TANGENT BUNDLE OF ORDER TWO
AND BIHARMONICITY

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Abstract. The problem studied in this paper is related to the biharmonicity of
a section from a Riemannian manifold \((M,g)\) to its tangent bundle \(T^2 M\) of order
two equipped with the diagonal metric \(g^D\). We show that a section on a compact
manifold is biharmonic if and only if it is harmonic. We also investigate the curva-
ture of \((T^2 M, g^D)\) and the biharmonicity of section of \(M\) as a map from \((M, g)\) to
\((T^2 M, g^D)\).

1. Introduction

Harmonic (resp., biharmonic) maps are critical points of energy (resp., bi-
energy) functional defined on the space of smooth maps between Riemannian
manifolds introduced by Eells and Sampson [4] (resp., Jiang [6]). In this pa-
per, we present some properties for biharmonic section between a Riemannian
manifold and its second tangent bundle which generalize the results of Ishihara
[5], Konderak [7], Oproiu [9] and Djaa-Ouakkas [3].

Consider a smooth map \(\phi: (M^n, g) \rightarrow (N^n, h)\) between two Riemannian mani-
folds, then the energy functional is defined by

\[
E(\phi) = \frac{1}{2} \int_M |d\phi|^2 dv_g
\]

(or over any compact subset \(K \subset M\)).

A map is called harmonic if it is a critical point of the energy functional \(E\) (or
\(E(K)\) for all compact subsets \(K \subset M\)). For any smooth variation \(\{\phi_t\}_{t \in I}\) of \(\phi\)
with \(\phi_0 = \phi\) and \(V = \frac{d\phi_t}{dt}|_{t=0}\), we have

\[
\frac{d}{dt} E(\phi_t)|_{t=0} = -\frac{1}{2} \int_M h(\tau(\phi), V) dv_g,
\]

where

\[
\tau(\phi) = tr_g \nabla d\phi
\]
is the tension field of \(\phi\). Then we have the following theorem.

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Theorem 1.1. A smooth map \( \phi: (M^m, g) \to (N^n, h) \) is harmonic if and only if
\[ \tau(\phi) = 0. \]

If \((x^i)_{1 \leq i \leq m}\) and \((y^\alpha)_{1 \leq \alpha \leq n}\) denote local coordinates on \(M\) and \(N\), respectively, then equation (4) takes the form
\[ \tau(\phi) = \left( \Delta \phi^\alpha + g^{ij} \Gamma^N_{\beta\gamma} \frac{\partial \phi^\beta}{\partial x^i} \frac{\partial \phi^\gamma}{\partial x^j} \right) = 0, \]
where \(\Delta \phi^\alpha = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left( \sqrt{|g|} g^{ij} \frac{\partial \phi^\alpha}{\partial x^j} \right)\) is the Laplace operator on \((M^m, g)\) and \(\Gamma^N_{\beta\gamma}\) are the Christoffel symbols on \(N\).

Definition 1.2. A map \( \phi: (M, g) \to (N, h) \) between Riemannian manifolds is called biharmonic if it is a critical point of bienergie functional
\[ E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 dv^g. \]

The Euler-Lagrange equation attached to bienergy is given by vanishing of the bitension field
\[ \tau_2(\phi) = -J_\phi(\tau(\phi)) = - (\Delta \phi \tau(\phi) + \text{tr}_g R^N (\tau(\phi), d\phi) d\phi), \]
where \(J_\phi\) is the Jacobi operator defined by
\[ J_\phi: \Gamma(\phi^{-1}(TN)) \to \Gamma(\phi^{-1}(TN)) \]
\[ V \mapsto \Delta \phi V + \text{tr}_g R^N (V, d\phi) d\phi. \]

Theorem 1.3. A smooth map \( \phi: (M^m, g) \to (N^n, h) \) is biharmonic if and only if
\[ \tau_2(\phi) = 0. \]

From Theorem 1.1 and formula (7), we have the following corollary.

Corollary 1.4. If \( \phi: (M^m, g) \to (N^n, h) \) is harmonic, then \( \phi \) is biharmonic.
(For more details see [6]).

2. Preliminary Notes

2.1. Horizontal and vertical lifts on \(TM\)

Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold and \((TM, \pi, M)\) be its tangent bundle. A local chart \((U, x^i)_{i=1,...,n}\) on \(M\) induces a local chart \((\pi^{-1}(U), x^i, y^\alpha)_{i,j=1,...,n}\) on \(TM\). Denote the Christoffel symbols of \(g\) by \(\Gamma^k_{ij}\) and the Levi-Civita connection of \(g\) by \(\nabla\).

We have two complementary distributions on \(TM\), the vertical distribution \(\mathcal{V}\) and the horizontal distribution \(\mathcal{H}\) defined by
\[ \mathcal{V}_{(x,u)} = \ker(d\pi_{(x,u)}) = \left\{ a^i \frac{\partial}{\partial y^i} \big|_{(x,u)} : a^i \in \mathbb{R} \right\}, \]
and
\[ \mathcal{H}_{(x,u)} = \left\{ a^i \frac{\partial}{\partial x^i} \big|_{(x,u)} - a^j w^i_{\alpha j} \Gamma^k_{ij} \frac{\partial}{\partial y^k} \big|_{(x,u)} : a^i \in \mathbb{R} \right\}, \]
where \((x, u) \in TM\), such that \(T_{(x, u)}TM = \mathcal{H}_{(x, u)} \oplus \mathcal{V}_{(x, u)}\).

Let \(X = X^i \frac{\partial}{\partial x^i}\) be a local vector field on \(M\). The vertical and the horizontal lifts of \(X\) are defined by

\[
X^V = X^i \frac{\partial}{\partial y^i},
\]

\[
X^H = X^i \frac{\delta}{\delta x^i} = X^i \left\{ \frac{\partial}{\partial x^i} - y^j \Gamma^k_{ij} \frac{\partial}{\partial y^k} \right\}.
\]

For consequences, we have:

1. \(\left( \frac{\partial}{\partial x^i} \right)^H = \frac{\delta}{\delta x^i}\) and \(\left( \frac{\partial}{\partial x^i} \right)^V = \frac{\partial}{\partial y^i}\).

