Acta Math. Univ. Comenianae
Vol. LXXXIII, 2 (2014), pp. 165–179

TANGENT BUNDLE OF ORDER TWO AND BIHARMONICITY

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Abstract. The problem studied in this paper is related to the biharmonicity of a section from a Riemannian manifold (M, g) to its tangent bundle T^2M of order two equipped with the diagonal metric g^D. We show that a section on a compact manifold is biharmonic if and only if it is harmonic. We also investigate the curvature of (T^2M, g^D) and the biharmonicity of section of M as a map from (M, g) to (T^2M, g^D).

1. Introduction

Harmonic (resp., biharmonic) maps are critical points of energy (resp., bienergy) functional defined on the space of smooth maps between Riemannian manifolds introduced by Eells and Sampson [**4**] (resp., Jiang [**6**]). In this paper, we present some properties for biharmonic section between a Riemannian manifold and its second tangent bundle which generalize the results of Ishihara [**5**], Konderak [**7**], Oproiu [**9**] and Djaa-Ouakkas [**3**].

Consider a smooth map $\phi\colon (M^n, g) \to (N^n, h)$ between two Riemannian manifolds, then the energy functional is defined by

$$E(\phi) = \frac{1}{2}\int_M |\mathrm{d}\phi|^2 \mathrm{d}v_g \tag{1}$$

(or over any compact subset $K \subset M$).

A map is called harmonic if it is a critical point of the energy functional E (or $E(K)$ for all compact subsets $K \subset M$). For any smooth variation $\{\phi\}_{t\in I}$ of ϕ with $\phi_0 = \phi$ and $V = \frac{\mathrm{d}\phi_t}{\mathrm{d}t}|_{t=0}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E(\phi_t)|_{t=0} = -\frac{1}{2}\int_M h(\tau(\phi), V)\mathrm{d}v_g, \tag{2}$$

where

$$\tau(\phi) = \mathrm{tr}_g \nabla \mathrm{d}\phi \tag{3}$$

is the tension field of ϕ. Then we have the following theorem.

Received February 24, 2013; revised January 28, 2014.
2010 *Mathematics Subject Classification.* Primary 53A45, 53C20, 58E20.
Key words and phrases. Diagonal metric; λ-lift; biharmonic section.
The authors was supported by LGACA and GMFAMI Laboratories.
The authors would like to thank the referee for his useful remarks and suggestions.

Theorem 1.1. *A smooth map* $\phi\colon (M^m, g) \to (N^n, h)$ *is harmonic if and only if*

$$\tau(\phi) = 0. \tag{4}$$

If $(x^i)_{1\leq i\leq m}$ and $(y^\alpha)_{1\leq\alpha\leq n}$ denote local coordinates on M and N, respectively, then equation (4) takes the form

$$\tau(\phi)^\alpha = \left(\Delta\phi^\alpha + g^{ij}\, \overset{N}{\Gamma}{}^\alpha_{\beta\gamma}\, \frac{\partial\phi^\beta}{\partial x^i}\frac{\partial\phi^\gamma}{\partial x^j}\right) = 0, \tag{5}$$

where $\Delta\phi^\alpha = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}(\sqrt{|g|}g^{ij}\frac{\partial\phi^\alpha}{\partial x^j})$ is the Laplace operator on (M^m, g) and $\overset{N}{\Gamma}{}^\alpha_{\beta\gamma}$ are the Christoffel symbols on N.

Definition 1.2. A map $\phi\colon (M, g) \to (N, h)$ between Riemannian manifolds is called biharmonic if it is a critical point of bienergie functional

$$E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 \mathrm{d}v^g. \tag{6}$$

The Euler-Lagrange equation attached to bienergy is given by vanishing of the bitension field

$$\tau_2(\phi) = -J_\phi(\tau(\phi)) = -(\Delta^\phi\tau(\phi) + \mathrm{tr}_g\, R^N(\tau(\phi), \mathrm{d}\phi)\mathrm{d}\phi), \tag{7}$$

where J_ϕ is the Jacobi operator defined by

$$\begin{aligned} J_\phi\colon \Gamma(\phi^{-1}(TN)) &\to \Gamma(\phi^{-1}(TN)) \\ V &\mapsto \Delta^\phi V + \mathrm{tr}_g\, R^N(V, \mathrm{d}\phi)\mathrm{d}\phi. \end{aligned} \tag{8}$$

Theorem 1.3. *A smooth map* $\phi\colon (M^m, g) \to (N^n, h)$ *is biharmonic if and only if*

$$\tau_2(\phi) = 0. \tag{9}$$

From Theorem 1.1 and formula (7), we have the following corollary.

Corollary 1.4. *If* $\phi\colon (M^m, g) \to (N^n, h)$ *is harmonic, then* ϕ *is biharmonic.*

(For more details see [**6**]).

