4].

In [16], K. Takano studied the statistical manifolds with an almost complex
structure. In what follows, we shall study the interference of an almost subtangent
structure on a statistical manifold. Recall that the almost tangent structures were
introduced by R. S. Clark and M. Bruckheimer [5], [6] and independently, by
H. A. Eliopoulos [14]. An \textit{almost tangent structure} on a $2n$-dimensional smooth
manifold $M$ is an endomorphism $J$ of the tangent bundle $TM$ of constant rank,
satisfying
\begin{equation}
\ker J = \text{Im} J.
\end{equation}
The pair $(M, J)$ is called the \textit{almost tangent manifold}. The name is motivated
by the fact that (1.3) implies the nilpotence $J^2 = 0$ exactly as the natural tangent
structure of tangent bundles. It is known that the most important $G$-structures
of the first type are those defined by linear operators satisfying certain algebraic
relations. Note that the almost tangent structures define a class of conjugate
$G$-structures on $M$, a group $G$ for a representative structure consisting of all
matrices of the form \begin{pmatrix} A & 0 \\ B & A \end{pmatrix}, where $A, B$ are matrices of order $n \times n$ and $A$ is
non-singular.

In addition, if we assume that $J$ is integrable, i.e.,
\begin{equation}
\end{equation}
then $J$ is called the \textit{tangent structure} and $(M, J)$ is called \textit{tangent manifold}.

Basic facts following directly from the definition are stated in [17]:
\begin{enumerate}
\item the distribution $\text{Im} J (= \ker J)$ defines a foliation denoted by $V(M)$ and
called the the \textit{vertical distribution}.
Example 1.1. [7] $M = \mathbb{R}^2$, $J_e(x, y) = (0, x)$ is a tangent structure with $\ker J_e$ the $Y$-axis, hence the name. The subscript $e$ comes from ”Euclidean”.

(ii) there exists an atlas on $M$ with local coordinates $(x, y) = (x^i, y^i)_{1 \leq i \leq n}$ such that $J = \frac{\partial}{\partial y^i} \otimes dx^i$, i.e.,

\[
J \left( \frac{\partial}{\partial x^i} \right) = \frac{\partial}{\partial y^i}, \quad J \left( \frac{\partial}{\partial y^i} \right) = 0.
\]

We call $(x, y)$ canonical coordinates and the change of canonical coordinates $(x, y) \to (\tilde{x}, \tilde{y})$ is given by

\[
\begin{align*}
\tilde{x}^i &= \tilde{x}^i(x) \\
\tilde{y}^i &= \frac{\partial \tilde{x}^i}{\partial x^a} y^a + B^i(x).
\end{align*}
\]

It results the description in terms of $G$-structures. Namely, a tangent structure is a $G$-structure with

\[
G = \left\{ C = \begin{pmatrix} A & O_n \\ B & A \end{pmatrix} \in GL(2n, \mathbb{R}) ; \ A \in GL(n, \mathbb{R}), \ B \in gl(n, \mathbb{R}) \right\}
\]

and $G$ is the invariance group of matrix $J = \begin{pmatrix} O_n & O_n \\ I_n & O_n \end{pmatrix}$, i.e., $C \in G$ if and only if $C \cdot J = J \cdot C$.

The natural almost tangent structure $J$ of $M = TN$ is an example of tangent structure having exactly the expression (1.5) if $(x^i)$ are the coordinates on $N$ and $(y^i)$ are the coordinates in the fibers of $TN \to N$. Also, $J_e$ in Example 1.1 has the above expression (1.5) with $n = 1$, whence it is integrable.

If the condition 1.3 is weakened, requiring that only $J$ squares to 0, we call $J$ an almost subtangent structure. In this case, $\text{Im} J \subset \ker J$ and for a non-degenerate metric $g$ on $M$, ker $J$ is the Lagrangian distribution for the almost symplectic structure $\omega_{1, J}(X, Y) := g(X, JY) - g(JX, Y)$, $X, Y \in \mathcal{X}(M)$. In this context we introduce the analogue notion of a holomorphic statistical manifold, namely, the special statistical manifold and give a construction of strong special statistical manifolds.

2. Subtangent-like statistical manifolds

Let $M$ be a smooth manifold, $g$ a non-degenerate metric and $J$ an almost subtangent structure on $M$.

Definition 2.1. We say that $(M, g, J)$ is an almost subtangent-like manifold if there exists an endomorphism of the tangent bundle $J^*$ satisfying

\[
g(JX, Y) + g(X, J^*Y) = 0
\]

for any $X, Y \in \mathcal{X}(M)$.