2. \(\left( \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j} \right)_{i,j=1,...,n}\) is a local frame on \(TM\)

3. If \(u = u^i \frac{\partial}{\partial x^i} \in T_xM\), then \(u^H = u^i \left\{ \frac{\partial}{\partial x^i} - y^j \Gamma^k_{ij} \frac{\partial}{\partial y^k} \right\}\) and \(u^V = u^i \frac{\partial}{\partial y^i}\).

**Definition 2.1.** Let \((M, g)\) be a Riemannian manifold and \(F: TM \to TM\) be a smooth bundle endomorphism of \(TM\). Then we define a vertical and horizontal vector fields \(VF, HF\) on \(TM\) by

\[
VF: TM \to TTM
(x, u) \mapsto (F(u))^V,
\]

\[
HF: TM \to TTM
(x, u) \mapsto (F(u))^H.
\]

Locally we have

\[
VF = y^i F^j_i \frac{\partial}{\partial y^j} = y^i \left( F \left( \frac{\partial}{\partial x^i} \right) \right)^V
\]

\[
HF = y^i F^j_i \frac{\partial}{\partial x^j} - y^j y^k F^i_j \Gamma^k_{ij} \frac{\partial}{\partial y^k} = y^i \left( F \left( \frac{\partial}{\partial x^i} \right) \right)^H.
\]

**Proposition 2.2** ([1]). Let \((M, g)\) be a Riemannian manifold and \(\hat{\nabla}\) be the Levi-Civita connection of the tangent bundle \((TM, g^i)\) equipped with the Sasaki metric. If \(F\) is a tensor field of type \((1, 1)\) on \(M\), then

\[
(\hat{\nabla}_X V F)_{(x, u)} = (F(X))^V_{(x, u)},
\]

\[
(\hat{\nabla}_X H F)_{(x, u)} = (F(X))^H_{(x, u)} + \frac{1}{2} (R_x(u, X_x) F(u))^H,
\]

\[
(\hat{\nabla}_X V F)_{(x, u)} = V(\nabla_X F)(x, u) + \frac{1}{2} (R_x(u, F_x(u)) X_x)^H,
\]

\[
(\hat{\nabla}_X H F)_{(x, u)} = H(\nabla_X F)(x, u) - \frac{1}{2} (R_x(X_x, F_x(u)) u)^V,
\]

where \((x, u) \in TM\) and \(X \in \Gamma(TM)\).
2.2. Second Tangent Bundle

Let \( M \) be an \( n \)-dimensional smooth differentiable manifold and \( (U_{\alpha}, \psi_{\alpha})_{\alpha \in I} \) a corresponding atlas. For each \( x \in M \), we define an equivalence relation on

\[
C_x = \{ \gamma: (-\varepsilon, \varepsilon) \to M \mid \gamma \text{ is smooth and } \gamma(0) = x, \varepsilon > 0 \}
\]

by

\[
\gamma \approx_x h \iff \gamma'(0) = h'(0) \text{ and } \gamma''(0) = h''(0),
\]

where \( \gamma' \) and \( \gamma'' \) denote the first and the second derivation of \( \gamma \), respectively.

\[
\gamma': (-\varepsilon, \varepsilon) \to TM; \quad t \mapsto [d\gamma(t)](1)
\]

\[
\gamma'': (-\varepsilon, \varepsilon) \to T(TM); \quad t \mapsto [d\gamma'(t)](1).
\]

**Definition 2.3.** We define the second tangent space of \( M \) at the point \( x \) to be the quotient

\[
T_x^2 M = C_x / \approx_x
\]

and the second tangent bundle of \( M \) the union of all second tangent space, \( T^2 M = \bigcup_{x \in M} T_x^2 M \). We denote the equivalence class of \( \gamma \) by \( j^2_x \gamma \) with respect to \( \approx_x \), and by \( j^2 \gamma \) an element of \( T^2 M \).

In the general case, the structure of higher tangent bundle \( T^r M \) is considered in [8, Chapters 1–2] and [2].

**Proposition 2.4 ([3]).** Let \( M \) be an \( n \)-dimensional manifold, then \( TM \) is subbundle of \( T^2 M \) and the map

\[
i: TM \to T^2 M
\]

\[
j^2_x f = j^2_x \tilde{f}
\]

is an injective homomorphism of natural bundles (not of vector bundles), where

\[
\tilde{f}^i = \int_0^1 f^i(s) ds - tf^i(0) + f^i(0) \quad i = 1 \ldots n.
\]

**Theorem 2.5.** Let \( (M, g) \) be a Riemannian manifold and \( \nabla \) be the Levi-Civita connection. If \( TM \oplus TM \) denotes the Whitney sum, then

\[
S: T^2 M \to TM \oplus TM
\]

\[
j^2 \gamma(0) \mapsto (\dot{\gamma}(0), (\nabla_{\gamma(0)} \dot{\gamma})(0))
\]

is a diffeomorphism of natural bundles. In the induced coordinate we have

\[
(x^i; y^i; z^i) \mapsto (x^i, y^i, z^i + y^j y^k \Gamma^i_{jk}).
\]

**Remark 2.6.** The diffeomorphism \( S \) determines a vector bundle structure on \( T^2 M \) by

\[
\alpha \Psi_1 + \beta \Psi_2 = S^{-1}(\alpha S(\Psi_1) + \beta S(\Psi_2)),
\]

where \( \Psi_1, \Psi_2 \in T^2 M \) and \( \alpha, \beta \in \mathbb{R} \), for which \( S \) is a linear isomorphism of vector bundles and \( i: TM \to T^2 M \) is an injective linear homomorphism of vector bundles (for more details see [2]).
**Definition 2.7** ([3]). Let \((M,g)\) be a Riemannian manifold and \(T^2M\) be its tangent bundle of order two endowed with the vectorial structure induced by the diffeomorphism \(S\). For any section \(\sigma \in \Gamma(T^2M)\), we define two vector fields on \(M\) by

\[
X_\sigma = P_1 \circ S \circ \sigma,
\]
(18)
\[
Y_\sigma = P_2 \circ S \circ \sigma,
\]
(19)
where \(P_1\) and \(P_2\) denote the first and the second projections from \(TM \oplus TM\) on \(TM\).

