SOME GENERALIZED HERMITE-HADAMARD TYPE INEQUALITIES INVOLVING FRACTIONAL INTEGRAL OPERATOR FOR FUNCTIONS WHOSE SECOND DERIVATIVES IN ABSOLUTE VALUE ARE S-CONVEX

E. SET, S. S. DRAGOMIR AND A. GÖZPINAR

Abstract. In this article, a general integral identity for twice differentiable mapping involving fractional integral operators is derived. Secondly by using this identity we obtain some new generalized Hermite-Hadamard type inequalities for functions whose absolute values of second derivatives are s-convex and concave. The main results generalize the existing Hermite-Hadamard type inequalities involving the Riemann-Liouville fractional integral. Also we point out, some results in this study in some special cases such as setting $s = 1, \lambda = \alpha, \sigma(0) = 1$ and $w = 0$, more reasonable than those obtained in [8].

1. Introduction and Preliminaries

Let $f: I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function on an interval $I$ in the set of real numbers $\mathbb{R}$. Then, for $a, b \in I$ with $a < b$, the following so-called Hermite-Hadamard inequality (see, e.g., [12])

\begin{equation}
\frac{f(a) + f(b)}{2} \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}
\end{equation}

holds true. Since its discovery in 1983, Hermite-Hadamard’s inequality has been considered the most useful inequality in mathematical analysis. A number of the papers have been written on this inequality providing new proofs, noteworthy extensions, generalizations and numerous applications, see and references therein [6, 7, 12, 15].

Two definitions of s-convexity ($0 < s \leq 1$) of real-valued functions are well known in the literature.

**Definition 1.1.** Let $0 < s \leq 1$. A function $f: [0, \infty) \rightarrow \mathbb{R}$ is said to be s-Orlicz convex or s-convex in the first sense if for every $x, y \in [0, \infty)$ and $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s = 1$, we have

\begin{equation}
f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y).
\end{equation}
We denote the set of all $s$-convex functions in the first sense by $K_1^s$. This definition of $s$-convexity was introduced by Orlicz in [14] and was used in the theory of Orlicz spaces. Then, $s$-convex function in the second sense was introduced in Breckner’s paper [4] and a number of properties and connections with $s$-convexity in the first sense are discussed in paper [10].

**Definition 1.2.** [4] A function $f: \mathbb{R}_+ \to \mathbb{R}$ is said to be $s$-convex in the second sense if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

for all $x, y \in \mathbb{R}_+$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

We denote this by $K_2^s$. It is obvious that the $s$-convexity means just the convexity when $s = 1$.

In [6] Dragomir and Fitzpatrick proved a variant of Hadamard’s inequality which holds for $s$-convex functions in the second sense as follows

**Theorem 1.1.** Suppose that $f: [0, \infty) \to [0, \infty)$ is an $s$-convex function in the second sense, where $s \in (0, 1]$ and let $a, b \in [0, \infty)$, $a < b$. If $f \in L_1[a,b]$ then the following inequality holds

$$2^{s-1} \left( \frac{a+b}{2} \right)^s \leq 1 - a \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{s+1}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.3). For more study, see ([2, 3, 6, 11]).

In the following, we give some necessary definitions and preliminary results which are used and referred to throughout this paper.

**Definition 1.3.** Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b,$$

respectively, where $\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1}du$. Here $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. Some recent results and properties concerning this operator can be found [5, 9, 27, 19, 16, 20, 21].

The beta function is defined as follows:

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1}(1-t)^{b-1}dt, \quad a, b > 0,$$

where $\Gamma(\alpha)$ is Gamma function. In [27], Sarikaya et al. gave a remarkable integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals as follows.
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Theorem 1.2. Let \( f : [a, b] \to \mathbb{R} \) be a positive function with \( 0 \leq a < b \) and \( f \in [a, b] \). If \( f \) is convex function on \([a, b]\), then the following inequality for fractional integrals holds.

\[
(1.4) \quad f\left(\frac{a + b}{2}\right) \leq \frac{\Gamma(\alpha + 1)}{2(b - a)\alpha} \left[ J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a) \right] \leq \frac{f(a) + f(b)}{2}.
\]

It is obviously seen that if we take \( \alpha = 1 \) in Theorem 1.2, then the inequality (1.4) reduces to well-known Hermite-Hadamard’s inequality as (1.1).

Hermite Hadamard type inequality for \( s \)-convex functions on Riemann-Liouville fractional integrals is given in [19] as follows.

Theorem 1.3. Let \( f : [a, b] \to \mathbb{R} \) be a positive function with \( 0 \leq a < b \) and \( f \in L_1[a, b] \). If \( f \) is \( s \)-convex mapping in the second sense on \([a, b]\), then the following inequality for fractional integral with \( \alpha > 0 \) and \( s \in (0, 1) \) hold

\[
(1.5) \quad 2^{s-1} f\left(\frac{a + b}{2}\right) \leq \frac{\Gamma(\alpha + 1)}{2(b - a)\alpha} \left[ J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a) \right] \leq \alpha \left[ \frac{1}{\alpha + s} + B(\alpha, s + 1) \right] \frac{f(a) + f(b)}{2},
\]

where \( B(a, b) \) is beta function.

