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SOME GENERALIZED INTEGRAL INEQUALITIES VIA FRACTIONAL INTEGRALS

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Abstract. The main goal of this paper is to introduce a new integral definition concerned with fractional calculus. Then we establish generalized Hermite-Hadamard type integral inequalities for convex function using proposed fractional integrals. The results presented in this paper provide extensions of those given in earlier works.

1. Introduction & Preliminaries

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very significant in the literature (see, e.g., [**11**, p. 137], [**5**]). These inequalities state that if $f\colon I\to\mathbb{R}$ is a convex function on the interval I of real numbers and $a,b\in I$ with $a<b$, then

$$f\Big(\frac{a+b}{2}\Big)\leq\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\leq\frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been studied (see, for example, [**1, 2, 5, 6, 11, 15, 16**]).

On the other hand, a number of mathematicians have studied the fractional integral inequalities and their applications using Riemann-Liouville fractional integrals. For results connected with Hermite-Hadamard type inequalities involving fractional integrals, one can see [**3, 4, 8, 12, 13, 14, 17, 18, 19**]. In the following, we present a brief synopsis of all necessary definitions and results that are required. More details, one can consult [**7, 9, 10**].

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Definition 1.1. Let $f \in L_1[a,b]$. The Riemann-Liouville fractional integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, \qquad x > a,$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\mathrm{d}t, \qquad x < b,$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$.

In [**13**], Sarikaya et al. proved a variant of Hermite-Hadamard's inequalities in Riemann-Liouville fractional integral forms as follows

Theorem 1.2. *Let $f\colon [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold*

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{2}$$

with $\alpha > 0$.

Remark. For $\alpha = 1$, inequality (2) reduces to inequality (1).

Meanwhile, Sarikaya et al. [**13**] presented the following important integral identity including the first-order derivative of f to establish many interesting Hermite-Hadamard-type inequalities for convex functions via Riemann-Liouville fractional integrals of the order $\alpha > 0$.

Lemma 1.3. *Let $f\colon [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L_1[a,b]$, then the following equality for fractional integrals holds:*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \\ &= \frac{b-a}{2} \int_0^1 \left[(1-t)^{\alpha} - t^{\alpha}\right] f'\left(ta + (1-t)b\right) \mathrm{d}t. \end{aligned}$$

Using this Lemma, the authors obtained the following fractional integral inequality in [**13**]

Theorem 1.4. *Let $f\colon [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality for fractional integrals holds*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right]\right| \\ &\leq \frac{b-a}{2\left(\alpha+1\right)} \Big(1 - \frac{1}{2^{\alpha}}\Big) \left[f'(a) + f'(b)\right]. \end{aligned} \tag{3}$$

The aim of this paper is to present a new definition concerned with fractional integrals and establish generalized Hermite-Hadamard type integral inequalities for convex function involving this fractional integrals.

2. Main Findings & Cumulative Results

In order to obtain our results, let us start with some notations given in [**8**]. Let $f\colon I^\circ \to \mathbb{R}$ be a function such that $a, b \in I^\circ$ and $0 < a < b < \infty$. Throughout this article, we suppose that $F(x) = f(x) + \widetilde{f}(x)$ and $\widetilde{f}(x) = f(a+b-x)$ for $x \in [a, b]$. Then it is easy to show that if f is a convex function, then F is also a convex function.

We are now give a new generalized definitions concerned with fractional integrals as follows

Definition 2.1. Let $u\colon [a, b] \to \mathbb{R}$ be an increasing and positive monotone function on (a, b) and $f, u \in L[a, b]$ with $a < b$. The generalized Riemann-Liouville fractional integrals $J^{\alpha,k}_{a+,u}f$ and $J^{\alpha,k}_{b-,u}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha,k}_{a+,u}\left(f\right)\left(x\right) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\left(u(x)-u(t)\right)^k f(t)\mathrm{d}t, \qquad x > a,$$

and

$$J^{\alpha,k}_{b-,u}\left(f\right)\left(x\right) = \frac{1}{\Gamma(\alpha)}\int_x^b (x-t)^{\alpha-1}\left(u(t)-u(x)\right)^k f(t)\mathrm{d}t, \qquad x < b,$$

provided that the integrals exist, respectively, $k \in N \cup \{0\}$.