From Remark 2.6 and Definition 2.7, we deduce the following.

**Proposition 2.8.** For all sections \(\sigma, \varpi \in \Gamma(T^2M)\) and \(\alpha \in \mathbb{R}\), we have

\[
X_{\alpha \sigma + \varpi} = \alpha X_\sigma + X_\varpi,
\]
(20)
\[
Y_{\alpha \sigma + \varpi} = \alpha Y_\sigma + Y_\varpi,
\]
where \(\alpha \sigma + \varpi = S^{-1}(\alpha S(\sigma) + S(\varpi))\).

**Definition 2.9** ([3]). Let \((M,g)\) be a Riemannian manifold and \(T^2M\) be its tangent bundle of order two endowed with the vectorial structure induced by the diffeomorphism \(S\). We define a connection \(\hat{\nabla}\) on \(\Gamma(T^2M)\) by

\[
\hat{\nabla}: \Gamma(TM) \times \Gamma(T^2M) \to \Gamma(T^2M)
\]
(21)
\[
(z, \sigma) \mapsto \hat{\nabla}_z \sigma = S^{-1}(\nabla_Z X_\sigma, \nabla_Z Y_\sigma)
\]
where \(\nabla\) is the Levi-Civita connection on \(M\).

**Proposition 2.10.** If \((U, x^i)\) is a chart on \(M\) and \((\sigma^i, \bar{\sigma}^i)\) are the components of section \(\sigma \in \Gamma(T^2M)\), then

\[
X_\sigma = \sigma^i \frac{\partial}{\partial x^i},
\]
(22)
\[
Y_\sigma = (\bar{\sigma}^k + \sigma^j \Gamma^i_{jk}) \frac{\partial}{\partial x^i}.
\]

**Proposition 2.11.** Let \((M,g)\) be a Riemannian manifold and \(T^2M\) be its tangent bundle of order two, then

\[
J: \Gamma(TM) \to \Gamma(T^2M)
\]
(23)
\[
Z \mapsto S^{-1}(Z, 0)
\]
is an injective homomorphism of vector bundles.

Locally if \((U; x^i)\) is a chart on \(M\) and \((U; x^i; y^j)\) and \((U; x^i; y^j; z^k)\) are the induced charts on \(TM\) and \(T^2M\). respectively, then we have

\[
J: (x^i, y^j) \mapsto (x^i, y^j, -y^j y^k \Gamma^i_{jk}).
\]
(24)

**Definition 2.12.** Let \((M,g)\) be a Riemannian manifold and \(X \in \Gamma(TM)\) be a vector field on \(M\). For \(\lambda = 0, 1, 2\), the \(\lambda\)-lift of \(X\) to \(T^2M\) is defined by

\[
X^0 = S_{-1}(X^H, X^H)
\]
(25)
\[
X^1 = S_{-1}(X^V, 0)
\]
(26)
\[ X^2 = S^{-1} \left( 0, X^V \right). \]

**Theorem 2.13** ([2]). Let \( (M, g) \) be a Riemannian manifold and \( R \) its tensor curvature, then for all vector fields \( X, Y \in \Gamma(TM) \) and \( p \in T^2M \), we have:
1. \([X^0, Y^0]_p = [X, Y]_p^0 - (R(X, Y)u)^1 - (R(X, Y)w)^2,\)
2. \([X^0, Y^1] = (\nabla_X Y)^1,\)
3. \([X^1, Y^j] = 0,\)
where \((u, w) = S(p)\) and \( i, j = 1, 2.\)

**Definition 2.14.** Let \((M, g)\) be a Riemannian manifold. For any section \( \sigma \in \Gamma(T^2M) \), we define the vertical lift of \( \sigma \) to \( T^2M \) by
\[ \sigma^V = S^{-1}(X^V_\sigma, Y^V_\sigma) \in \Gamma(T(T^2M)). \]

**Remark 2.15.** From Definition 2.7 and the formulae (14), (23) and (28), we obtain
\[ \sigma^V = X^1_\sigma + Y^2_\sigma, \]
\[ (\hat{\nabla}_Z \sigma)^V = (\nabla_Z X_\sigma)^1 + (\nabla_Z Y_\sigma)^2, \]
\[ Z^1 = J(Z)^V, \]
\[ Z^2 = i(Z)^V \]
for all \( \sigma \in \Gamma(T^2M) \) and \( Z \in \Gamma(TM) \).

### 2.3. Diagonal metric

**Theorem 2.16** ([3]). Let \((M, g)\) be a Riemannian manifold and \( TM \) its tangent bundle equipped with the Sasakian metric \( g^s \), then
\[ g^D = S^{-1}(\tilde{g}, \tilde{g}) \]
is the only metric that satisfies the following formulae
\[ g^D(X^i, Y^j) = \delta_{ij} \cdot g(X, Y) \circ \pi_2 \]
for all vector fields \( X, Y \in \Gamma(TM) \) and \( i, j = 0, \ldots, 2 \), where \( \tilde{g} \) is the metric defined by
\[ \tilde{g}(X^H, Y^H) = \frac{1}{2}g^s(X^H, Y^H), \]
\[ \tilde{g}(X^H, Y^V) = g^s(X^H, Y^V), \]
\[ \tilde{g}(X^V, Y^V) = g^s(X^V, Y^V). \]

\( g^D \) is called the diagonal lift of \( g \) to \( T^2M \).