2. Preliminary Notes

2.1. Horizontal and vertical lifts on TM

Let (M, g) be an n-dimensional Riemannian manifold and (TM, π, M) be its tangent bundle. A local chart $(U, x^i)_{i=1\ldots n}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^j)_{i,j=1,\ldots,n}$ on TM. Denote the Christoffel symbols of g by Γ^k_{ij} and the Levi-Civita connection of g by ∇.

We have two complementary distributions on TM, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$ defined by

$$\begin{aligned} \mathcal{V}_{(x,u)} &= \mathrm{Ker}(\mathrm{d}\pi_{(x,u)}) \\ &= \big\{a^i \frac{\partial}{\partial y^i}\big|_{(x,u)};\quad a^i \in \mathbb{R}\big\}, \\ \mathcal{H}_{(x,u)} &= \big\{a^i \frac{\partial}{\partial x^i}\big|_{(x,u)} - a^i u^j \Gamma^k_{ij}\frac{\partial}{\partial y^k}\big|_{(x,u)};\quad a^i \in \mathbb{R}\big\}, \end{aligned}$$

where $(x,u) \in TM$, such that $T_{(x,u)}TM = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}$.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$X^V = X^i \frac{\partial}{\partial y^i} \tag{10}$$

$$X^H = X^i \frac{\delta}{\delta x^i} = X^i \Big\{ \frac{\partial}{\partial x^i} - y^j \Gamma_{ij}^k \frac{\partial}{\partial y^k} \Big\}. \tag{11}$$

For consequences, we have:

1. $\Big(\frac{\partial}{\partial x^i}\Big)^H = \frac{\delta}{\delta x^i}$ and $\Big(\frac{\partial}{\partial x^i}\Big)^V = \frac{\partial}{\partial y^i}$.
2. $\Big(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\Big)_{i,j=1,\dots,n}$ is a local frame on TM
3. If $u = u^i \frac{\partial}{\partial x^i} \in T_xM$, then $u^H = u^i \Big\{ \frac{\partial}{\partial x^i} - y^j \Gamma_{ij}^k \frac{\partial}{\partial y^k} \Big\}$ and $u^V = u^i \frac{\partial}{\partial y^i}$.

Definition 2.1. Let (M,g) be a Riemannian manifold and $F\colon TM \to TM$ be a smooth bundle endomorphism of TM. Then we define a vertical and horizontal vector fields VF, HF on TM by

$$\begin{aligned} VF\colon TM &\to TTM \\ (x,u) &\mapsto (F(u))^V, \end{aligned}$$

$$\begin{aligned} HF\colon TM &\to TTM \\ (x,u) &\mapsto (F(u))^H. \end{aligned}$$

Locally we have

$$VF = y^i F_i^j \frac{\partial}{\partial y^j} = y^i \Big(F(\frac{\partial}{\partial x^i}) \Big)^V \tag{12}$$

$$HF = y^i F_i^j \frac{\partial}{\partial x^j} - y^i y^k F_i^j \Gamma_{jk}^s \frac{\partial}{\partial y^s} = y^i \Big(F(\frac{\partial}{\partial x^i}) \Big)^H. \tag{13}$$

Proposition 2.2 ([**1**])**.** *Let (M,g) be a Riemannian manifold and $\widehat{\nabla}$ be the Levi-Civita connection of the tangent bundle (TM, g^s) equipped with the Sasaki metric. If F is a tensor field of type $(1,1)$ on M, then*

$$\begin{aligned} (\widehat{\nabla}_{X^V} VF)_{(x,u)} &= (F(X))^V_{(x,u)}, \\ (\widehat{\nabla}_{X^V} HF)_{(x,u)} &= (F(X))^H_{(x,u)} + \frac{1}{2}(R_x(u, X_x)F(u))^H, \\ (\widehat{\nabla}_{X^H} VF)_{(x,u)} &= V(\nabla_X F)(x,u) + \frac{1}{2}(R_x(u, F_x(u))X_x)^H, \\ (\widehat{\nabla}_{X^H} HF)_{(x,u)} &= H(\nabla_X F)(x,u) - \frac{1}{2}(R_x(X_x, F_x(u))u)^V, \end{aligned}$$

where $(x,u) \in TM$ and $X \in \Gamma(TM)$.