Example 1. If we choose $u(t) = t$ and $f(t) = 1$, it follows that

$$J^{\alpha,k}_{a+,t}(1)(x) = \frac{(x-a)^{\alpha+k}}{(\alpha+k)\Gamma(\alpha)} \tag{4}$$

and

$$J^{\alpha,k}_{b-,t}(1)(x) = \frac{(b-x)^{\alpha+k}}{(\alpha+k)\Gamma(\alpha)}. \tag{5}$$

First, we present a new Hermite-Hadamard type of inequalities for new generalized fractional integrals in the following theorem

Theorem 2.2. *Let $f\colon [a, b] \to \mathbb{R}$ be a convex function on $[a, b]$ and $u\colon [a, b] \to \mathbb{R}$ be an increasing and positive monotone function on (a, b), and $f, u \in L[a, b]$ with $a < b$. Then F is also integrable and the following inequalities for fractional integral operators hold*

$$\begin{aligned} & f\Big(\frac{a+b}{2}\Big)\left[J^{\alpha,k}_{a+,u}(1)(b) + J^{\alpha,k}_{b^-,u}(1)(a)\right] \\ & \leq \frac{1}{2}\left[J^{\alpha,k}_{a+,u}\left(F\right)\left(b\right) + J^{\alpha,k}_{b^-,u}\left(F\right)\left(a\right)\right] \\ & \leq \left[J^{\alpha,k}_{a+,u}(1)(b) + J^{\alpha,k}_{b^-,u}(1)(a)\right]\frac{f(a)+f(b)}{2} \end{aligned} \tag{6}$$

with $\alpha > 0$ and $k \in N \cup \{0\}$.

Proof. Since f is an convex mapping on $[a,b]$, we have

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \tag{7}$$

for $x,y \in [a,b]$. Now, for $t \in [0,1]$, let $x = ta+(1-t)b$ and $y = (1-t)a+tb$. Then we find that

$$2f\left(\frac{a+b}{2}\right) \leq f\left(ta+(1-t)b\right) + f\left((1-t)a+tb\right). \tag{8}$$

Then multiplying both sides of (8) by $t^{\alpha-1}\left(u(b)-u\left(ta+(1-t)b\right)\right)^{k}$ and integrating the resulting inequality with respect to t over $[0,1]$, we deduce that

$$\begin{aligned}
&2f\left(\frac{a+b}{2}\right)\int_0^1 t^{\alpha-1}\left(u(b)-u\left(ta+(1-t)b\right)\right)^{k}\mathrm{d}t \\
&\leq \int_0^1 t^{\alpha-1}\left(u(b)-u\left(ta+(1-t)b\right)\right)^{k} f\left(ta+(1-t)b\right)\mathrm{d}t \\
&\quad + \int_0^1 t^{\alpha-1}\left(u(b)-u\left(ta+(1-t)b\right)\right)^{k} f\left((1-t)a+tb\right)\mathrm{d}t.
\end{aligned}$$

Using the change of variable $y = ta+(1-t)b$, we have

$$2f\left(\frac{a+b}{2}\right)J_{a+}^{\alpha,k}(1)(b) \leq J_{a+}^{\alpha,k}\left(\widetilde{f}\right)(b) + J_{a+}^{\alpha,k}\left(f\right)(b),$$

i.e.,

$$2f\left(\frac{a+b}{2}\right)J_{a+}^{\alpha,k}(1)(b) \leq J_{a+}^{\alpha,k}\left(F\right)(b). \tag{9}$$