In [8] Dragomir et al proved the following identity and by using this identity they established new results involving Riemann-Liouville fractional integrals for twice differentiable convex mappings.

Lemma 1.1. [8] Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be a twice differentiable function on \( I^\circ \), the interior of \( I \). Assume that \( a, b \in I^\circ \) with \( a < b \) and \( f'' \in L[a, b] \), then the following identity for fractional integral with \( \alpha > 0 \) holds

\[
(1.6) \quad \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)\alpha} \left[ J_{b^-}^\alpha f(a) + J_{a^+}^\alpha f(b) \right] = \frac{(b - a)^2}{2(\alpha + 1)} \int_0^1 t(1 - t^\alpha) \left[ f''(ta + (1 - t)b) + f''(tb + (1 - t)a) \right] dt.
\]

In [25], Raina introduced a class of functions defined formally by

\[
(1.7) \quad \mathcal{F}_{\rho,\lambda}^\sigma(x) = \mathcal{F}_{\rho,\lambda}^{\sigma^{(0)},\sigma^{(1)},...}(x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k \quad (\rho, \lambda > 0; \ x \in \mathbb{R}),
\]

where the coefficients \( \sigma(k) \) \( (k \in \mathbb{N} = \mathbb{N} \cup \{0\}) \) are a bounded sequence of positive real numbers and \( \mathbb{R} \) is the set of real numbers. With the help of (1.7), Raina [25] and Agarwal et al. [1] defined the following left-sided and right-sided fractional integral operators respectively, as follows

\[
(1.8) \quad \left( \mathcal{J}_{\rho,\lambda,a}^{\sigma,\omega,\varphi}(x) \right) = \int_a^x (x - t)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^{\sigma}[w(x - t)^\rho] \varphi(t) dt \quad (x > a > 0),
\]

\[
(1.9) \quad \left( \mathcal{J}_{\rho,\lambda,b}^{\sigma,\omega,\varphi}(x) \right) = \int_x^b (t - x)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^{\sigma}[w(t - x)^\rho] \varphi(t) dt \quad (0 < x < b),
\]
where $\lambda, \rho > 0$, $w \in \mathbb{R}$ and $\varphi(t)$ is such that the integral on the right side exists. Recently some new integral inequalities this operator have appeared in the literature (see, e.g., [23, 22, 24, 1, 17, 18, 29]).

It is easy to verify that $J_{\sigma, \rho, \lambda, a+; w} \varphi(x)$ and $J_{\sigma, \rho, \lambda, b-; w} \varphi(x)$ are bounded integral operators on $L(a,b)$, if

\begin{equation}
\mathcal{M} := \mathcal{F}_{\rho, \lambda+1}^\sigma [w(b-a)^\rho] < \infty.
\end{equation}

In fact, for $\varphi \in L(a,b)$, we have

\begin{equation}
\|J_{\sigma, \rho, \lambda, a+; w} \varphi(x)\|_1 \leq \mathcal{M}(b-a)^\lambda \|\varphi\|_1
\end{equation}

and

\begin{equation}
\|J_{\sigma, \rho, \lambda, b-; w} \varphi(x)\|_1 \leq \mathcal{M}(b-a)^\lambda \|\varphi\|_1,
\end{equation}

where

\begin{equation}
\|\varphi\|_p := \left( \int_a^b |\varphi(t)|^p dt \right)^{\frac{1}{p}}.
\end{equation}

Here, many useful fractional integral operators can be obtained by specializing the coefficient $\sigma(k)$. For instance the classical Riemann-Liouville fractional integrals $J_{\alpha}^\alpha$ and $J_{\alpha}^\alpha$ of order $\alpha$ follow easily by setting $\lambda = \alpha$, $\sigma(0) = 1$ and $w = 0$ in (1.8) and (1.9).

In [26] generalized Hermite-Hadamard’s inequality for $s$-convex mapping fractional integral operators as follows.

\textbf{Theorem 1.4.} Let $f : [a,b] \to \mathbb{R}$ be a function with $0 \leq a < b$ and $f \in L_1[a,b]$. If $f$ is an $s$-convex function on $[a,b]$ then we have the following inequalities for generalized fractional integral operators

\begin{equation}
2^s f \left( \frac{a+b}{2} \right) \leq \frac{1}{(b-a)^\lambda} \mathcal{F}_{\rho, \lambda}^\sigma [w(b-a)^\rho] \left[ (J_{\sigma, \rho, \lambda, b-; w} f)(a) + (J_{\sigma, \rho, \lambda, a+; w} f)(b) \right]
\end{equation}

\begin{equation}
\leq \frac{1}{\mathcal{F}_{\rho, \lambda+1}^\sigma [w(b-a)^\rho]} \left[ A_1(\lambda, s) + \mathcal{F}_{\rho, \lambda}^\sigma [w(b-a)^\rho] \right] [f(a) + f(b)],
\end{equation}

where

\begin{equation}
\sigma_{0,s}(k) = \frac{\sigma(k)}{\lambda + \rho k + s}, \quad k = 0,1,2\ldots \quad \text{and}
\end{equation}

\begin{equation}
A_1(\lambda, s) = \int_0^t t^{\lambda-1}(1-t)^s \mathcal{F}_{\rho, \lambda}^\sigma [w(b-a)^\rho] dt.
\end{equation}