In this case, $(J^*)^2 = 0$, $(J^*)^* = J$ and $J^*$ is called an conjugate (or the dual) almost subtangent structure of $J$. 
If $J$ and $J^*$ are two conjugate almost subtangent structures, then $J - J^*$ and $J + J^*$, respectively, are symmetric and skew-symmetric with respect to $g$. Also, $JJ^* + J^*J$ is symmetric with respect to $g$.

**Example 2.1.** Regarding the Example 1.1 (i), we get the metric $g = \text{diag}(1, -1)$ and then $(\mathbb{R}^2, g, J_e)$ is an almost subtangent-like statistical manifold with

$$J_e^*(x, y) = (y, 0)$$

or equivalently

$$J_e^* \left( \frac{\partial}{\partial x} \right) = 0, \quad J_e^* \left( \frac{\partial}{\partial y} \right) = \frac{\partial}{\partial x}.$$ 

**Definition 2.2.** We say that $(M, \nabla, g, J)$ is an almost subtangent-like statistical manifold if $(M, \nabla, g)$ is statistical manifold and $(M, g, J)$ is an almost subtangent-like manifold. Moreover, if $\nabla J = 0$, we drop the adjective almost. From (2.1), we get that $\text{Im} J^* \subset \ker J^* \perp g \text{Im} J \subset \ker J$. Note that on a subtangent-like statistical manifold $(M, \nabla, g, J)$, the linear connection $\nabla$ restricts to the distribution $\ker J$, which means that for $Y \in \ker J$, it follows $\nabla_X Y \in \ker J$ for any $X \in \mathfrak{X}(M)$.

Concerning the conjugate structures of an almost subtangent-like statistical manifold, we can state the following result

**Proposition 2.3.** Let $(M, \nabla, g, J)$ be an almost subtangent-like statistical manifold, $\nabla^*$ the conjugate connection of $\nabla$ and $J^*$ a conjugate almost subtangent structure of $J$. Then

1. $(M, \nabla^*, g, J^*)$ is an almost subtangent-like statistical manifold;
2. $g((\nabla_X J)Y, Z) + g(Y, (\nabla_X J^*)Z) = 0$ for any $X, Y, Z \in \mathfrak{X}(M)$;
3. if $(M, \nabla, g, J)$ is a subtangent-like statistical manifold, then $(M, \nabla^*, g, J^*)$ is a subtangent-like statistical manifold, too.

**Proof.** 1. From the previous considerations. 2. From a direct computation. 3. From 1 and 2. □

3. **J-conjugate connections and the statistical structure**

Let $g$ be a non-degenerate metric on $M$, $\nabla$ an affine connection, $\nabla^*$ its conjugate w.r.t. $g$, $J$ an almost subtangent structure and $J^*$ its conjugate w.r.t. $g$.

Define the $J$-conjugate connections of $\nabla$ and $\nabla^*$ by

$$\nabla^{(J)} := \nabla - J \circ \nabla, \quad \nabla^{*(J)} := \nabla^* - J \circ \nabla^*$$

and the $J^*$-conjugate connections of $\nabla$ and $\nabla^*$ by

$$\nabla^{(J^*)} := \nabla - J^* \circ \nabla^*, \quad \nabla^{*(J^*)} := \nabla^* - J^* \circ \nabla^*.$$

W.r.t. $g$, the conjugate connection of $\nabla^{(J)}$ is $\nabla^{*(J^*)}$ and the conjugate connection of $\nabla^{(J^*)}$ is $\nabla^{*(J)}$, i.e.,

$$(\nabla^{(J)})^* = \nabla^{*(J^*)}, \quad (\nabla^{(J^*)})^* = \nabla^{*(J)}.$$
The following propositions establish properties of the $J$-(and $J^*$-)conjugate connections of an affine connection and its dual and provide necessary and sufficient conditions for these connections to give rise to statistical or almost subtangent-like statistical manifolds.

**Proposition 3.1.** Let $(M,\nabla, g)$ be a statistical manifold and $J$ an almost subtangent structure. Then $(M,\nabla(J), g)$ is a statistical manifold if and only if $\nabla_XJY - \nabla_YJX \in \ker J$ and $g(X, (\nabla_ZJ)Y) = g(Z, (\nabla_XJ)Y)$ for any $X, Y, Z \in \mathfrak{X}(M)$.