Similarly, multiplying both sides of (8) by $t^{\alpha-1}\left(u\left((1-t)a+tb\right)-u(a)\right)^{k}$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}
&2f\left(\frac{a+b}{2}\right)\int_0^1 t^{\alpha-1}\left(u\left((1-t)a+tb\right)-u(a)\right)^{k}\mathrm{d}t \\
&\leq \int_0^1 t^{\alpha-1}\left(u\left((1-t)a+tb\right)-u(a)\right)^{k} f\left(ta+(1-t)b\right)\mathrm{d}t \\
&\quad + \int_0^1 t^{\alpha-1}\left(u\left((1-t)a+tb\right)-u(a)\right)^{k} f\left((1-t)a+tb\right)\mathrm{d}t.
\end{aligned}$$

Using the change of variable $y = (1-t)a+tb$, we have

$$2f\left(\frac{a+b}{2}\right)J_{b^-,u}^{\alpha,k}(1)(a) \leq J_{b^-,u}^{\alpha,k}\left(F\right)(a). \tag{10}$$

Summing the inequalities (9) and (10), we get

$$f\Big(\frac{a+b}{2}\Big)\left[J^{\alpha,k}_{a+,u}(1)(b)+J^{\alpha,k}_{b^-,u}(1)(a)\right]\le\frac{1}{2}\left[J^{\alpha,k}_{a+,u}(F)(b)+J^{\alpha,k}_{b^-,u}(F)(a)\right].$$

This completes the proof of first inequality in (6).

For the proof of the second inequality in (6), since f is convex, we have

$$f(ta+(1-t)b)+f((1-t)a+tb)\le[f(a)+f(b)]. \tag{11}$$

Multiplying both sides of (11) by $t^{\alpha-1}(u(b)-u(ta+(1-t)b))^k$ and integrating the resulting inequality with respect to t over $[0,1]$, we have

$$\begin{aligned}\int_0^1 & t^{\alpha-1}(u(b)-u(ta+(1-t)b))^k f(ta+(1-t)b)\,\mathrm{d}t\\ &+\int_0^1 t^{\alpha-1}(u(b)-u(ta+(1-t)b))^k f((1-t)a+tb)\,\mathrm{d}t\\ \le&[f(a)+f(b)]\int_0^1 t^{\alpha-1}(u(b)-u(ta+(1-t)b))^k\,\mathrm{d}t.\end{aligned}$$

Then, we get

$$J^{\alpha,k}_{a+,u}(F)(b)\le[f(a)+f(b)]\,J^{\alpha,k}_{a+,u}(1)(b). \tag{12}$$

Similarly, multiplying both sides of (11) by $t^{\alpha-1}(u((1-t)a+tb)-u(a))^k$ and integrating the resulting inequality with respect to t over $[0,1]$, we get

$$J^{\alpha,k}_{b^-,u}(F)(a)\le[f(a)+f(b)]\,J^{\alpha,k}_{b^-,u}(1)(a). \tag{13}$$

By adding the inequalities (12) and (13), we have

$$\frac{1}{2}\left[J^{\alpha,k}_{a+,u}(F)(b)+J^{\alpha,k}_{b^-,u}(F)(a)\right]\le\left[J^{\alpha,k}_{a+,u}(1)(b)+J^{\alpha,k}_{b^-,u}(1)(a)\right]\frac{f(a)+f(b)}{2},$$

which completes the proof. □

Remark. If we choose $k=0$ in Theorem 2.2, then the inequality (6) reduces to inequality (2).