In this paper, first aim is to establish a new integral identity for a twice differentiable function via fractional integral operators. Using this new identity, we present some Hermite-Hadamard type inequalities for functions whose second-order derivatives absolute values are $s$-convex and concave in the second sense.
2. Main Results

Lemma 2.1. Let \( f: [a, b] \to \mathbb{R} \) be a twice differentiable mapping on \((a, b)\) \((a < b)\). Also, \( \rho, \lambda > 0 \) and \( w \in \mathbb{R} \). If \( f'' \in L[a, b] \), then the following equality for generalized fractional integrals holds

\[
\mathcal{F}_{\rho, \lambda+1}[w(b-a)\rho] \frac{f(a) + f(b)}{2} - \frac{1}{2(b-a)^{\lambda}} \left( (\mathcal{J}_{\rho, \lambda, w}^{\rho} f)(a) + (\mathcal{J}_{\rho, \lambda, w}^{\rho} f)(b) \right)
= \frac{(b-a)^{2}}{2} \times \int_{0}^{1} \{ t \mathcal{F}_{\rho, \lambda+2}[w(b-a)\rho] \\
- t^{\lambda+1} \mathcal{F}_{\rho, \lambda+2}[w(b-a)^{\rho} \rho] f'' (ta + (1-t)b) \} \, dt.
\]

Proof. Integrating by parts and changing variables with \( x = (ta + (1-t)b) \) we get

\[
I_1 = \mathcal{F}_{\rho, \lambda+2}[w(b-a)\rho] \int_{0}^{1} t f'' (ta + (1-t)b) \, dt
= \mathcal{F}_{\rho, \lambda+2}[w(b-a)\rho] \left\{ \frac{1}{a-b} \int_{0}^{1} t f' (ta + (1-t)b) \, dt \right\}
\]

by using same method

\[
I_2 = \mathcal{F}_{\rho, \lambda+2}[w(b-a)\rho] \int_{0}^{1} t f'' ((1-t)a + tb) \, dt
= \mathcal{F}_{\rho, \lambda+2}[w(b-a)\rho] \left\{ \frac{f'(b)}{b-a} - \frac{f(b) - f(a)}{(b-a)^{2}} \right\},
\]

analogously

\[
I_3 = \int_{0}^{1} t^{\lambda+1} \mathcal{F}_{\rho, \lambda+2}[w(b-a)^{\rho} \rho] f'' (ta + (1-t)b) \, dt
= \frac{1}{a-b} \int_{0}^{1} t^{\lambda} \mathcal{F}_{\rho, \lambda+1}[w(b-a)^{\rho} \rho] f' (ta + (1-t)b) \, dt
- \frac{1}{a-b} \int_{0}^{1} t^{\lambda} \mathcal{F}_{\rho, \lambda+1}[w(b-a)^{\rho} \rho] f' (ta + (1-t)b) \, dt
= \frac{1}{a-b} \int_{0}^{1} t^{\lambda-1} \mathcal{F}_{\rho, \lambda}[w(b-a)^{\rho} \rho] f (ta + (1-t)b) \, dt
+ \frac{1}{a-b} \int_{0}^{1} t^{\lambda-1} \mathcal{F}_{\rho, \lambda}[w(b-a)^{\rho} \rho] f (ta + (1-t)b) \, dt
\]
\[ \begin{align*}
&= \frac{F_{\rho,\lambda+1}^{\sigma}(w(b-a)^{\rho})f'(a)}{a-b} - \frac{F_{\rho,\lambda+1}^{\sigma}(w(b-a)^{\rho})f(a)}{(b-a)^{2}} + \frac{1}{(b-a)^{2}} \int_{a}^{b} \left( \frac{b-x}{b-a} \right)^{\lambda-1} F_{\rho,\lambda}^{\sigma} \left[ w(b-a)^{\rho} \left( \frac{b-x}{b-a} \right)^{\rho} \right] f(x) \frac{dx}{b-a} \\
&= \frac{F_{\rho,\lambda+1}^{\sigma}(w(b-a)^{\rho})f'(a)}{a-b} - \frac{F_{\rho,\lambda+1}^{\sigma}(w(b-a)^{\rho})f(a)}{(b-a)^{2}} + \left( J_{\rho,\lambda,a+w}^{\sigma}f \right)(b) - \frac{\rho}{\lambda+2} \frac{F_{\rho,\lambda+1}^{\sigma}(w(b-a)^{\rho})f(a)}{(b-a)^{2}} \frac{f''(a)}{(b-a)^{2}}. \\
\end{align*} \]

and

\[ I_{4} = \int_{0}^{1} t^{\lambda+1} \frac{F_{\rho,\lambda+1}^{\sigma}(w(b-a)^{\rho})f''(a)}{a-b} + \left( J_{\rho,\lambda,a+w}^{\sigma}f \right)(b) \frac{dt}{(b-a)^{2}}. \]  