**Proof.** Notice that $$(\nabla^{(J)}_X g)(Y, Z) = (\nabla_X g)(Y, Z) + g(J(\nabla_XJ)Y, Z) + g(Y, J(\nabla_XJ)Z)$$ and the relation between the torsion of $\nabla^{(J)}$ and the torsion of $\nabla$ is $$T_{\nabla(J)}(X, Y) = T_{\nabla}(X, Y) - J(\nabla_XJY - \nabla_YJX)$$ for any $X, Y, Z \in \mathfrak{X}(M)$. □

In particular, if $J$ is a subtangent structure, then $(M, \nabla(J), g)$ is a statistical manifold.

**Proposition 3.2.** Let $(M,\nabla, g, J)$ be an almost subtangent-like statistical manifold, $\nabla^*$ the conjugate connection of $\nabla$ and $J^*$ a conjugate almost subtangent structure of $J$.

1. (a) $(M,\nabla(J), g, J)$ is an almost subtangent-like statistical manifold if and only if $\nabla_XJY - \nabla_YJX \in \ker J$ and $g(X, (\nabla_ZJ)Y) = g(Z, (\nabla_XJ)Y)$ for any $X, Y, Z \in \mathfrak{X}(M)$.

(b) If $(M,\nabla(J), g, J)$ is an almost subtangent-like statistical manifold, then $(M, (\nabla(J))^*, g, J^*)$ is an almost subtangent-like statistical manifold or equivalent, $(M, \nabla^*(J^*), g, J^*)$ is almost subtangent-like statistical manifold.

2. (a) $(M,\nabla^{(J^*)}, g, J^*)$ is an almost subtangent-like statistical manifold if and only if $\nabla_XJ^*Y - \nabla_YJ^*X \in \ker J^*$ and $g(X, (\nabla_ZJ^*)Y) = g(Z, (\nabla_XJ^*)Y)$ for any $X, Y, Z \in \mathfrak{X}(M)$.

(b) If $(M,\nabla^{(J^*)}, g, J^*)$ is an almost subtangent-like statistical manifold, then $(M, (\nabla^{(J^*)})^*, g, J)$ is an almost subtangent-like statistical manifold or equivalent, $(M, \nabla^*(J), g, J)$ is an almost subtangent-like statistical manifold.

**Proof.** (a) follows from Proposition 2.3 and (b) from Proposition 2.3 and the observation that $(\nabla^{(J^*)})^* = \nabla^{(J)}$. □

4. Special statistical manifolds

An analogue of the notion of holomorphic statistical manifold defined in [15] can be here considered, namely, the special statistical manifold (s.s.m.).
4.1. Weak special statistical manifold

**Definition 4.1.** We say that the subtangent-like statistical manifold \((M, \nabla, g, J)\) is weak s.s.m. if the 2-form \(\omega_J(X, Y) := g(X, JY)\) is \(\nabla\)-parallel.

A necessary and sufficient condition for a subtangent-like statistical manifold to be weak s.s.m. is given in the following proposition

**Proposition 4.2.** The subtangent-like statistical manifold \((M, \nabla, g, J)\) is weak s.s.m. if and only if

\[
(\nabla_X g) \circ (J \times I) = -g \circ (\nabla_X J \times I)
\]

for any \(X \in \mathfrak{X}(M)\).

**Proof.** Computing \((\nabla_X \omega_J)(Y, Z) = -(\nabla_X g)(JY, Z) - g((\nabla_X J)Y, Z)\) for any \(X, Y, Z \in \mathfrak{X}(M)\) and from the condition \(\nabla \omega_J = 0\), we get the required relation. □

**Proposition 4.3.** Under the hypothesis above,

1. \((M, \nabla, g, J)\) is weak s.s.m. if and only if \((M, \nabla, g, J^*)\) is weak s.s.m..
2. \((M, \nabla^*, g, J)\) is weak s.s.m. if and only if \((M, \nabla^*, g, J^*)\) is weak s.s.m..

**Proof.** It is a consequence of the previous relations. □

4.2. Strong special statistical manifold

**Definition 4.4.** We say that the subtangent-like statistical manifold \((M, \nabla, g, J)\) is strong s.s.m. if the 2-form \(\omega_{JJ}(X, Y) := g(X, JY) - g(JX, Y)\) is \(\nabla\)-parallel.