Corollary 2.3. *If we choose $u(t)=t$ in* Theorem 2.2, *then we have the inequality for Riemann-Liouville fractional integrals*

$$\begin{aligned}&f\Big(\frac{a+b}{2}\Big)\left[\frac{(x-a)^{\alpha+k}+(b-x)^{\alpha+k}}{(\alpha+k)\Gamma(\alpha)}\right]\\ &\le\frac{\Gamma(\alpha+k)}{2\Gamma(\alpha)}\left[J^{\alpha+k}_{a+}(F)(b)+J^{\alpha+k}_{b^-}(F)(a)\right]\\ &\le\left[\frac{(x-a)^{\alpha+k}+(b-x)^{\alpha+k}}{(\alpha+k)\Gamma(\alpha)}\right]\frac{f(a)+f(b)}{2}.\end{aligned}$$

Proof. From Definition 2.1 with $u(t) = t$, we have

$$J_{a+,t}^{\alpha,k}(F)(b) = \frac{1}{\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} (x-t)^k F(t)\mathrm{d}t = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} J_{a+}^{\alpha+k}(F)(b) \tag{14}$$

and similarly,

$$J_{b-,t}^{\alpha,k}(F)(a) = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} J_{b-}^{\alpha+k}(F)(b). \tag{15}$$

Using the equalities (4), (5), (14), and (15), we obtain the desired result. □

Now, we give an important identity for new generalized fractional integrals in the following theorem

Lemma 2.4. *Let $f\colon [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) and $u\colon [a,b] \to \mathbb{R}$ be an increasing and positive monotone function on (a,b) with $a < b$. If $f', u \in L[a,b]$, then F is also differentiable and $F \in L[a,b]$, and the following equality holds*

$$\begin{aligned} &\left[J_{a+,u}^{\alpha,k}(1)(b) + J_{b^-,u}^{\alpha,k}(1)(a)\right] \frac{f(a)+f(b)}{2} - \frac{1}{2}\left[J_{a+,u}^{\alpha,k}(F)(b) + J_{b^-,u}^{\alpha,k}(F)(a)\right] \\ &= \frac{(b-a)^\alpha}{2\Gamma(\alpha)} \int_a^b G(y)F'(y)\mathrm{d}y, \end{aligned} \tag{16}$$

where $F'(y) = f'(y) - f'(a+b-y)$ and

$$\begin{aligned} G(y) = &\left[\int_0^{\frac{y-a}{b-a}} s^{\alpha-1}\left(u(b) - u\left(sa + (1-s)b\right)\right)^k \mathrm{d}s\right] \\ &+ \left[\int_{\frac{y-a}{b-a}}^1 (1-s)^{\alpha-1}\left(u(sa+(1-s)b) - u(a)\right)^k \mathrm{d}s\right]. \end{aligned} \tag{17}$$

Proof. Integrating by parts, we have

$$\begin{aligned} I_1 &= \int_0^1 \left[\int_0^t s^{\alpha-1}\left(u(b) - u\left(sa+(1-s)b\right)\right)^k \mathrm{d}s\right] f'\left((1-t)a+tb\right)\mathrm{d}t \\ &= \frac{\Gamma(\alpha)}{(b-a)^{\alpha+1}} J_{a+}^{\alpha,k}(1)(b)f(b) - \frac{\Gamma(\alpha)}{(b-a)^{\alpha+1}} J_{a+}^{\alpha,k}\big(\widetilde{f}\big)(b). \end{aligned}$$

Similarly, using the integration by parts,

$$\begin{aligned} I_2 &= \int_0^1 \left[\int_0^t s^{\alpha-1}\left(u(b) - u\left(sa+(1-s)b\right)\right)^k \mathrm{d}s\right] f'\left(ta+(1-t)b\right)\mathrm{d}t \\ &= -\frac{\Gamma(\alpha)}{(b-a)^{\alpha+1}} J_{a+,u}^{\alpha,k}(1)(b)f(a) + \frac{\Gamma(\alpha)}{(b-a)^{\alpha+1}} J_{a+,u}^{\alpha,k}(f)(b), \end{aligned}$$

$$I_3 = \int_0^1 \left[\int_t^1 (1-s)^{\alpha-1} \left(u(sa+(1-s)b) - u(a)\right)^k \mathrm{d}s \right] f'\left(at+(1-t)b\right) \mathrm{d}t$$
$$= \frac{\Gamma(\alpha)}{(b-a)^{\alpha+1}} J^{\alpha,k}_{b^-,u}(1)(a) f(a) - \frac{\Gamma(\alpha)}{(b-a)^{\alpha+1}} J^{\alpha,k}_{b^-,u}\left(f\right)(a),$$