(2.5)

Thus combining (2.2), (2.3), (2.4) and (2.5) as \( I_{1} + I_{2} - I_{3} - I_{4} \) and multiplying both sides of the obtained equality with \( \frac{(b-a)^{2}}{2} \), which proof is completed. \( \square \)

**Remark 2.1.** Setting \( \lambda = \alpha, \sigma(0) = 1 \) and \( w = 0 \) in Lemma 2.1, we find the same identity as [8, Lemma 1].

Using this lemma, we can get the following results via generalized fractional integral operator for twice differentiable function whose absolute value is s-convex and s-concave.

**Theorem 2.1.** Let \( f: [a, b] \to \mathbb{R} \) be a twice differentiable function on \( (a, b) \) \((a < b)\). Also, \( \rho, \lambda > 0 \) and \( w \in \mathbb{R} \). If \( |f''| \) is s-convex in the second sense on \( (a, b) \) then the following inequality for generalized fractional integral operators holds

\[ \begin{align*}
&\left| \frac{F_{\rho,\lambda+1}^{\sigma}(w(b-a)^{\rho})f(a) + f(b)}{2} - \frac{1}{2(b-a)^{\lambda}} \left[ \left( J_{\rho,\lambda,b-w}^{\sigma}f \right)(a) + \left( J_{\rho,\lambda,a+w}^{\sigma}f \right)(b) \right] \right| \\
&\leq \frac{(b-a)^{2}}{2} \frac{F_{\rho,\lambda+2}^{\sigma}(w(b-a)^{\rho})}{2} \left[ |f''(a)| + |f''(b)| \right],
\end{align*} \]

where \( s \in (0, 1], B(\ldots, \ldots) \) is Euler beta function and

\[ \sigma_{1,s}(k) = \sigma(k) \left[ \frac{(\lambda + \rho k)}{(2 + s)(\lambda + \rho k + s + 2)} + B(2, s + 1) - B(\lambda + \rho k + 2, s + 1) \right]. \]
Proof. From Lemma 2.1 with properties of modulus, we get
\[(2.6)\]
\[
\left| \mathcal{F}_{\rho,\lambda+1}^\sigma[w(b-a)^\rho] \right| \frac{f(a) + f(b)}{2} \leq \frac{1}{2(b-a)^\lambda} \left[ (\mathcal{J}_{\rho,\lambda,b^{-},w}^\sigma f)(a) + (\mathcal{J}_{\rho,\lambda,a^{+},w}^\sigma f)(b) \right] \\
\leq \frac{(b-a)^2}{2} \int_0^1 \left\{ t \mathcal{F}_{\rho,\lambda+2}^\sigma[w(b-a)^\rho] - t^{\lambda+1} \mathcal{F}_{\rho,\lambda+2}^\sigma[w(b-a)^\rho t^\rho] \right\} dt \\
\times \left[ \left| f''(ta + (1-t)b) + f''(1-t)a + tb) \right| \right] dt \\
\leq \frac{(b-a)^2}{2} \sum_{k=0}^\infty \frac{\sigma(k)|w|^k(b-a)^{\rho k}}{\Gamma(\rho k + \lambda + 2)} \int_0^1 \left| t - t^{\lambda+\rho k+1} \right| \left| f''(ta + (1-t)b) \right| dt \\
+ \frac{(b-a)^2}{2} \sum_{k=0}^\infty \frac{\sigma(k)|w|^k(b-a)^{\rho k}}{\Gamma(\rho k + \lambda + 2)} \int_0^1 \left| t - t^{\lambda+\rho k+1} \right| \left| f''(1-t)a + tb) \right| dt.
\]
Since \(|f''|\) is \(s\)-convex, we have
\[(2.7)\]
\[
\int_0^1 (t - t^{\lambda+\rho k+1}) \left| f''(ta + (1-t)b) \right| dt + \int_0^1 (t - t^{\lambda+\rho k+1}) \left| f''(1-t)a + tb) \right| dt \\
\leq \left[ \int_0^1 t^{1+s}(1 - t^{\lambda+\rho k}) dt + \int_0^1 t(1-t)^s(1 - t^{\lambda+\rho k}) dt \right] \left[ |f''(a)| + |f''(b)| \right] \\
= \frac{(\lambda + \rho k)}{(2+s)(\lambda + \rho k + s+2)} + B(2, s+1) - B(\lambda + \rho k + 2, s+1) \left[ |f''(a)| + |f''(b)| \right].
\]
Thus combining the inequalities (2.6) and (2.7), the requested result is obtained.

\[\square\]