2.2. Second Tangent Bundle

Let M be an n-dimensional smooth differentiable manifold and $(U_\alpha, \psi_\alpha)_{\alpha\in I}$ a corresponding atlas. For each $x \in M$, we define an equivalence relation on

$$C_x = \{\gamma\colon (-\varepsilon, \varepsilon) \to M \ ; \ \gamma \text{ is smooth and } \gamma(0) = x, \varepsilon > 0\}$$

by

$$\gamma \approx_x h \Leftrightarrow \gamma'(0) = h'(0) \quad \text{and} \quad \gamma''(0) = h''(0),$$

where γ' and γ'' denote the first and the second derivation of γ, respectively,

$$\begin{aligned} \gamma'&\colon (-\varepsilon, \varepsilon) \to TM; && t \mapsto [d\gamma(t)](1) \\ \gamma''&\colon (-\varepsilon, \varepsilon) \to T(TM); && t \mapsto [d\gamma'(t)](1). \end{aligned}$$

Definition 2.3. We define the second tangent space of M at the point x to be the quotient $T_x^2 M = C_x/\approx_x$ and the second tangent bundle of M the union of all second tangent space, $T^2M = \bigcup_{x\in M} T_x^2 M$. We denote the equivalence class of γ by $j_x^2\gamma$ with respect to $\approx_x$, and by $j^2\gamma$ an element of T^2M.

In the general case, the structure of higher tangent bundle T^rM is considered in [**8**, Chapters 1–2] and [**2**].

Proposition 2.4 ([**3**]). *Let M be an n-dimensional manifold, then TM is subbundle of T^2M and the map*

$$\begin{aligned} i\colon TM &\to T^2M \\ j_x^1 f &= j_x^2 \tilde{f} \end{aligned} \tag{14}$$

is an injective homomorphism of natural bundles (not of vector bundles), where

$$\tilde{f}^i = \int_0^t f^i(s)\mathrm{d}s - tf^i(0) + f^i(0) \qquad i = 1 \ldots n. \tag{15}$$

Theorem 2.5. *Let (M, g) be a Riemannian manifold and ∇ be the Levi-Civita connection. If $TM \oplus TM$ denotes the Whitney sum, then*

$$\begin{aligned} S\colon T^2M &\to TM \oplus TM \\ j^2\gamma(0) &\mapsto (\dot\gamma(0), (\nabla_{\dot{\gamma(0)}}\dot\gamma)(0)) \end{aligned} \tag{16}$$

is a diffeomorphism of natural bundles. In the induced coordinate we have

$$(x^i; y^i; z^i) \mapsto (x^i, y^i, z^i + y^j y^k \Gamma^i_{jk}). \tag{17}$$

Remark 2.6. The diffeomorphism S determines a vector bundle structure on T^2M by

$$\alpha.\Psi_1 + \beta \cdot \Psi_2 = S^{-1}(\alpha S(\Psi_1) + \beta S(\Psi_2)),$$

where $\Psi_1, \Psi_2 \in T^2M$ and $\alpha, \beta \in \mathbb{R}$, for which S is a linear isomorphism of vector bundles and $i\colon TM \to T^2M$ is an injective linear homomorphism of vector bundles (for more details see [**2**]).

Definition 2.7 ([**3**]). Let (M,g) be a Riemannian manifold and T^2M be its tangent bundle of order two endowed with the vectorial structure induced by the diffeomorphism S. For any section $\sigma \in \Gamma(T^2M)$, we define two vector fields on M by

$$X_\sigma = P_1 \circ S \circ \sigma, \tag{18}$$

$$Y_\sigma = P_2 \circ S \circ \sigma, \tag{19}$$

where P_1 and P_2 denote the first and the second projections from $TM \oplus TM$ on TM.

From Remark 2.6 and Definition 2.7, we deduce the following.

Proposition 2.8. *For all sections $\sigma, \varpi \in \Gamma(T^2M)$ and $\alpha \in \mathbb{R}$, we have*

$$X_{\alpha\sigma+\varpi} = \alpha X_\sigma + X_\varpi,$$
$$Y_{\alpha\sigma+\varpi} = \alpha Y_\sigma + Y_\varpi,$$

where $\alpha\sigma + \varpi = S^{-1}(\alpha S(\sigma) + S(\varpi))$.

Definition 2.9 ([**3**]). Let (M,g) be a Riemannian manifold and T^2M be its tangent bundle of order two endowed with the vectorial structure induced by the diffeomorphism S. We define a connection $\widehat{\nabla}$ on $\Gamma(T^2M)$ by

$$\begin{aligned} \widehat{\nabla} \colon \Gamma(TM) \times \Gamma(T^2M) &\to \Gamma(T^2M) \\ (z,\sigma) &\mapsto \widehat{\nabla}_Z \sigma = S^{-1}(\nabla_Z X_\sigma, \nabla_Z Y_\sigma) \end{aligned} \tag{20}$$

where ∇ is the Levi-Civita connection on M.

Proposition 2.10. *If (U,x^i) is a chart on M and $(\sigma^i, \bar{\sigma}^i)$ are the components of section $\sigma \in \Gamma(T^2M)$, then*

$$X_\sigma = \sigma^i \frac{\partial}{\partial x^i} \tag{21}$$

$$Y_\sigma = (\bar{\sigma}^k + \sigma^i \sigma^j \Gamma^k_{ij}) \frac{\partial}{\partial x^k}\,. \tag{22}$$

Proposition 2.11. *Let (M,g) be a Riemannian manifold and T^2M be its tangent bundle of order two, then*

$$\begin{aligned} J \colon \Gamma(TM) &\to \Gamma(T^2M) \\ Z &\mapsto S^{-1}(Z,0) \end{aligned} \tag{23}$$

is an injective homomorphism of vector bundles.