and

$$I_4 = \int_0^1 \left[\int_t^1 (1-s)^{\alpha-1} \left(u((sa+(1-s)b) - u(a)\right)^k \mathrm{d}s \right] f'\left((1-t)a+tb\right) \mathrm{d}t$$
$$= -\frac{\Gamma(\alpha)}{(b-a)^{\alpha+1}} J^{\alpha,k}_{b^-,u}(1)(a) f(b) + \frac{\Gamma(\alpha)}{(b-a)^{\alpha+1}} J^{\alpha,k}_{b^-,u}\left(\widetilde{f}\right)(a).$$

Then it follows that

$$I_1 - I_2 + I_3 - I_4 = \frac{\Gamma(\alpha)}{(b-a)^{\alpha+1}} \left[J^{\alpha,k}_{a+,u}(1)(b) + J^{\alpha,k}_{b^-,u}(1)(a) \right] \left[f(a)+f(b) \right] - \frac{\Gamma(\alpha)}{(b-a)^{\alpha+1}} \left[J^{\alpha,k}_{a+,u}\left(F\right)(b) + J^{\alpha,k}_{b^-,u}\left(F\right)(a) \right]. \tag{18}$$

If we multiply both sides of (18) by $\frac{(b-a)^{\alpha+1}}{2\Gamma(\alpha)}$, then we obtain the following result

$$\begin{aligned} &\left[J^{\alpha,k}_{a+,u}(1)(b) + J^{\alpha,k}_{b^-,u}(1)(a) \right] \frac{f(a)+f(b)}{2} - \frac{1}{2} \left[J^{\alpha,k}_{a+,u}\left(F\right)(b) + J^{\alpha,k}_{b^-,u}\left(F\right)(a) \right] \\ &= \frac{(b-a)^{\alpha+1}}{2\Gamma(\alpha)} \Bigg\{ \int_0^1 \left[\int_0^t s^{\alpha-1} \left(u(b) - u\left(sa+(1-s)b\right)\right)^k \mathrm{d}s \right] \\ &\qquad \times \left[f'\left((1-t)a+tb\right) - f'\left(at+(1-t)b\right) \right] \mathrm{d}t \\ &\quad + \int_0^1 \left[\int_t^1 (1-s)^{\alpha-1} \left(u(sa+(1-s)b) - u(a)\right)^k \mathrm{d}s \right] \\ &\qquad \times \left[f'\left((1-t)a+tb\right) - f'\left(at+(1-t)b\right) \right] \mathrm{d}t \Bigg\}. \end{aligned} \tag{19}$$

Using the change of variable $y = (1-t)a + tb$ in (19), since

$$G(y) = \left[\int_0^{\frac{y-a}{b-a}} s^{\alpha-1} \left(u(b) - u\left(sa+(1-s)b\right)\right)^k \mathrm{d}s \right] + \left[\int_{\frac{y-a}{b-a}}^1 (1-s)^{\alpha-1} \left(u(sa+(1-s)b) - u(a)\right)^k \mathrm{d}s \right],$$

then it follows that

$$\left[J^{\alpha,k}_{a+,u}(1)(b) + J^{\alpha,k}_{b^-,u}(1)(a)\right]\frac{f(a)+f(b)}{2} - \frac{1}{2}\left[J^{\alpha,k}_{a+,u}(F)(b) + J^{\alpha,k}_{b^-,u}(F)(a)\right]$$
$$= \frac{(b-a)^\alpha}{2\Gamma(\alpha)}\int_a^b G(y)F'(y)\mathrm{d}y.$$