**Proposition 2.17.** Let \((M, g)\) be a Riemannian manifold and \( \hat{\nabla} \) be the Levi-Civita connection of the tangent bundle of order two equipped with the diagonal metric \( g^D \). Then:
1. \((\nabla_{X^0} Y^0)_p = (\nabla_X Y)^0 - \frac{1}{2}(R(X, Y)u)^1 - \frac{1}{2}(R(X, Y)w)^2,\)
2. \((\nabla_{X^0} Y^1)_p = (\nabla_X Y)^1 + \frac{1}{2}(R(u, Y)X)^0,\)
3. \((\nabla_{X^0} Y^2)_p = (\nabla_X Y)^2 + \frac{1}{2}(R(w, Y)X)^0,\)
4. \((\tilde{\nabla}_X Y)_p = \frac{1}{2} (R_u (u, X) Y)_p^0\),
5. \((\tilde{\nabla}_X Y)_p = \frac{1}{2} (R_w (w, X) Y)_p^0\),
6. \((\tilde{\nabla}_X Y)_p = 0\)
for all vector fields \(X, Y \in \Gamma(TM)\) and \(p \in \Gamma(T^2M)\), where \(i, j = 1, 2\) and \((u, w) = S(p)\).

3. Biharmonicity Of section

3.1. The Curvature Tensor

**Definition 3.1.** Let \((M, g)\) be a Riemannian manifold and \(F: TM \rightarrow TM\) be a smooth bundle endomorphism of \(TM\). For \(\lambda = 0, 1, 2\), the \(\lambda\)-lift of \(F\) to \(T^2M\) is defined by
\[
F^0 = S^{-1}_x (HF, HF),
F^1 = S^{-1}_x (VF, 0),
F^2 = S^{-1}_x (0, VF).
\]

From Proposition 2.17, we obtain the following lemma.

**Lemma 3.2.** Let \(F: TM \rightarrow TM\) be a smooth bundle endomorphism of \(TM\), then we have
\[
(\tilde{\nabla}_X F^0)_p = F(X)_p^0 + \frac{1}{4} (R(u, X) F(u))_p^0,
(\tilde{\nabla}_X F^0)_p = F(X)_p^0 + \frac{1}{4} (R(w, X) F(w))_p^0,
(\tilde{\nabla}_X F^j)_p = F(X)_p^j \quad i, j = 1, 2,
(\tilde{\nabla}_X F^1)_p = (\nabla X F)_p^1 + \frac{1}{4} (R(u, F_x (u)) X)_p^0,
(\tilde{\nabla}_X F^2)_p = (\nabla X F)_p^2 + \frac{1}{2} (R(w, F_x (w)) X)_p^0,
(\tilde{\nabla}_X F^0)_p = (\nabla X F)_p^0 - \frac{1}{2} (R(X, F_x (u)) u)_p^1 - \frac{1}{2} (R(X, F_x (w)) w)_p^2
\]
for any \(p \in T^2M\), \(i, j = 1, 2\) and \(X \in \Gamma(TM)\).

Using the formula of curvature and Lemma 3.2, we have the following.

**Proposition 3.3.** Let \(R\) be a curvature tensor of \((M, g)\), and \(\bar{R}\) be curvature tensor of \((T^2M, g^D)\) equipped with the diagonal lift of \(g\). Then we have the following
\[
1. \bar{R}(X^0, Y^0) Z^0 = \left( \bar{R}(X, Y) Z + \frac{1}{4} R(u, R(Z, Y) u) X + \frac{1}{4} R(w, R(Z, Y) w) X \right)_p^0
+ \left( \frac{1}{4} R(u, R(X, Z) u) Y + \frac{1}{4} R(w, R(X, Z) w) Y \right)_p^0
+ \left( \frac{1}{2} R(u, R(X, Y) u) Z + \frac{1}{2} R(w, R(X, Y) w) Z \right)_p^0
\]
\[ + \frac{1}{2} \left( \nabla_Z R(X, Y) u \right)^2, \]

2. \( \tilde{R}(X^0, Y^0) Z^i = \left( R(X, Y) Z + \frac{1}{4} R(R(u, Z) Y, X) u + \frac{1}{4} R(R(w, Z) Y, X) w \right) \]

\[ - \frac{1}{4} R(R(u, Z) X, Y) u - \frac{1}{4} R(R(w, Z) X, Y) w \right)^i \]

\[ + \frac{1}{2} \left( \nabla_X R(u, Z) Y + \nabla_X R(w, Z) Y - \nabla_Y R(u, Z) X \right) \]

\[ - \left( \nabla_Y R(w, Z) X \right), \]

3. \( \tilde{R}(X^1, Y^1) Z^0 = \left( \frac{1}{4} R(R(u, X) R(u, Y) Z + \frac{1}{4} R(w, X) R(w, Y) Z \right) \]

\[ - \frac{1}{4} R(R(u, Y) R(u, X) Z - \frac{1}{4} R(w, Y) R(w, X) Z) \right) \]

4. \( \tilde{R}(X^i, Y^2) Z^0 = \left( \frac{1}{4} R(R(u, X) R(u, Y) Z + \frac{1}{4} R(w, X) R(w, Y) Z \right) \]

\[ - \frac{1}{4} R(R(u, Y) R(u, X) Z - \frac{1}{4} R(w, Y) R(w, X) Z) \right) \]

5. \( \tilde{R}(X^i, Y^0) Z^0 = \left( \frac{1}{4} R(R(u, X) Z, X) u + \frac{1}{4} R(w, Y) Z, X) w + \frac{1}{2} R(X, Z) Y \right) \]

\[ + \frac{1}{2} \left( \nabla_X R(u, Y) Z + \nabla_X R(w, Y) Z \right), \]

6. \( \tilde{R}(X^i, Y^2) Z^i = \tilde{R}(X^1, Y^1) Z^i = \tilde{R}(X^2, Y^2) Z^i = 0 \)

Lemma 3.4. Let \((M, g)\) be a Riemannian manifold and \(T^2 M\) be the tangent bundle equipped with the diagonal metric. If \(Z \in \Gamma(TM)\) and \(\sigma \in \Gamma(T^2 M)\), then

\[ d_x \sigma(Z_x) = Z^0_p + \left( \nabla_Z \sigma \right)_p^V, \]

where \(p = \sigma(x)\).