Locally if $(U;x^i)$ is a chart on M and $(U;x^i;y^i)$ and $(U;x^i;y^i;z^i)$ are the induced charts on TM and T^2M. respectively, then we have

$$J \colon (x^i, y^i) \mapsto (x^i, y^i, -y^j y^k \Gamma^i_{jk}). \tag{24}$$

Definition 2.12. Let (M,g) be a Riemannian manifold and $X \in \Gamma(TM)$ be a vector field on M. For $\lambda = 0,1,2$, the λ-lift of X to T^2M is defined by

$$X^0 = S_*^{-1}(X^H, X^H) \tag{25}$$

$$X^1 = S_*^{-1}(X^V, 0) \tag{26}$$

$$X^2 = S_*^{-1}(0, X^V). \tag{27}$$

Theorem 2.13 ([**2**]). *Let (M, g) be a Riemannian manifold and R its tensor curvature, then for all vector fields $X, Y \in \Gamma(TM)$ and $p \in T^2M$, we have:*

1. $[X^0, Y^0]_p = [X, Y]^0_p - (R(X, Y)u)^1 - (R(X, Y)w)^2$,
2. $[X^0, Y^i] = (\nabla_X Y)^i$,
3. $[X^i, Y^j] = 0$,

where $(u, w) = S(p)$ and $i, j = 1, 2$.

Definition 2.14. Let (M, g) be a Riemannian manifold. For any section $\sigma \in \Gamma(T^2M)$, we define the vertical lift of σ to T^2M by

$$\sigma^V = S_*^{-1}(X_\sigma^V, Y_\sigma^V) \in \Gamma(T(T^2M)). \tag{28}$$

Remark 2.15. From Definition 2.7 and the formulae (14), (23) and (28), we obtain

$$\begin{aligned} \sigma^V &= X^1_\sigma + Y^2_\sigma, \\ (\widehat{\nabla}_Z \sigma)^V &= (\nabla_Z X_\sigma)^1 + (\nabla_Z Y_\sigma)^2, \\ Z^1 &= J(Z)^V, \\ Z^2 &= i(Z)^V \end{aligned}$$

for all $\sigma \in \Gamma(T^2M)$ and $Z \in \Gamma(TM)$.

2.3. Diagonal metric

Theorem 2.16 ([**3**]). *Let (M, g) be a Riemannian manifold and TM its tangent bundle equipped with the Sasakian metric g^s, then*

$$g^D = S_*^{-1}(\tilde{g}, \tilde{g})$$

is the only metric that satisfies the following formulae

$$g^D(X^i, Y^j) = \delta_{ij} \cdot g(X, Y) \circ \pi_2 \tag{29}$$

for all vector fields $X, Y \in \Gamma(TM)$ and $i, j = 0, \ldots, 2$, where $\tilde{g}$ is the metric defined by

$$\begin{aligned} \tilde{g}(X^H, Y^H) &= \frac{1}{2} g^s(X^H, Y^H), \\ \tilde{g}(X^H, Y^V) &= g^s(X^H, Y^V), \\ \tilde{g}(X^V, Y^V) &= g^s(X^V, Y^V). \end{aligned}$$

g^D is called the diagonal lift of g to T^2M.

Proposition 2.17. *Let (M, g) be a Riemannian manifold and $\widetilde{\nabla}$ be the Levi-Civita connection of the tangent bundle of order two equipped with the diagonal metric g^D. Then:*

1. $(\widetilde{\nabla}_{X^0} Y^0)_p = (\nabla_X Y)^0 - \frac{1}{2}(R(X, Y)u)^1 - \frac{1}{2}(R(X, Y)w)^2$,
2. $(\widetilde{\nabla}_{X^0} Y^1)_p = (\nabla_X Y)^1 + \frac{1}{2}(R(u, Y)X)^0$,
3. $(\widetilde{\nabla}_{X^0} Y^2)_p = (\nabla_X Y)^2 + \frac{1}{2}(R(w, Y)X)^0$,

4. $(\widetilde{\nabla}_{X^1}Y^0)_p = \frac{1}{2}(R_x(u,X)Y)^0$,
5. $(\widetilde{\nabla}_{X^2}Y^0)_p = \frac{1}{2}(R_x(w,X)Y)^0$,
6. $(\widetilde{\nabla}_{X^i}Y^j)_p = 0$

for all vector fields $X, Y \in \Gamma(TM)$ *and* $p \in \Gamma(T^2M)$*, where* $i, j = 1, 2$ *and* $(u, w) = S(p)$*.*

3. Biharmonicity Of section

3.1. The Curvature Tensor

Definition 3.1. Let (M, g) be a Riemannian manifold and $F\colon TM \to TM$ be a smooth bundle endomorphism of TM. For $\lambda = 0, 1, 2$, the λ-lift of F to T^2M is defined by

$$F^0 = S_*^{-1}(HF, HF),$$
$$F^1 = S_*^{-1}(VF, 0),$$
$$F^2 = S_*^{-1}(0, VF).$$

From Proposition 2.17, we obtain the following lemma.