Thus the desired equality (16) has been obtained. □

Theorem 2.5. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b), $u:[a,b]\to\mathbb{R}$ be an increasing and positive monotone function on (a,b), and $f', u \in L[a,b]$ with $a<b$. Then F is also differentiable and $F\in L[a,b]$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds*

$$\begin{aligned}(20)\qquad &\left|\left[J^{\alpha,k}_{a+,u}(1)(b) + J^{\alpha,k}_{b^-,u}(1)(a)\right]\frac{f(a)+f(b)}{2(b-a)^\alpha}\right. \\ &\left.-\frac{1}{2(b-a)^\alpha}\left[J^{\alpha,k}_{a+,u}(F)(b) + J^{\alpha,k}_{b^-,u}(F)(a)\right]\right| \\ &\le \frac{1}{2\Gamma(\alpha)}\int_a^b |G(y)|\,\mathrm{d}y\left(|f'(a)|+|f'(b)|\right).\end{aligned}$$

Proof. Notice that $F'(y)=f'(y)-f'(a+b-y)$. By the convexity of $|f'|$, it follows

$$\begin{aligned}(21)\qquad |F'(y)| &= \left|f'\left(\frac{b-y}{b-a}a+\frac{y-a}{b-a}b\right) - f'\left(\frac{b-y}{b-a}b+\frac{y-a}{b-a}a\right)\right| \\ &\le |f'(a)|+|f'(b)|.\end{aligned}$$

By inequalities (16) and (21), we obtain

$$\left|\left[J^{\alpha,k}_{a+,u}(1)(b) + J^{\alpha,k}_{b^-,u}(1)(a)\right]\frac{f(a)+f(b)}{2} - \frac{1}{2}\left[J^{\alpha,k}_{a+,u}(F)(b) + J^{\alpha,k}_{b^-,u}(F)(a)\right]\right|$$
$$\le \frac{(b-a)^\alpha}{2\Gamma(\alpha)}\int_a^b |G(y)|\,|F'(y)|\,\mathrm{d}y$$
$$\le \frac{(b-a)^\alpha}{2\Gamma(\alpha)}\int_a^b |G(y)|\,\mathrm{d}y\left(|f'(a)|+|f'(b)|\right).$$

Thus the proof is completed. □

Remark. If we choose $k=0$ in Theorem 2.5, then

$$G(y)=\frac{[(y-a)^\alpha-(b-y)^\alpha]}{\alpha(b-a)^\alpha}.$$

Thus, the ineuality (20) reduces to the ineuality (3).

Corollary 2.6. *If we choose* $k = 1$ *in* Theorem 2.5, $|u|$ *is convex on* $[a, b]$, *then we have the inequality*

$$\left|\left[J^{\alpha,1}_{a+,u}(1)(b) + J^{\alpha,1}_{b-,u}(1)(a)\right]\frac{f(a)+f(b)}{2(b-a)^{\alpha}} - \frac{1}{2(b-a)^{\alpha}}\left[J^{\alpha,1}_{a+,u}(F)(b) + J^{\alpha,1}_{b-,u}(F)(a)\right]\right|$$
$$\leq \frac{(b-a)}{\Gamma(\alpha+2)}\left(|u(a)| + |u(b)|\right)\left(|f'(a)| + |f'(b)|\right).$$

Proof. Using (17) with $k = 1$, we have

$$\begin{aligned} G(y) = & \int_0^{\frac{y-a}{b-a}} s^{\alpha-1}\left(u(b) - u\left(sa + (1-s)b\right)\right) \mathrm{d}s \\ & + \int_{\frac{y-a}{b-a}}^1 (1-s)^{\alpha-1}\left(u(sa + (1-s)b) - u(a)\right) \mathrm{d}s \\ = & \; u(b)\int_0^{\frac{y-a}{b-a}} s^{\alpha-1}\mathrm{d}s - u(a)\int_{\frac{y-a}{b-a}}^1 (1-s)^{\alpha-1}\mathrm{d}s \\ & - \int_0^{\frac{y-a}{b-a}} s^{\alpha-1}u\left(sa + (1-s)b\right)\mathrm{d}s + \int_{\frac{y-a}{b-a}}^1 (1-s)^{\alpha-1}u(sa + (1-s)b)\mathrm{d}s. \end{aligned}$$