A necessary and sufficient condition for a subtangent-like statistical manifold to be strong s.s.m. is given in the following proposition

**Proposition 4.5.** The subtangent-like statistical manifold \((M, \nabla, g, J)\) is strong s.s.m. if and only if

\[
(\nabla_X g) \circ (J \times I) - (\nabla_X g) \circ (J \times I) = g \circ (\nabla_X J \times I) - g \circ (I \times \nabla_X J),
\]

**Proof.** Follows from Proposition 4.2. □

Consider \((M, \nabla, g, J)\) an almost subtangent-like statistical manifold, \(\nabla^*\) the conjugate connection of \(\nabla\) and \(J^*\) a conjugate almost subtangent structure of \(J\) and define

\[
\omega_{JJ}(X, Y) := g(X, JY) - g(JX, Y), \\
\omega_{J^* J}(X, Y) := g(X, J^* Y) - g(JX, Y), \\
\omega_{J^* J^*}(X, Y) := g(X, J^* Y) - g(J^* X, Y).
\]

A simple computation shows that on a subtangent-like statistical manifold \((M, \nabla, g, J)\), the relation between \(\omega_J\), \(\omega_{J^*}\) and the 2-forms are defined above

\[
\omega_{JJ}(X, Y) = \omega_J(X, Y) - \omega_J(Y, X),
\]

\[
\omega_{J^* J} = 2\omega_J, \quad \omega_{J^* J} = 2\omega_{J^*}, \quad \omega_{J^* J} = -\omega_{JJ}
\]
and

\[
(\nabla_X \omega_{J^*})(Y, Z) = -(\nabla_X \omega_{J^*})(Z, Y), \\
(\nabla_X \omega_{J^*})(Y, Z) = -(\nabla_X \omega_{J^*})(Z, Y), \\
(\nabla_X \omega_{J^*})(Y, Z) + (\nabla_X \omega_{J^*})(Y, Z) = -g((\nabla_X J^*)Y + (\nabla_X J^*)Y, Z)
\]

for any \(X, Y, Z \in \mathfrak{X}(M)\).

**Proposition 4.6.** Under the hypothesis above,
1. \((M, \nabla, g, J)\) is strong s.s.m. if and only if \((M, \nabla, g, J^*)\) is strong s.s.m.
2. \((M, \nabla^*, g, J)\) is strong s.s.m. if and only if \((M, \nabla^*, g, J^*)\) is strong s.s.m.

**Proof.** It is a consequence of the fact that \(\omega_{J^*} = -\omega_{J^*}\). \(\square\)

Remark that the notion of weak s.s.m. implies the strong s.s.m., but conversely it’s not always true. The notions are equivalent if and only if

\[
(\nabla_X \omega_{J})(Y, Z) = (\nabla_X \omega_{J})(Z, Y),
\]

for any \(X, Y, Z \in \mathfrak{X}(M)\).

Now using the 2-form \(\omega_{J^*}\), we can associate to any affine connection \(\nabla\) another affine connection \(\nabla^*\) such that \(\omega_{J^*}\) is \(\nabla^*\)-parallel

\[
(\nabla^*_X \omega_{J})(Y, Z) = (\nabla^*_X \omega_{J})(Z, Y) + \frac{1}{2}((\nabla^*_X \omega_{J})(Y, Z) + (\nabla^*_X \omega_{J})(Z, Y)) = 0.
\]

Therefore, this procedure gives a way to construct strong special statistical manifolds.

**Theorem 4.7.** Let \((M, \nabla, g, J)\) be an almost subtangent-like statistical manifold and \(J^*\) a conjugate almost subtangent structure of \(J\). If \((\nabla J^*_X g)(Y, Z) = g((\nabla Y J^*)X, Z)\) for any \(X, Y, Z \in \mathfrak{X}(M)\), where \(J := J + J^*\), then the affine connection \(\nabla^*\) defined by relation (4.1) is the conjugate connection of \(\nabla\). In this case, \((M, \nabla^*, g, J)\) is a strong special statistical manifold.

**Proof.** Replacing \(\omega_{J^*}\) in (4.1) and considering the symmetry of \(\nabla g\), we obtain the required relation. \(\square\)

5. **Projectively equivalent statistical manifolds**

Geometrically, two torsion-free affine connections are projectively equivalent if they have the same geodesics as unparameterized curves. Thus, they determine a class of equivalence on a given manifold called the *projective structure*.

We say that two affine connections \(\nabla\) and \(\nabla^*\) on \(M\) are

1. **projectively equivalent** if there exists a 1-form \(\eta\) on \(M\) such that

\[
\nabla_X Y = \nabla_X Y + \eta(X)Y + \eta(Y)X
\]

for any \(X, Y \in \mathfrak{X}(M)\).
2. dual-projectively equivalent if there exists a 1-form $\eta$ on $M$ such that
\begin{equation}
\nabla_X^* Y = \nabla_X Y - g(X,Y)\eta^{\sharp},
\end{equation}
for any $X, Y \in \mathfrak{X}(M)$, where $g(\eta^{\sharp}, X) = \eta(X), X \in \mathfrak{X}(M)$.

Note that if two connections are projectively equivalent or dual-projectively equivalent, their conjugate connections may not be projectively or dual-projectively equivalent, respectively.