Lemma 3.2. *Let* $F\colon TM \to TM$ *be a smooth bundle endomorphism of* TM*, then we have*

$$(\widetilde{\nabla}_{X^1}F^0)_p = F(X)^0_p + \frac{1}{2}(R(u,X)F(u))^0_p,$$
$$(\widetilde{\nabla}_{X^2}F^0)_p = F(X)^0_p + \frac{1}{2}(R(w,X)F(w))^0_p,$$
$$(\widetilde{\nabla}_{X^i}F^j)_p = F(X)^j_p \qquad i, j = 1, 2,$$
$$(\widetilde{\nabla}_{X^0}F^1)_p = (\nabla_X F)^1_p + \frac{1}{2}(R(u, F_x(u))X_x)^0,$$
$$(\widetilde{\nabla}_{X^0}F^2)_p = (\nabla_X F)^2_p + \frac{1}{2}(R(w, F_x(w))X_x)^0,$$
$$(\widetilde{\nabla}_{X^0}F^0)_p = (\nabla_X F)^0_p - \frac{1}{2}(R(X_x, F_x(u))u)^1 - \frac{1}{2}(R(X_x, F_x(w))w)^2$$

for any $p \in T^2M$, $\quad i, j = 1, 2 \quad$ *and* $\quad X \in \Gamma(TM)$*.*

Using the formula of curvature and Lemma 3.2, we have the following.

Proposition 3.3. *Let* R *be a curvature tensor of* (M, g)*, and* $\widetilde{R}$ *be curvature tensor of* (T^2M, g^D) *equipped with the diagonal lift of* g*. Then we have the following*

$$1.\ \widetilde{R}(X^0, Y^0)Z^0 = \Big(R(X,Y)Z + \frac{1}{4}R(u, R(Z,Y)u)X + \frac{1}{4}R(w, R(Z,Y)w)X\Big)^0$$
$$+ \Big(\frac{1}{4}R(u, R(X,Z)u)Y + \frac{1}{4}R(w, R(X,Z)w)Y\Big)^0$$
$$+ \Big(\frac{1}{2}R(u, R(X,Y)u)Z + \frac{1}{2}R(w, R(X,Y)w)Z\Big)^0$$

$$
\begin{aligned}
&&& + \frac{1}{2}\Big(\nabla_Z R)(X,Y)u\Big)^1 + \frac{1}{2}\Big(\nabla_Z R)(X,Y)w\Big)^2,\\
&2.\ \widetilde{R}(X^0,Y^0)Z^i &=& \Big(R(X,Y)Z + \frac{1}{4}R(R(u,Z)Y,X)u + \frac{1}{4}R(R(w,Z)Y,X)w\\
&&& - \frac{1}{4}R(R(u,Z)X,Y)u - \frac{1}{4}R(R(w,Z)X,Y)w\Big)^i\\
&&& + \frac{1}{2}\Big((\nabla_X R)(u,Z)Y + (\nabla_X R)(w,Z)Y - (\nabla_Y R)(u,Z)X\\
&&& - (\nabla_Y R)(w,Z)X\Big)^0,\\
&3.\ \widetilde{R}(X^1,Y^1)Z^0 &=& \Big(R(X,Y)Z + \frac{1}{4}R(u,X)R(u,Y)Z + \frac{1}{4}R(w,X)R(w,Y)Z\\
&&& - \frac{1}{4}R(u,Y)R(u,X)Z) - \frac{1}{4}R(w,Y)R(w,X)Z)\Big)^0,\\
&4.\ \widetilde{R}(X^i,Y^2)Z^0 &=& \Big(R(X,Y)Z + \frac{1}{4}R(u,X)R(u,Y)Z + \frac{1}{4}R(w,X)R(w,Y)Z\\
&&& - \frac{1}{4}R(u,Y)R(u,X)Z - \frac{1}{4}R(w,Y)R(w,X)Z\Big)^0,\\
&5.\ \widetilde{R}(X^i,Y^0)Z^0 &=& -\Big(\frac{1}{4}R(u,Y)Z,X)u + \frac{1}{4}R(w,Y)Z,X)w + \frac{1}{2}R(X,Z)Y\Big)^i\\
&&& + \frac{1}{2}\Big((\nabla_X R)(u,Y)Z + (\nabla_X R)(w,Y)Z\Big)^0,\\
&6.\ \widetilde{R}(X^i,Y^0)Z^j &=& \Big(\frac{1}{2}R(Y,Z)X + \frac{1}{4}R(u,Y)R(u,X)Z + \frac{1}{4}R(w,Y)R(w,X)Z\Big)^0\\
&7.\ \widetilde{R}(X^1,Y^2)Z^i &=& \widetilde{R}(X^1,Y^1)Z^i = \widetilde{R}(X^2,Y^2)Z^i = 0
\end{aligned}
$$

for any $\xi = (p,u,w) \in T^2M$, $\quad i,j = 1,2 \quad$ *and* $\quad X,Y,Z \in \Gamma(TM)$.