**Corollary 2.1.** If we take \(s = 1\) in Theorem 2.1, we get the following inequality
\[
\left| \mathcal{F}_{\rho,\lambda+1}^\sigma[w(b-a)^\rho] \right| \frac{f(a) + f(b)}{2} \leq \frac{1}{2(b-a)^\lambda} \left[ (\mathcal{J}_{\rho,\lambda,b^{-},w}^\sigma f)(a) + (\mathcal{J}_{\rho,\lambda,a^{+},w}^\sigma f)(b) \right] \\
\leq \frac{(b-a)^2}{2} \mathcal{F}_{\rho,\lambda+2}[w(b-a)^\rho] \left[ |f''(a)| + |f''(b)| \right],
\]
where
\[
\sigma_{1,1}(k) = \sigma(k) \left[ \frac{(\lambda + \rho k)}{2(\lambda + \rho k + 2)} \right], \quad \rho, \lambda > 0, \quad w \in \mathbb{R}.
\]

**Corollary 2.2.** If we take \(\lambda = \alpha, \sigma(0) = 1\) and \(w = 0\) in Corollary 2.1, we get the following inequality
\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[ J_0^\alpha f(a) + J_1^\alpha f(b) \right] \right| \\
\leq \frac{(b-a)^2 \alpha}{4(\alpha + 1)(\alpha + 2)} \left[ |f''(a)| + |f''(b)| \right],
\]
which is more reasonable than the result obtained [8, Theorem 2] under the same assumptions.
Remark 2.2. Setting \( s = 1, \lambda = \alpha = 1, \sigma(0) = 1 \) and \( w = 0 \) in Theorem 2.1 we come to the same result as [28, Proposition 2].

**Theorem 2.2.** Let \( f: [a, b] \to \mathbb{R} \) be a twice differentiable function on \( (a, b) \) \( (a < b) \). Also, \( \rho, \lambda \geq 0 \) and \( w \in \mathbb{R} \). If \( |f''|^{q} \) is \( s \)-convex in the second sense and \( q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), then the following inequality for generalized fractional integral operators holds

\[
\left| \mathcal{I}_{\rho,\lambda}^{\alpha}[w(b-a)^{\alpha}] \frac{f(a) + f(b)}{2} - \frac{1}{2(b-a)^{\lambda}} \left[ (\mathcal{J}_{\rho,\lambda}^{\alpha}[w^{\alpha}]f)(a) + (\mathcal{J}_{\rho,\lambda}^{\alpha}[w^{\alpha}]f)(b) \right] \right| \leq \frac{(b-a)^{2}}{2} \int_{0}^{1} \left| t \mathcal{I}_{\rho,\lambda}^{\alpha}[w(b-a)^{\alpha}] - t^{\lambda+1} \mathcal{I}_{\rho,\lambda}^{\alpha}[w(b-a)^{\alpha}] t^{\lambda+1} \mathcal{I}_{\rho,\lambda}^{\alpha}[w(b-a)^{\alpha}] \right| dt
\]

\[
\leq \frac{(b-a)^{2}}{2} \left| t \mathcal{I}_{\rho,\lambda}^{\alpha}[w(b-a)^{\alpha}] - t^{\lambda+1} \mathcal{I}_{\rho,\lambda}^{\alpha}[w(b-a)^{\alpha}] \right| dt
\]

\[
\leq \frac{(b-a)^{2}}{2} \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^{k}(b-a)^{\rho}k}{(\rho k + \lambda + 2)} \int_{0}^{1} \left| t - t^{\lambda+\rho+1} \right| \left| f''(ta + (1-t)b) \right| dt
\]

\[
+ \frac{(b-a)^{2}}{2} \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^{k}(b-a)^{\rho}k}{(\rho k + \lambda + 2)} \int_{0}^{1} \left| t - t^{\lambda+\rho+1} \right| \left| f''((1-t)a + tb) \right| dt.
\]

Using Hölder Inequality and the \( s \)-convexity of \( |f''|^{q} \) we get the following inequality

\[
\int_{0}^{1} t(1-t^{\lambda+\rho}) \left[ |f''(ta + (1-t)b)| + |f''((1-t)a + tb)| \right] dt
\]

\[
\leq \left[ \int_{0}^{1} t(1-t^{\lambda+\rho}) \right]^{\frac{1}{p}} \left[ \left( \int_{0}^{1} |f''(ta + (1-t)b)|^{q} dt \right)^{\frac{1}{q}} + \left( \int_{0}^{1} |f''((1-t)a + tb)|^{q} dt \right)^{\frac{1}{q}} \right]^{\frac{1}{p}}
\]

\[
\leq \left[ \int_{0}^{1} t^{p}(1-t^{\lambda+\rho}) \right]^{\frac{1}{p}} \left( \frac{1}{s+1} \right)^{\frac{1}{p}} \left[ |f''(a)|^{q} + |f''(b)|^{q} \right]^{\frac{1}{p}}
\]

\[
= \left[ \frac{1}{\lambda + \rho} B\left( \frac{p+1}{\lambda + \rho}, p + 1 \right) \right]^{\frac{1}{p}} \left( \frac{1}{s+1} \right)^{\frac{1}{p}} \left[ |f''(a)|^{q} + |f''(b)|^{q} \right]^{\frac{1}{p}}.
\]
By changing \( x = t^{\lambda + \rho k} \) and simple calculating we get
\[
\int_0^1 t^p (1 - t^{(\lambda + \rho k)})^p \, dt = \frac{1}{\lambda + \rho k} B \left( \frac{p + 1}{\lambda + \rho k}, p + 1 \right).
\]
Thus combining (2.8) and (2.9), the desired result is obtained. \( \square \)