**Proposition 5.1.** If $(M, \nabla, g)$ is a statistical manifold and $\nabla^*$ is the conjugate connection of $\nabla$, then $\nabla$ and $\nabla^*$ are $\eta$-projective equivalent or $\eta$-dual-projective equivalent if and only if $\eta \otimes I = I \otimes \eta$.

**Proof.** Replacing the expression of $\nabla^*$ in 1.1 and taking into account that $\nabla g$ is symmetric, we obtain the required relation. Q.E.D.

Remark that if $\eta \otimes I = I \otimes \eta$, then for any endomorphism $J$ of the tangent bundle, $\eta \otimes J = (\eta \circ J) \otimes I$ and $J \otimes \eta = I \otimes (\eta \circ J)$.

**Proposition 5.2.** Let $(M, \nabla, g)$ be a statistical manifold and $\nabla^*$ the conjugate connection of $\nabla$. Consider $\omega$ a 2-form on $M$. 

1. If $\nabla$ and $\nabla^*$ are $\eta$-projective equivalent, then $\nabla \omega = \nabla^* \omega + 4 \eta \otimes \omega$.

2. If $\nabla$ and $\nabla^*$ are $\eta$-dual-projective equivalent, then
\begin{equation}
(\nabla_X \omega)(Y, Z) = (\nabla_X^* \omega)(Y, Z) - g(X,Y)\omega(\eta^{\sharp}, Z) - g(X,Z)\omega(Y, \eta^{\sharp})
\end{equation}
for any $X, Y, Z \in \mathfrak{X}(M)$.

**Proof.** From the previous proposition. Q.E.D.

**Corollary 5.3.** Let $(M, \nabla, g, J)$ be an almost subtangent-like statistical manifold and $\nabla^*$ the conjugate connection of $\nabla$.

1. If $\nabla$ and $\nabla^*$ are $\eta$-projective equivalent, then $\nabla J = \nabla^* J$ and $\nabla \eta = \nabla^* \eta + 2\eta \otimes \eta$.

2. If $\nabla$ and $\nabla^*$ are $\eta$-dual-projective equivalent, then
\begin{equation}
(\nabla_X J)Y = (\nabla_X^* J)Y - g(X,Y)J(\eta^{\sharp}) + g(X,Y)\eta^{\sharp}
\end{equation}
and
\begin{equation}
(\nabla_X \eta)Y = (\nabla_X^* \eta)Y - g(X,Y)\eta(\eta^{\sharp})
\end{equation}
for any $X, Y \in \mathfrak{X}(M)$.

**Proof.** From the previous propositions. Q.E.D.

**Corollary 5.4.** Let $(M, \nabla, g, J)$ be a special almost subtangent-like statistical manifold, $\nabla^*$ the conjugate connection of $\nabla$ and $\omega_J$ the 2-form defined by $(g, J)$.

1. If $\nabla$ and $\nabla^*$ are $\eta$-projective equivalent, then $\nabla J = \nabla^* J$, $\nabla \eta = \nabla^* \eta + 2\eta \otimes \eta$ and $0 = \nabla^* \omega_J + 4 \eta \otimes \omega$. 

2. If $\nabla$ and $\nabla^*$ are $\eta$-dual-projective equivalent, then
\[
(\nabla_X J)Y = (\nabla_X^* J)Y - g(X,Y)J(\eta^\perp) + g(X,JY)\eta^\perp,
\]
\[
(\nabla_X \eta)Y = (\nabla_X^* \eta)Y - g(X,Y)\eta(\eta^\perp),
\]
\[
0 = (\nabla_X^* \omega)(Y,Z) - g(X,JY)\eta(Z) - g(JX,Z)\eta(Y)
\]
for any $X, Y, Z \in \mathfrak{X}(M)$.

Proof. From the previous propositions and from the fact that $\eta \otimes J = (\eta \circ J^*) \otimes I$ and $J \otimes \eta = I \otimes (\eta \circ J)$. \qed

Remark 5.1. To the class of pairs $(\nabla, \nabla^*)$ which are solutions of the nonlinear system
\begin{align*}
\nabla J &= \nabla^* J \\
\nabla \eta &= \nabla^* \eta + 2\eta \otimes \eta \\
\nabla \omega &= \nabla^* \omega + 4\eta \otimes \omega
\end{align*}
(5.3)
for $J, \eta$ and $\omega$ apriori given, for $\omega_{J}(X,Y) = g(X,JY)$, the $\eta$-projective equivalent conjugate connections on the special almost subtangent-like statistical manifold $(M, \nabla, g, J)$ belong.

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