Lemma 3.4. *Let* (M,g) *be a Riemannian manifold and* T^2M *be the tangent bundle equipped with the diagonal metric. If* $Z \in \Gamma(TM)$ *and* $\sigma \in \Gamma(T^2M)$*, then*

$$d_x\sigma(Z_x) = Z^0_p + (\widehat{\nabla}_Z\sigma)^V_p, \tag{30}$$

where $p = \sigma(x)$.

Proposition 3.5 ([**3**])**.** *Let* (M,g) *be a Riemannian manifold and* T^2M *be its tangent bundle of order two equipped with the diagonal metric. Then the tension field associated with* $\sigma \in \Gamma(T^2M)$ *is*

$$
\begin{aligned}
\tau(\sigma) &= (\operatorname{trace}_g \nabla^2 X_\sigma)^1 + (\operatorname{trace}_g \nabla^2 Y_\sigma)^2\\
&\quad + \Big(\operatorname{trace}_g(R(X_\sigma,\nabla_* X_\sigma)* + R(Y_\sigma,\nabla_* Y_\sigma)*)\Big)^0 \qquad (31)\\
&= (\operatorname{trace}_g \widehat{\nabla}^2\sigma)^V + \Big(\operatorname{trace}_g(R(X_\sigma,\nabla_* X_\sigma)* + R(Y_\sigma,\nabla_* Y_\sigma)*)\Big)^0,
\end{aligned}
$$

where $-\operatorname{trace}_g \nabla^2$ *(resp.,* $-\operatorname{trace}_g \widehat{\nabla}^2$*) denotes the Laplacian attached to* ∇ *(resp.,* $\widehat{\nabla}$*).*

4. Biharmonicity of Section $\sigma\colon (M,g) \to (T^2M, g^D)$

For a section $\sigma \in \Gamma(T^2M)$, we denote

$$\tau^0(\sigma) = \tau^0(X_\sigma) + \tau^0(Y_\sigma), \tag{32}$$

$$\tau^V(\sigma) = \tau^1(X_\sigma) + \tau^2(Y_\sigma), \tag{33}$$

$$\overline{\tau}^0\Big(\sigma\Big) = \Big(\tau^0(X_\sigma) + \tau^0(Y_\sigma)\Big)^0, \tag{34}$$

$$\overline{\tau}^V(\sigma) = \Big(\tau^1(X_\sigma)\Big)^1 + \Big(\tau^2(Y_\sigma)\Big)^2, \tag{35}$$

where

$$\begin{aligned}
\tau^0(X_\sigma) &= \operatorname{trace}_g(R(X_\sigma, \nabla_* X_\sigma)*),\\
\tau^0(Y_\sigma) &= \operatorname{trace}_g(R(Y_\sigma, \nabla_* Y_\sigma)*),\\
\tau^1(X_\sigma) &= \operatorname{trace}_g \nabla^2 X_\sigma,\\
\tau^2(Y_\sigma) &= \operatorname{trace}_g \nabla^2 Y_\sigma.
\end{aligned}$$

From these notations, we have

$$\tau(\sigma) = \overline{\tau}^V + \overline{\tau}^0. \tag{36}$$

Theorem 4.1. *Let (M,g) be a Riemannian compact manifold and (T^2M, g^D) be its tangent bundle of order two equipped with the diagonal metric and a vector bundle structure via the diffeomorphism S between T^2 and $TM \oplus TM$. Then $\sigma\colon M \to T^2M$ is a biharmonic section if and only if σ is harmonic.*

Proof. First, if σ is harmonic, then from Corollary 1.4, we deduce that σ is biharmonic.

Conversely, assuming that σ is biharmonic. Let σ_t be a compactly supported variation of σ defined by $\sigma_t = (1+t)\sigma$. Using Proposition 2.8, we have

$$X_{\sigma_t} = (1+t)X_\sigma \quad \text{and} \quad Y_{\sigma_t} = (1+t)Y_\sigma. \tag{37}$$

Substituting (37) in (32) to (35), we obtain

$$\tau^0(\sigma_t) = (1+t)^2\tau^0(\sigma) \quad \text{and} \quad \tau^V(\sigma_t) = (1+t)\tau^V(\sigma) \tag{38}$$

$$\overline{\tau}^0(\sigma_t) = (1+t)^2\overline{\tau}^0(\sigma) \quad \text{and} \quad \overline{\tau}^V(\sigma_t) = (1+t)\overline{\tau}^V(\sigma). \tag{39}$$