**Remark 2.3.** Setting \( \lambda = \alpha = 1, \sigma(0) = 1 \) and \( w = 0 \) in Theorem 2.2 the same result as in [11, Theorem 10].

**Corollary 2.3.** Taking \( s = 1 \) in Theorem 2.2, the following inequality holds
\[
\left| \mathcal{F}_{\rho,\lambda+1}^s \left[ w(b-a)^\rho \frac{f(a) + f(b)}{2} \right] - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[ (\mathcal{J}_{\rho,\lambda,b^-;w}^s f)(a) + (\mathcal{J}_{\rho,\lambda,a^+;w}^s f)(b) \right] \right|
\leq \frac{(b-a)^2}{2} \mathcal{F}_{\rho,\lambda+2}^{s\rho} \left[ \frac{|f''(a)|^q + |f''(b)|^q}{2} \right]^\frac{1}{q},
\]
where \( \rho, \lambda > 0, w \in \mathbb{R}, B(\ldots,\ldots) \) is Euler beta function and
\[
\sigma_2(k) = 2\sigma(k) \left[ \frac{1}{(\lambda + \rho k)} B \left( \frac{p + 1}{\lambda + \rho k}, p + 1 \right) \right]^\frac{1}{p}.
\]

**Corollary 2.4.** Taking \( \lambda = \alpha, \sigma(0) = 1 \) and \( w = 0 \) in Corollary 2.3, the following inequality holds
\[
\left| \mathcal{F}_{\rho,\lambda+1}^s \left[ w(b-a)^\rho \frac{f(a) + f(b)}{2} \right] - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[ (\mathcal{J}_{\rho,\lambda,b^-;w}^s f)(a) + (\mathcal{J}_{\rho,\lambda,a^+;w}^s f)(b) \right] \right|
\leq \frac{(b-a)^2}{(\alpha + 1)} \left[ \frac{1}{\alpha} B \left( \frac{p + 1}{\alpha}, p + 1 \right) \right]^\frac{1}{p} \left[ \frac{|f''(a)|^q + |f''(b)|^q}{2} \right]^\frac{1}{q},
\]
which is more reasonable than obtained [8, Theorem 3] under the same assumptions.

**Theorem 2.3.** Let \( f : [a, b] \to \mathbb{R} \) be a twice differentiable function on \( (a, b) \) \( (a < b) \). Also, \( \rho, \lambda > 0 \) and \( w \in \mathbb{R} \). If \( |f''|^q \) is \( s \)-convex in the second sense and \( q \geq 1 \), then the following inequality for generalized fractional integral operators holds
\[
\left| \mathcal{F}_{\rho,\lambda+1}^s \left[ w(b-a)^\rho \frac{f(a) + f(b)}{2} \right] - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[ (\mathcal{J}_{\rho,\lambda,b^-;w}^s f)(a) + (\mathcal{J}_{\rho,\lambda,a^+;w}^s f)(b) \right] \right|
\leq \frac{(b-a)^2}{2} \mathcal{F}_{\rho,\lambda+2}^{s\rho} \left[ ||w||(b-a)^\rho \right],
\]
where \( s \in (0, 1] \), \( B(\ldots) \) is Euler beta function and

\[
\sigma_{\lambda, s}(k) = \sigma(k) \left[ \frac{\lambda + \rho k}{2(\lambda + \rho k + 2)} \right]^{1-\frac{s}{2}}
\]

\[
\times \left\{ \left[ \frac{\lambda + \rho}{(s+2)(\lambda + \rho k + s + 2)} |f''(a)|^q + (B(2, s+1) - B(\lambda + \rho k + 2, s+1)) |f''(b)|^q \right]^{\frac{s}{2}} + \left[ (B(2, s+1) - B(\lambda + \rho k + 2, s+1)) |f''(a)|^q + \frac{\lambda + \rho k}{(s+2)(\lambda + \rho k + s + 2)} |f''(b)|^q \right]^{\frac{s}{2}} \right\}
\]

**Proof.** From Lemma 2.1 with properties of modulus we get

\[
(b-a)^2 \int_{0}^{1} |f''(ta + (1-t)b)|^q \, dt < \left[ \frac{\lambda + \rho}{(s+2)(\lambda + \rho k + s + 2)} |f''(a)|^q + (B(2, s+1) - B(\lambda + \rho k + 2, s+1)) |f''(b)|^q \right]^{\frac{s}{2}}
\]

\[
\times \left\{ \left[ (B(2, s+1) - B(\lambda + \rho k + 2, s+1)) |f''(a)|^q + \frac{\lambda + \rho k}{(s+2)(\lambda + \rho k + s + 2)} |f''(b)|^q \right]^{\frac{s}{2}} \right\}
\]