By using the convexity of $|u|$, we obtain

$$\begin{aligned} |G(y)| \leq & \frac{|u(b)|(y-a)^{\alpha} + |u(a)|(b-y)^{\alpha}}{\alpha(b-a)^{\alpha}} + \int_0^{\frac{y-a}{b-a}} s^{\alpha-1}\left|u\left(sa + (1-s)b\right)\right| \mathrm{d}s \\ & + \int_{\frac{y-a}{b-a}}^1 (1-s)^{\alpha-1}\left|u(sa + (1-s)b)\right| \mathrm{d}s \\ \leq & \frac{|u(b)|(y-a)^{\alpha} + |u(a)|(b-y)^{\alpha}}{\alpha(b-a)^{\alpha}} + \int_0^{\frac{y-a}{b-a}} s^{\alpha-1}\left[s|u(a)| + (1-s)|u(b)|\right] \mathrm{d}s \\ & + \int_{\frac{y-a}{b-a}}^1 (1-s)^{\alpha-1}\left[s|u(a)| + (1-s)|u(b)|\right] \mathrm{d}s \end{aligned}$$

$$= \frac{|u(b)|(y-a)^{\alpha}+|u(a)|(b-y)^{\alpha}}{\alpha(b-a)^{\alpha}}+|u(a)|\left[\frac{(y-a)^{\alpha+1}-(b-y)^{\alpha+1}}{(\alpha+1)(b-a)^{\alpha+1}}+\frac{(b-y)^{\alpha}}{\alpha(b-a)^{\alpha}}\right]$$
$$+|u(b)|\left[\frac{-(y-a)^{\alpha+1}+(b-y)^{\alpha+1}}{(\alpha+1)(b-a)^{\alpha+1}}+\frac{(y-a)^{\alpha}}{\alpha(b-a)^{\alpha}}\right]$$
$$= |u(a)|\left[\frac{(y-a)^{\alpha+1}-(b-y)^{\alpha+1}}{(\alpha+1)(b-a)^{\alpha+1}}+\frac{2(b-y)^{\alpha}}{\alpha(b-a)^{\alpha}}\right]$$
$$+|u(b)|\left[\frac{-(y-a)^{\alpha+1}+(b-y)^{\alpha+1}}{(\alpha+1)(b-a)^{\alpha+1}}+\frac{2(y-a)^{\alpha}}{\alpha(b-a)^{\alpha}}\right].$$

Thus, we have the inequality

$$\frac{1}{2\Gamma(\alpha)}\int_a^b |G(y)|\,\mathrm{d}y \le \frac{(b-a)}{\Gamma(\alpha+2)}\left(|u(a)|+|u(b)|\right),$$

which completes the proof. □

Corollary 2.7. *If we choose* $u(t)=t$ *in* Theorem 2.5, *then we have the inequality for Riemann-Liouville fractional integrals,*

$$\left|\left[\frac{(x-a)^{\alpha+k}+(b-x)^{\alpha+k}}{(\alpha+k)\Gamma(\alpha)}\right]\frac{f(a)+f(b)}{2(b-a)^{\alpha}} - \frac{\Gamma(\alpha+k)}{2\Gamma(\alpha)(b-a)^{\alpha}}\left[J_{a+}^{\alpha+k}(F)(b)+J_{b-}^{\alpha+k}(F)(a)\right]\right|$$
$$\le \frac{(b-a)^{k+1}}{(\alpha+k)(\alpha+k+1)\Gamma(\alpha)}\left(|f'(a)|+|f'(b)|\right).$$

Proof. Using (17) with $u(t)=t$, we have

$$|G(y)| = (b-a)^k \int_0^{\frac{y-a}{b-a}} s^{\alpha+k-1}\mathrm{d}s + \int_{\frac{y-a}{b-a}}^1 (1-s)^{\alpha+k-1}\mathrm{d}s$$
$$= \frac{(y-a)^{\alpha+k}+(b-y)^{\alpha+k}}{(\alpha+k)(b-a)^{\alpha}}.$$