Then

$$\begin{aligned}
E_2(\sigma_t) &= \frac{1}{2}\int |\tau(\sigma_t)|^2_{g^D} v_g = \frac{1}{2}\int |\overline{\tau}^0(\sigma_t)|^2_{g^D} v_g + \frac{1}{2}\int |\overline{\tau}^V(\sigma_t)|^2_{g^D} v_g\\
&= \frac{(1+t)^4}{2}\int |\overline{\tau}^0(\sigma)|^2_{g^D} v_g + \frac{(1+t)^2}{2}\int |\overline{\tau}^V(\sigma)|^2_{g^D} v_g.
\end{aligned}$$

Since the section σ is biharmonic, then for the variation σ_t, we have

$$0 = \frac{d}{\mathrm{d}t}E_2(\sigma_t)|_{t=0} = 2\int |\overline{\tau}^0(\sigma)|^2_{g^D} v_g + \int |\overline{\tau}^V(\sigma)|^2_{g^D} v_g.$$

Hence

$$\overline{\tau}^0(\sigma) = 0 \quad \text{and} \quad \overline{\tau}^V(\sigma) = 0, \quad \text{then} \quad \tau(\sigma) = 0.$$

□

In the case where M is not compact, the characterization of biharmonic sections requires the following two lemmas.

Lemma 4.2. *Let (M,g) be a Riemannian manifold and (T^2M, g^D) be its tangent bundle of order two equipped with the diagonal metric. If $\sigma \in \Gamma(T^2M)$ is a smooth section, then the Jacobi tensor $J_\sigma(\tau^V(\sigma))$ is given by*

$$J_\sigma(\overline{\tau}^V(\sigma)) = \Big\{ \operatorname{trace}_g \nabla^2(\tau^V(\sigma))\Big\}^V + \Big\{ \operatorname{trace}_g (R(u, \nabla_*\tau^1(X_\sigma) * + R(w, \nabla_*\tau^2(Y_\sigma)) * + R(\tau^V(\sigma), \nabla_*\sigma)* + \frac{1}{2}R(u,\tau^1(X_\sigma))R(u,\nabla_* X_\sigma) * + \frac{1}{2}R(w,\tau^2(Y_\sigma))R(w,\nabla_* Y_\sigma)*)\Big\}^0.$$

Proof. Let $p \in T^2M$ and $\{e_i\}_{i=1}^m$ be a local orthonormal frame on M such that $(\nabla_{e_i}e_i)_x = 0$. If we denote $F_i(x,u,w) = \frac{1}{2}R(u,\tau^1(X_\sigma))e_i + \frac{1}{2}R(w,\tau^2(Y_\sigma))e_i$, then we have

$$\begin{aligned}\widetilde{\nabla}^\sigma_{e_i}\overline{\tau}^V(\sigma)_p &= (\widetilde{\nabla}^\sigma_{e_i^0 + (\nabla_{e_i}X_\sigma)^1 + (\nabla_{e_i}Y_\sigma)^2}(\tau^1(X_\sigma))^1 + (\tau^2(Y_\sigma))^2)_p \\ &= (\nabla_{e_i}(\tau^V(\sigma))^V_p + \frac{1}{2}(R(u,\tau^1(X_\sigma))e_i + R(w,\tau^2(Y_\sigma))e_i)^0 \\ &= (\nabla_{e_i}(\tau^V(\sigma))^V_p + (F_i(x,u,w))^0,\end{aligned}$$

hence

$$\begin{aligned}(\operatorname{trace}_g \widetilde{\nabla}^2\overline{\tau}^V(\sigma)_p &= \sum_{i=1}^m \Big\{\widetilde{\nabla}^\sigma_{e_i}\widetilde{\nabla}^\sigma_{e_i}(\overline{\tau}^V(\sigma))\Big\}_p \\ &= \sum_{i=1}^m \Big\{\widetilde{\nabla}^\sigma_{e_i^0+(\nabla_{e_i}X_\sigma)^1+(\nabla_{e_i}Y_\sigma)^2}((\nabla_{e_i}(\tau^V(\sigma))^V + (F_i)^0\Big\}_p \\ &= \sum_{i=1}^m \Big\{\widetilde{\nabla}_{e_i^0}(\nabla_{e_i}\tau^1(X_\sigma))^1 + \widetilde{\nabla}_{e_i^0}(\nabla_{e_i}\tau^1(Y_\sigma))^2 \\ &\qquad + \widetilde{\nabla}_{e_i^0}F_i^0 + \widetilde{\nabla}_{(\nabla_{e_i}X_\sigma)^1}F_i^0 + \widetilde{\nabla}_{(\nabla_{e_i}Y_\sigma)^2}F_i^0\Big\}_p.\end{aligned}$$