Using Power-mean Inequality and \( s \)-convexity of \( |f''|^q \), we obtain the following inequality

\[
\int_{0}^{1} \left( t(1 - t^{\lambda + \rho k}) \right) \, dt \leq \left[ \frac{\lambda + \rho k}{2(\lambda + \rho k + 2)} \right]^{1-\frac{s}{2}} \left[ \int_{0}^{1} (t - t^{\lambda + \rho k + 1}) |f''(ta + (1-t)b)|^q \, dt \right]^{\frac{s}{2}}
\]

\[
\times \left\{ \left[ \frac{\lambda + \rho}{(s+2)(\lambda + \rho k + s + 2)} |f''(a)|^q + (B(2, s+1) - B(\lambda + \rho k + 2, s+1)) |f''(b)|^q \right]^{\frac{s}{2}} + \left[ (B(2, s+1) - B(\lambda + \rho k + 2, s+1)) |f''(a)|^q + \frac{\lambda + \rho k}{(s+2)(\lambda + \rho k + s + 2)} |f''(b)|^q \right]^{\frac{s}{2}} \right\}
\]
Combining the inequalities (2.10) and (2.11) we have

\[
\left| F_{p,λ+1}^σ[w(b-a)^p] \frac{f(a) + f(b)}{2} - \frac{1}{2(b-a)^λ} [J_{p,λ+1}^σ(a) + J_{p,λ+1}^σ(b)] \right| \\
\leq \frac{(b-a)^2}{2} \sum_{k=0}^{\infty} σ(k)w^k(b-a)^{pk} \frac{λ + pk}{2(λ + pk + 2)} \left[ \left( \frac{λ + pk}{2(λ + pk + 2)} \right)^{1-\frac{p}{q}} \right]
\times \left\{ \left[ \frac{λ + pk}{2(λ + pk + 2)} \right] |f''(a)|^q + \frac{(λ + pk)(λ + pk + 5)}{6(λ + pk + 2)(λ + pk + 3)} |f''(b)|^q \right\} \}
\leq \frac{(b-a)^2}{2} F_{p,λ+2}^σ[w(b-a)^p],
\]

Thus the proof is completed. □

**Corollary 2.5.** Taking \( s = 1 \) with \( ρ,λ > 0 \) and \( w ∈ \mathbb{R} \) in Theorem 2.3, the following inequality holds

\[
\left| F_{p,λ+1}^σ[w(b-a)^p] \frac{f(a) + f(b)}{2} - \frac{1}{2(b-a)^λ} [J_{p,λ+1}^σ(a) + J_{p,λ+1}^σ(b)] \right| \\
\leq \frac{(b-a)^2}{2} F_{p,λ+2}^σ[w(b-a)^p],
\]

where

\[
σ_{3,1}(k) = σ(k) \left[ \frac{λ + pk}{2(λ + pk + 2)} \right]^{1-\frac{p}{q}}
\times \left\{ \left[ \frac{λ + pk}{2(λ + pk + 2)} \right] |f''(a)|^q + \frac{(λ + pk)(λ + pk + 5)}{6(λ + pk + 2)(λ + pk + 3)} |f''(b)|^q \right\} \}
\]

**Remark 2.4.** Setting \( λ = α = 1, σ(0) = 1 \) and \( w = 0 \) in Theorem 2.3 the same result as [11, Theorem 8].

**Remark 2.5.** Setting \( s = 1, λ = α, σ(0) = 1 \) and \( w = 0 \) in Theorem 2.3 the same result as [8, Theorem 4].

**Theorem 2.4.** Let \( f: [a, b] → \mathbb{R} \) be a twice differentiable function on \( (a, b) \) \((a < b)\). Also, \( ρ,λ > 0 \) and \( w ∈ \mathbb{R} \). If \( |f''|^q \) is \( s \)-concave in the second sense and \( q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), then the following inequality for generalized fractional
integral operators holds
\[
\left| \mathcal{F}_{\rho,\lambda+1}^\sigma[w(b-a)^p] f(a) + f(b) \right| \leq \frac{1}{2(b-a)^\lambda} \left[ (\mathcal{J}_{\rho,\lambda,b^-w}^\sigma)(a) + (\mathcal{J}_{\rho,\lambda,a+w}^\sigma)(b) \right]
\]
\[
\leq \frac{(b-a)^2}{2} \mathcal{F}_{\rho,\lambda+2}^{\sigma_{4,s}}[w(b-a)^p] f'' \left( \frac{a+b}{2} \right),
\]
where \( s \in (0,1] \) and \( \sigma_{4,s}(k) = \sigma(k)2^k \left[ \frac{2}{\lambda + p} B \left( \frac{p+1}{\lambda + p}, p+1 \right) \right] \).