Then it follows that

$$\frac{1}{2\Gamma(\alpha)}\int_a^b |G(y)|\,\mathrm{d}y\left(|f'(a)|+|f'(b)|\right)$$
$$= \frac{(b-a)^{k+1}}{(\alpha+k)(\alpha+k+1)\Gamma(\alpha)}\left(|f'(a)|+|f'(b)|\right). \tag{22}$$

This completes the proof. □

3. Concluding remarks

In this study, a new definitions for fractional integrals have been proposed and tested. Then the introduced fractional integral operators have been applied to Hermite Hadamard type integral inequalities to validate their generalized properties.

References

1. Azpeitia A. G., *Convex functions and the Hadamard inequality*, Rev. Colombiana Math. **28** (1994), 7–12.

2. Bakula M. K. and Pečarić J., *Note on some Hadamard-type inequalities,* J. Inequal. Pure Appl. Math. **5**(3) (2004), Art. ID 74.

3. Dahmani Z., *On Minkowski and Hermite-Hadamard integral inequalities via fractional integration*, Ann. Funct. Anal. **1**(1) (2010), 51–58.

4. Deng J. and Wang J., *Fractional Hermite-Hadamard inequalities for (α, m)-logarithmically convex functions*, J. Inequal. Appl. **2013** (2013), Art. ID 364.

5. Dragomir S. S. and Pearce C. E. M., *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.

6. Dragomir S. S. and Agarwal R. P., *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. lett. **11**(5) (1998), 91–95.

7. Gorenflo R. and Mainardi F., *Fractional Calculus: Integral and Differential Equations of Fractional Order*. Springer Verlag, Wien (1997), 223–276.

8. Jleli M. and Samet B., *On Hermite-Hadamard type inequalities via fractional integrals of a function with respect to another function*, J. Nonlinear Sci. Appl. **9** (2016), 1252–1260.

9. Kilbas A. A., Srivastava H. M. and Trujillo J. J., *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Sci. B.V., Amsterdam, 2006.

10. Miller S. and Ross B., *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, USA, 1993.

11. Pečarić J. E., Proschan F. and Tong Y. L., *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, Boston, 1992.

12. Sarıkaya M. Z. and Ögunmez H., *On new inequalities via Riemann-Liouville fractional integration*, Abstr. Appl. Anal. **2012** (2012), Art. ID 428983.

13. Sarıkaya M. Z., Set E., Yaldiz H. and Başak N., *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*, Math. Comput. Model. **57** (2013), 2403–2407.

14. Sarıkaya M. Z., Filiz H. and Kiriş M. E., *On some generalized integral inequalities for Riemann-Liouville fractional integrals*, Filomat **29**6 (2015), 1307–1314.

15. Sarıkaya M. Z. and Budak H., *Generalized Hermite-Hadamard type integral inequalities for functions whose 3rd derivatives are s-convex,* Tbilisi Math. J. **7**(2) (2014), 41–49.

16. Sarıkaya M. Z. and Kiriş M. E., *Some new inequalities of Hermite-Hadamard type for s-convex functions*, Miskolc Math. Notes **16**(1) (2015), 491–501.

17. Sarkaya M. Z. and Budak H., *Some Hermite-Hadamard type integral inequalities for twice differentiable mappings via fractional integrals*, Facta Univ. Ser. Math. Inform. **29**(4) (2014), 371–384.

18. Set E., Sarıkaya M. Z., Özdemir M. E. and Yıldırım H., *The Hermite-Hadamard's inequality for some convex functions via fractional integrals and related results*, JAMSI **10**(2) (2014), 69–83.

19. Zhang Y. and Wang J., *On some new Hermite-Hadamard inequalities involving Riemann-Liouville fractional integrals*, J. Inequal. Appl. **2013** (2013), Art. ID 220.

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