Proposition 3.5 ([3]). Let \((M, g)\) be a Riemannian manifold and \(T^2 M\) be its tangent bundle of order two equipped with the diagonal metric. Then the tension field associated with \(\sigma \in \Gamma(T^2 M)\) is

\[ \tau(\sigma) = (\text{trace}_g \nabla^2 X_\sigma)^1 + (\text{trace}_g \nabla^2 Y_\sigma)^2 \]

\[ + \left( \text{trace}_g (R(X_\sigma, \nabla_\sigma X_\sigma) \ast + R(\gamma_\sigma, \nabla_\sigma \gamma_\sigma) \ast) \right)^0 \]

\[ = (\text{trace}_g \nabla^2 \sigma)^V + \left( \text{trace}_g (R(X_\sigma, \nabla_\sigma X_\sigma) \ast + R(\gamma_\sigma, \nabla_\sigma \gamma_\sigma) \ast) \right)^0, \]

where \(\text{trace}_g \nabla^2\) (resp., \(\text{trace}_g \nabla^2\)) denotes the Laplacian attached to \(\nabla\) (resp., \(\nabla\)).
4. Biharmonicity of Section $\sigma: (M, g) \rightarrow (T^2M, g^D)$

For a section $\sigma \in \Gamma(T^2M)$, we denote

\[
\tau^0(\sigma) = \tau^0(X_\sigma) + \tau^0(Y_\sigma),
\]
\[
\tau^V(\sigma) = \tau^1(X_\sigma) + \tau^2(Y_\sigma),
\]
\[
\varpi^0(\sigma) = \left(\tau^0(X_\sigma) + \tau^0(Y_\sigma)\right)^0,
\]
\[
\varpi^V(\sigma) = \left(\tau^1(X_\sigma)\right)^1 + \left(\tau^2(Y_\sigma)\right)^2,
\]

where
\[
\tau^0(X_\sigma) = \text{trace}_g(R(X_\sigma, \nabla X_\sigma)\ast),
\]
\[
\tau^0(Y_\sigma) = \text{trace}_g(R(Y_\sigma, \nabla Y_\sigma)\ast),
\]
\[
\tau^1(X_\sigma) = \text{trace}_g \nabla^2 X_\sigma,
\]
\[
\tau^2(Y_\sigma) = \text{trace}_g \nabla^2 Y_\sigma.
\]

From these notations, we have

\[
\tau(\sigma) = \varpi^V + \varpi^0.
\]

**Theorem 4.1.** Let $(M, g)$ be a Riemannian compact manifold and $(T^2M, g^D)$ be its tangent bundle of order two equipped with the diagonal metric and a vector bundle structure via the diffeomorphism $S$ between $T^2$ and $TM \oplus TM$. Then $\sigma: M \rightarrow T^2M$ is a biharmonic section if and only if $\sigma$ is harmonic.

**Proof.** First, if $\sigma$ is harmonic, then from Corollary 1.4, we deduce that $\sigma$ is biharmonic.

Conversely, assuming that $\sigma$ is biharmonic. Let $\sigma_t$ be a compactly supported variation of $\sigma$ defined by $\sigma_t = (1 + t)\sigma$. Using Proposition 2.8, we have

\[
X_{\sigma_t} = (1 + t)X_\sigma \quad \text{and} \quad Y_{\sigma_t} = (1 + t)Y_\sigma.
\]

Substituting (37) in (32) to (35), we obtain

\[
\tau^0(\sigma_t) = (1 + t)^2 \tau^0(\sigma) \quad \text{and} \quad \tau^V(\sigma_t) = (1 + t)\tau^V(\sigma)
\]
\[
\varpi^0(\sigma_t) = (1 + t)^2 \varpi^0(\sigma) \quad \text{and} \quad \varpi^V(\sigma_t) = (1 + t)\varpi^V(\sigma).
\]

Then
\[
E_2(\sigma_t) = \frac{1}{2} \int |\tau(\sigma_t)|^2 g^D v_g = \frac{1}{2} \int |\tau^0(\sigma_t)|^2 g^D v_g + \frac{1}{2} \int |\tau^V(\sigma_t)|^2 g^D v_g = \frac{(1 + t)^4}{2} \int |\varpi^0(\sigma)|^2 g^D v_g + \frac{(1 + t)^2}{2} \int |\varpi^V(\sigma)|^2 g^D v_g.
\]

Since the section $\sigma$ is biharmonic, then for the variation $\sigma_t$, we have
\[
0 = \frac{d}{dt}E_2(\sigma_t)|_{t=0} = 2 \int |\varpi^0(\sigma)|^2 g^D v_g + \int |\varpi^V(\sigma)|^2 g^D v_g.
\]

Hence

\[
\varpi^0(\sigma) = 0 \quad \text{and} \quad \varpi^V(\sigma) = 0, \quad \text{then} \quad \tau(\sigma) = 0.
\]

□
In the case where \( M \) is not compact, the characterization of biharmonic sections requires the following two lemmas.

**Lemma 4.2.** Let \( (M, g) \) be a Riemannian manifold and \( (T^2M, g^D) \) be its tangent bundle of order two equipped with the diagonal metric. If \( \sigma \in \Gamma(T^2M) \) is a smooth section, then the Jacobi tensor \( J_\sigma(\tau^V(\sigma)) \) is given by

\[
J_\sigma(\tau^V(\sigma)) = \left\{ \text{trace}_g \nabla^2(\tau^V(\sigma)) \right\}^V + \left\{ \text{trace}_g (R(u, \nabla_\sigma \tau^1(X_\sigma)) \ast + R(w, \nabla_\sigma \tau^2(Y_\sigma)) \ast + R(\tau^V(\sigma), \nabla_\sigma \sigma) \ast + \frac{1}{2} R(u, \tau^1(X_\sigma)) R(u, \nabla_\sigma \sigma) \ast + \frac{1}{2} R(w, \tau^2(Y_\sigma)) R(w, \nabla_\sigma \sigma) \ast \right\}^0.
\]