**Proof.** From Lemma 2.1 and H"older inequality with properties of modulus, we have
\[
\left| \mathcal{F}_{\rho,\lambda+1}^\sigma[w(b-a)^p] f(a) + f(b) \right| \leq \frac{1}{2(b-a)^\lambda} \left[ (\mathcal{J}_{\rho,\lambda,b^-w}^\sigma)(a) + (\mathcal{J}_{\rho,\lambda,a+w}^\sigma)(b) \right]
\]
\[
\leq \frac{(b-a)^2}{2} \int_0^1 |\mathcal{F}_{\rho,\lambda+2}^{\sigma_{4,s}}[w(b-a)^p] - t^{\lambda+1} \mathcal{F}_{\rho,\lambda+2}^{\sigma_{4,s}}[w(b-a)^p] t^p| dt
\]
\[
\leq \frac{(b-a)^2}{2} \sum_{k=0}^{\infty} \sigma(k)w^k(b-a)^p k \Gamma(\rho k + \lambda + 2)
\]
\[
\times \left\{ \int_0^1 \left[ f''(ta + (1-t)b) \right]^q dt \right\}^\frac{1}{q}
\]
\[
\leq \frac{(b-a)^2}{2} \sum_{k=0}^{\infty} \sigma(k)w^k(b-a)^p k \Gamma(\rho k + \lambda + 2)
\]
\[
\times \left\{ \int_0^1 \left[ f''((1-t)a + tb) \right]^q dt \right\}^\frac{1}{q}
\]
\[
\times \left\{ \int_0^1 \left[ f''((1-t)a + tb) \right]^q dt \right\}^\frac{1}{q}
\]
\[
\bigg\}
\]
Since \( |f''|^q \) is \( s \)-concave, we can write
\[
\int_0^1 |f''((1-t)a + tb)|^q dt \leq 2^{s-1} \int_0^1 f'' \left( \frac{a+b}{2} \right)^q dt,
\]
\[
\int_0^1 |f''((1-t)a + tb)|^q dt \leq 2^{s-1} \int_0^1 f'' \left( \frac{a+b}{2} \right)^q dt.
\]
On the other hand, by simple calculating we establish
\[
\int_0^1 (t - t^{\lambda+p+1})^p dt = \frac{1}{\lambda + p k} B \left( \frac{p+1}{\lambda + p}, p+1 \right).
\]
Thus combining (2.13), (2.14) and (2.12) the requested result is obtained. \( \square \)
Corollary 2.6. Taking $s = 1$ in Theorem 2.4, the following inequality holds
\[
\begin{align*}
&\left| F_{\rho, \lambda+1}^{\sigma} \left[ w(b-a)^{\sigma} \right] - \frac{1}{2(b-a)^{\lambda}} \left[ (J_{\rho, \lambda}^{\sigma} b^{-w,f})(a) + (J_{\rho, \lambda}^{\sigma} a^{w,f})(b) \right] \right| \\
&\leq \frac{(b-a)^{2}}{2} F_{\rho, \lambda+2} w |(b-a)^{\sigma}| f'' \left( \frac{a+b}{2} \right),
\end{align*}
\]
where $\rho, \lambda > 0$, $w \in \mathbb{R}$, $\sigma 4^{\sigma,1}(k) = \sigma(k)2^{\frac{1}{\lambda + \rho k}} B \left( \frac{p+1}{\lambda + \rho k}, p + 1 \right) \right)^{1/2}$.

Corollary 2.7. If we take $\lambda = \alpha$, $\sigma(0) = 1$ and $w = 0$ in Corollary 2.6, the following inequality holds
\[
\begin{align*}
&\left| f(a) + f(b) - \frac{\Gamma(\alpha + 1)}{2(b-a)^{\lambda}} \left[ J_{\rho, \lambda}^{\alpha} f(a) + J_{\rho, \lambda}^{\alpha} f(b) \right] \right| \\
&\leq \frac{(b-a)^{2}}{2(\alpha + 1)} \left[ \frac{1}{\alpha} B \left( \frac{p+1}{\alpha}, p + 1 \right) \right]^{1/2} \left| f'' \left( \frac{a+b}{2} \right) \right|,
\end{align*}
\]
which is more reasonable than [8, Theorem 5] under the same assumptions.

Remark 2.6. Setting $s = 1$, $\lambda = \alpha = 1$, $\sigma(0) = 1$ and $w = 0$ in Theorem 2.3 the same result as [11, Theorem 9].

References


E. Set, Department of Mathematics, Faculty of Science and Arts, Ordu University, Ordu, Turkey, e-mail: erhanset@yahoo.com

S. S. Dragomir, Mathematics, College of Engineering and Science, Victoria University, Melbourne City, Australia, e-mail: sever.dragomir@vu.edu.au

A. Gözpınar, Department of Mathematics, Faculty of Science and Arts, Ordu University, Ordu, Turkey, e-mail: abdurrahmangozpinar79@gmail.com