**Proof.** Let \( p \in T^2M \) and \( \{e_i\}_{i=1}^m \) be a local orthonormal frame on \( M \) such that \( (\nabla_{e_i} e_i)_x = 0 \). If we denote \( F_i(x, u, w) = \frac{1}{2} R(u, \tau^1(X_\sigma)) e_i + \frac{1}{2} R(w, \tau^2(Y_\sigma)) e_i, \) then we have

\[
(\nabla_{e_i} \tau^V(\sigma)_p) = \left( \nabla_{e_i} \tau^2(\tau^V(\sigma)) \right)_p = \left( \nabla_{e_i} \tau^1(\tau^V(\sigma)) \right)_p = \left( \nabla_{e_i} \tau^0(\tau^V(\sigma)) \right)_p = \left( \nabla_{e_i} \tau^V(\sigma) \right)_p + (F_i(x, u, w))^0,
\]

hence

\[
(\text{trace}_g \nabla^2 \tau^V(\sigma)_p) = \sum_{i=1}^m \left\{ \nabla_{e_i} \tau^V(\sigma) \right\}_p = \sum_{i=1}^m \left\{ \nabla_{e_i} \tau^2(\tau^V(\sigma)) + (\nabla_{e_i} \tau^1(\tau^V(\sigma)) + (\nabla_{e_i} \tau^0(\tau^V(\sigma)) \right\}_p = \sum_{i=1}^m \left\{ \nabla_{e_i} \tau^2(\tau^V(\sigma)) + (F_i)^0 \right\}_p.
\]

Using Proposition 2.17, we obtain

\[
(\text{trace}_g \nabla^2 \tau^V(\sigma)_p) = \sum_{i=1}^m \left\{ (\nabla_{e_i} \nabla_{e_i} \tau^1(X_\sigma)) - \frac{1}{4} R(e_i, R(u, \tau^1(X_\sigma)) e_i) u \right\}^1_p + \sum_{i=1}^m \left\{ (\nabla_{e_i} \nabla_{e_i} \tau^2(Y_\sigma)) - \frac{1}{4} R(e_i, R(w, \tau^2(Y_\sigma)) e_i) w \right\}^2_p + \sum_{i=1}^m \left\{ \frac{1}{2} R(u, \nabla_{e_i} \tau^1(X_\sigma)) e_i + \frac{1}{2} R(u, \nabla_{e_i} \tau^1(X_\sigma)) e_i + \frac{1}{2} R(w, \nabla_{e_i} \tau^2(Y_\sigma)) e_i + \frac{1}{2} R(w, \nabla_{e_i} \tau^2(Y_\sigma)) e_i + \frac{1}{2} R(u, \nabla_{e_i} X_\sigma) R(u, \nabla_{e_i} X_\sigma) + \frac{1}{2} R(u, \nabla_{e_i} X_\sigma) R(u, \nabla_{e_i} X_\sigma) + \frac{1}{2} R(\nabla_{e_i} X_\sigma, \tau^2(Y_\sigma)) e_i \right\}^0.
\]
From proposition 3.3, we have

\[
\text{trace}_g(\tilde{R}(\tau^V(\sigma), d\sigma)d\sigma) = \sum_{i=1}^{m} \left\{ \tilde{R}(\tau^1(X_\sigma))^1, e_i^0, e_i^0 + \tilde{R}(\tau^1(X_\sigma))^1, (\nabla_{e_i} X_\sigma)^1) e_i^0 \right. \\
+ \tilde{R}(\tau^1(X_\sigma))^1, (\nabla_{e_i} Y_\sigma)^2) e_i^0 + \tilde{R}(\tau^1(X_\sigma))^1, e_i^0)(\nabla_{e_i} X_\sigma)^1 \\
+ \tilde{R}(\tau^1(X_\sigma))^1, e_i^0)(\nabla_{e_i} Y_\sigma)^2 + \tilde{R}(\tau^2(Y_\sigma))^2, e_i^0) e_i^0 \\
+ \tilde{R}(\tau^2(Y_\sigma))^2, (\nabla_{e_i} X_\sigma)^1) e_i^0 + \tilde{R}(\tau^2(Y_\sigma))^2, (\nabla_{e_i} Y_\sigma)^2) e_i^0 \\
+ \tilde{R}(\tau^2(Y_\sigma))^2, e_i^0)(\nabla_{e_i} X_\sigma)^1 + \tilde{R}(\tau^2(Y_\sigma))^2, e_i^0)(\nabla_{e_i} Y_\sigma)^2 \}. 
\]

By calculating at point \( p \in T^2M \), we obtain

\[
\text{trace}_g(\tilde{R}(\tau^V(\sigma), d\sigma)d\sigma)_p = \sum_{i=1}^{m} \left\{ -\frac{1}{4} R(R(u, \tau^1(X_\sigma))e_i, e_i)u \right\} - \left\{ \frac{1}{4} R(R(w, \tau^2(Y_\sigma))e_i, e_i)w \right\} \\
+ \sum_{i=1}^{m} \left\{ R(\tau^1(X_\sigma), \nabla_{e_i} X_\sigma)e_i + R(\tau^2(Y_\sigma), \nabla_{e_i} Y_\sigma)e_i \\
+ \frac{1}{4} R(R(u, \tau^1(X_\sigma)) R(u, \nabla_{e_i} X_\sigma)e_i - \frac{1}{4} R(R(w, \nabla_{e_i} Y_\sigma) R(w, \tau^2(Y_\sigma))e_i \\
+ \frac{1}{4} R(R(w, \tau^2(Y_\sigma)) R(w, \nabla_{e_i} Y_\sigma)e_i - \frac{1}{4} R(R(u, \nabla_{e_i} X_\sigma) R(u, \tau^1(X_\sigma))e_i \\
+ \frac{1}{2} R(\tau^1(X_\sigma), \nabla_{e_i} X_\sigma)e_i + \frac{1}{2} R(\tau^2(Y_\sigma), \nabla_{e_i} Y_\sigma)e_i \\
+ \frac{1}{4} R(R(u, \tau^1(X_\sigma)) R(u, \nabla_{e_i} X_\sigma)e_i + \frac{1}{4} R(R(u, \tau^2(Y_\sigma)) R(w, \nabla_{e_i} Y_\sigma)e_i \\
- \frac{1}{2} (\nabla_{e_i} R(u, \tau^1(X_\sigma))e_i - \frac{1}{2} (\nabla_{e_i} R(w, \tau^2(Y_\sigma))e_i \right\}^0. 
\]

Considering the formula (8), we deduce

\[
J_\sigma(\tau^V(\sigma)) = \left\{ \text{trace}_g \nabla(\tau^V(\sigma)) \right\}^V + \left\{ \text{trace}_g (R(u, \nabla_{\tau^1(X_\sigma)}) + R(w, \nabla_{\tau^2(Y_\sigma)}) + R(u, \tau^1(X_\sigma)) R(u, \nabla_{e_i} X_\sigma)e_i + \frac{1}{2} R(R(w, \tau^2(Y_\sigma)) R(w, \nabla_{e_i} Y_\sigma)e_i \right\}^0. 
\]

**Lemma 4.3.** Let \((M, g)\) be a Riemannian manifold and \((T^2M, g^D)\) be its tangent bundle of order two equipped with the diagonal metric. If \(\sigma \in \Gamma(T^2M)\) is a
smooth section, then the Jacobi tensor $J_\sigma(\tau^0(\sigma))$ is given by

\[ J_\sigma(\tau^0(\sigma))_p = \text{trace}_g \left\{ 2R(\tau^0(X_p), \ast)\nabla\ast \tau^0(X_p) - R(\ast, \nabla\ast \tau^0(X_p)) u + \frac{1}{2} R(R(u, \nabla\ast \nabla\ast \ast \tau^0(X_p)) u \right\}^1 \]

\[ + \text{trace}_g \left\{ 2R(\tau^0(Y_p), \ast)\nabla\ast \tau^0(Y_p) - R(\ast, \nabla\ast \tau^0(Y_p)) w + \frac{1}{2} R(R(w, \nabla\ast \ast \tau^0(Y_p)) w \right\}^2 \]

\[ + \text{trace}_g \left\{ \nabla\ast \tau^0(\ast) + R(u, \nabla\ast\nabla\ast X_p) \nabla\ast \tau^0(X_p) + R(w, \nabla\ast\nabla\ast \ast \tau^0(Y_p) \right. \]

\[ + \frac{1}{2} R(u, \nabla\ast \tau^0(X_p)) \nabla\ast \tau^0(X_p) + \frac{1}{2} R(w, \nabla\ast \tau^0(Y_p)) \nabla\ast \tau^0(Y_p) \]

\[ + R(u, R(\tau^0(Y_p), \ast) w) + R(\ast, \ast) \ast \ast \nabla\ast \tau^0(\ast) \ast w + \psi \ast \ast \nabla\ast (R(u, \nabla\ast X_p)) \ast w \]

\[ + (\nabla\ast X_p)(R(u, \nabla\ast Y_p)) \ast w \}_{p}^0 \]

for all $p = (x, u, w) \in T^2 M$.

**Proof.** Let $p = (x, u, w) \in T^2 M$ and $\{e_i\}_{i=1}^m$ be a local orthonormal frame on $M$ such that $(\nabla(e_i)e_i)_x = 0$, denoted by

\[ F_i = F_{iX} + F_{iY} = \frac{1}{2} R(e_i, \tau^0(X_p)) \ast + \frac{1}{2} R(e_i, \tau^0(Y_p)) \ast \]

\[ G = G_X + G_Y = \frac{1}{2} R(\ast, \nabla\ast X_p) \tau^0(X_p) + \frac{1}{2} R(\ast, \nabla\ast Y_p) \tau^0(Y_p). \]

First, using Lemma 3.4 and Proposition 2.17, we calculate

\[ \text{trace}_g \tilde{\nabla}^2(\tau^0(\sigma))_p = \sum_{i=1}^m \left\{ \tilde{\nabla}^0_e \tilde{\nabla}^0_e (\tau^0(\sigma))^0 \right\} \]

\[ = \sum_{i=1}^m \left\{ (\nabla\ast X_p) \nabla\ast \nabla\ast \ast \tau^0(\ast)^0 - F_{iX} - F_{iY} + G_i \right\}^p. \]

From Proposition 2.17, we have

\[ \text{trace}_g \tilde{\nabla}^2(\tau^0(\sigma))_p = \sum_{i=1}^m \left\{ (\nabla\ast X_p) \nabla\ast \tau^0(\ast)^0 + \frac{1}{2} R(u, \nabla\ast X_p) \nabla\ast \tau^0(X_p) \right\}

\[ + \frac{1}{2} R(w, \nabla\ast Y_p) \nabla\ast \tau^0(Y_p)^0 - (\nabla\ast X_p) \nabla\ast \tau^0(Y_p)^0 - (\nabla\ast X_p) \nabla\ast \tau^0(Y_p)^0 \]

\[ - \frac{1}{2} R(e_i, \nabla\ast X_p) \nabla\ast \tau^0(X_p) u - \left( \frac{1}{2} R(e_i, \nabla\ast Y_p) \nabla\ast \tau^0(Y_p) w \right) \]

\[ - \frac{1}{2} R(e_i, \nabla\ast X_p) F_{iX} - \left( \frac{1}{2} R(e_i, \nabla\ast Y_p) F_{iY} \right) \]

\[ - \frac{1}{2} R(e_i, G_{iX}(u)) - \left( \frac{1}{2} R(e_i, G_{iY}(w)) \right) \]

\[ + (G_{iX}(\nabla\ast X_p))^0 + (G_{iY}(\nabla\ast Y_p))^0 \]

\[ + \frac{1}{2} R(u, \nabla\ast X_p) G_{iX}(u) \]
Substituting (40) and (41) in (42), we arrive at

\[
\text{trace}_p \nabla^2 (\tau^0(\sigma))_p \\
= \sum_{i=1}^{m} \left\{ \nabla_{e_i} \nabla_{e_i} \tau^0(\sigma) + R(u, \nabla_{e_i} X_{\sigma}) \nabla_{e_i} \tau^0(\sigma) \right\}
+ \frac{1}{2} R(u, \nabla_{e_i} \nabla_{e_i} X_{\sigma}) \tau^0(\sigma)
+ \frac{1}{2} R(u, \nabla_{e_i} \nabla_{e_i} X_{\sigma}) \tau^0(\sigma) + \frac{1}{4} R(u, \nabla_{e_i} X_{\sigma}) \tau^0(\sigma)
+ \frac{1}{4} R(u, \nabla_{e_i} X_{\sigma}) \tau^0(\sigma) - \frac{1}{4} R(u, \nabla_{e_i} X_{\sigma}) \tau^0(\sigma) u e_i
\]

(43)

On the other hand, we have

\[
\text{trace}_p \left\{ R(\tau^0(\sigma), \omega \sigma) d\sigma \right\}_p \\
= \sum_{i=1}^{m} \left\{ R(\tau^0(\sigma), e_i) e_i + \frac{3}{4} R(u, R(\tau^0(X_{\sigma}), e_i) u) e_i \right\}
+ \frac{3}{4} R(u, R(\tau^0(\sigma), e_i) e_i + \nabla_{e_i} (X_{\sigma} R)(u, \nabla_{e_i} X_{\sigma}) e_i
+ \nabla_{e_i} (X_{\sigma} R)(u, \nabla_{e_i} X_{\sigma}) e_i - \frac{1}{2} (\nabla_{e_i} R)(u, \nabla_{e_i} X_{\sigma}) \tau^0(\sigma)
+ \frac{1}{4} R(u, \nabla_{e_i} X_{\sigma}) \tau^0(\sigma) \tau^0(\sigma) u e_i
+ \frac{1}{4} R(u, \nabla_{e_i} X_{\sigma}) \tau^0(\sigma) \tau^0(\sigma) u e_i
\]

(44)
By summing (43) and (44), the proof of Lemma 4.3 is completed.

From Lemma 4.2 and 4.3, we deduce the following theorems

**Theorem 4.4.** Let $(M, g)$ be a Riemannian manifold and $(T^2M, g^D)$ be its tangent bundle of order two equipped with the diagonal metric. If $\sigma: M \to T^2M$ is a smooth section, then the bitension field of $\sigma$ is given by

$$
\tau_2(\sigma)_p = \text{trace}_g \left\{ \nabla^2 \tau^1(X_\sigma) + 2R(\tau^0(X_\sigma), \ast)\nabla_\ast X_\sigma - R(\ast, \nabla_\ast \tau^0(X_\sigma))u \\
+ \frac{1}{2}R(R(u, \nabla_\ast \ast, \tau^0(X_\sigma))u \right\}_p
$$

for all $p \in T^2M$.

**Theorem 4.5.** Let $(M, g)$ be a Riemannian manifold and $(T^2M, g^D)$ be its tangent bundle of order two equipped with the diagonal metric. A section $\sigma: M \to T^2M$ is biharmonic if and only if the following conditions are verified:

1) $0 = \text{trace}_g \left\{ \nabla^2 \tau^1(X_\sigma) + 2R(\tau^0(X_\sigma), \ast)\nabla_\ast X_\sigma - R(\ast, \nabla_\ast \tau^0(X_\sigma))u \\
+ \frac{1}{2}R(R(u, \nabla_\ast \ast, \tau^0(X_\sigma))u \right\}_p$,

2) $0 = \text{trace}_g \left\{ \nabla^2 \tau^2(Y_\sigma) + 2R(\tau^0(Y_\sigma), \ast)\nabla_\ast Y_\sigma - R(\ast, \nabla_\ast \tau^0(Y_\sigma))w \\
+ \frac{1}{2}R(R(w, \nabla_\ast \ast, \tau^0(Y_\sigma))w \right\}_p$.
3) \[ 0 = \text{trace}_g \left\{ R(u, \nabla_x \tau^1(X_\sigma)) \ast + R(w, \nabla_x \tau^2(Y_\sigma)) \ast + R(\tau^1(X_\sigma), \nabla_x X_\sigma) \ast + R(\tau^2(Y_\sigma), \nabla_x Y_\sigma) \ast + \frac{1}{2} R(u, \tau^1(X_\sigma)) R(u, \nabla_x X_\sigma \ast) + \frac{1}{2} R(w, \tau^2(Y_\sigma)) R(w, \nabla_x Y_\sigma) \ast + \nabla_x \nabla \tau^0(\sigma) + R(u, \nabla_x X_\sigma) \nabla_x \tau^0(X_\sigma) + R(w, \nabla_x Y_\sigma) \nabla_x \tau^0(Y_\sigma) \right\} \]

\[ + \frac{1}{2} R(u, \nabla_x \tau^0(X_\sigma)) R(u, \nabla_x \tau^0(Y_\sigma)) \ast + \frac{1}{2} R(w, \nabla_x \tau^0(Y_\sigma)) R(w, \nabla_x \tau^0(Y_\sigma)) \ast + R(u, R(\tau^0(X_\sigma), \ast) w) \ast + R(w, R(\tau^0(Y_\sigma), \ast) w) \ast + R(\tau^0(\sigma), \ast) \ast + (\nabla \tau^0(\sigma) R)(u, \nabla_x X_\sigma) \ast + (\nabla \tau^0(\sigma) R)(w, \nabla_x Y_\sigma) \ast \right\}_p \]

for all \( p = S^{-1}(x, u, w) \in T^2 M \).

**Corollary 4.6.** Let \((M, g)\) be a Riemannian manifold and \((T^2 M, g^D)\) be its tangent bundle of order two equipped with the diagonal metric. If \( \sigma : M \to T^2 M \) is a section such that \( X_\sigma \) and \( Y_\sigma \) are biharmonic vector fields, then \( \sigma \) is biharmonic.

(For biharmonic vector see [1]).

**References**


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