Using Proposition 2.17, we obtain

$$\begin{aligned}(\operatorname{trace}_g \widetilde{\nabla}^2\overline{\tau}^V(\sigma))_p &= \sum_{i=1}^m \Big\{(\nabla_{e_i}\nabla_{e_i}\tau^1(X_\sigma)) - \frac{1}{4}R(e_i, R(u,\tau^1(X_\sigma)e_i)u\Big\}^1_p \\ &+ \sum_{i=1}^m \Big\{(\nabla_{e_i}\nabla_{e_i}\tau^2(Y_\sigma)) - \frac{1}{4}R(e_i,R(w,\tau^2(Y_\sigma)e_i)w\Big\}^2_p + \sum_{i=1}^m\Big\{\frac{1}{2}R(u,\nabla_{e_i}\tau^1(X_\sigma))e_i \\ &+ \frac{1}{2}R(w,\nabla_{e_i}\tau^2(Y_\sigma))e_i + \frac{1}{2}(\nabla_{e_i}R(u,\tau^1(X_\sigma))e_i) + \frac{1}{2}(\nabla_{e_i}R(w,\tau^2(Y_\sigma))e_i) \\ &+ \frac{1}{2}R(\tau^1(X_\sigma),\nabla_{e_i}u)e_i + \frac{1}{2}R(\tau^2(Y_\sigma),\nabla_{e_i}w)e_i + \frac{1}{4}R(u,\nabla_{e_i}X_\sigma)R(u,\tau^1(X_\sigma))e_i \\ &+ \frac{1}{4}R(w,\nabla_{e_i}Y_\sigma)R(w,\tau^2(Y_\sigma))e_i + \frac{1}{2}R(\nabla_{e_i}X_\sigma,\tau^1(X_\sigma))e_i + \frac{1}{2}R(\nabla_{e_i}X_\sigma,\tau^2(Y_\sigma))e_i\Big\}^0.\end{aligned}$$

From proposition 3.3, we have

$$\begin{aligned}
&\operatorname{trace}_g(\widetilde{R}(\overline{\tau}^V(\sigma), \mathrm{d}\sigma)\mathrm{d}\sigma) \\
&= \sum_{i=1}^{m} \Big\{ \widetilde{R}((\tau^1(X_\sigma))^1, e_i^0)e_i^0 + \widetilde{R}((\tau^1(X_\sigma))^1, (\nabla_{e_i}X_\sigma)^1)e_i^0 \\
&\quad + \widetilde{R}((\tau^1(X_\sigma))^1, (\nabla_{e_i}Y_\sigma)^2)e_i^0 + \widetilde{R}((\tau^1(X_\sigma))^1, e_i^0)(\nabla_{e_i}X_\sigma)^1 \\
&\quad + \widetilde{R}((\tau^1(X_\sigma))^1, e_i^0)(\nabla_{e_i}Y_\sigma)^2 + \widetilde{R}((\tau^2(Y_\sigma))^2, e_i^0)e_i^0 \\
&\quad + \widetilde{R}((\tau^2(Y_\sigma))^2, (\nabla_{e_i}X_\sigma)^1)e_i^0 + \widetilde{R}((\tau^2(Y_\sigma))^2, (\nabla_{e_i}Y_\sigma)^2)e_i^0 \\
&\quad + \widetilde{R}((\tau^2(Y_\sigma))^2, e_i^0)(\nabla_{e_i}X_\sigma)^1 + \widetilde{R}((\tau^2(Y_\sigma))^2, e_i^0)(\nabla_{e_i}Y_\sigma)^2 \Big\}.
\end{aligned}$$

By calculating at point $p \in T^2M$, we obtain

$$\begin{aligned}
&\operatorname{trace}_g(\widetilde{R}(\overline{\tau}^V(\sigma), d\sigma)d\sigma)_p \\
&= \sum_{i=1}^{m} \Big\{ -\frac{1}{4}R(R(u, \tau^1(X_\sigma))e_i, e_i)u \Big\}^1 - \Big\{ \frac{1}{4}R(R(w, \tau^2(Y_\sigma)e_i), e_i)w \Big\}^2 \\
&\quad + \sum_{i=1}^{m} \Big\{ R(\tau^1(X_\sigma), \nabla_{e_i}X_\sigma)e_i + R(\tau^2(Y_\sigma), \nabla_{e_i}Y_\sigma)e_i \\
&\quad + \frac{1}{4}R(u, \tau^1(X_\sigma))R(u, \nabla_{e_i}X_\sigma)e_i - \frac{1}{4}R(w, \nabla_{e_i}Y_\sigma)R(w, \tau^2(Y_\sigma))e_i \\
&\quad + \frac{1}{4}R(w, \tau^2(Y_\sigma))R(w, \nabla_{e_i}Y_\sigma)e_i - \frac{1}{4}R(u, \nabla_{e_i}X_\sigma)R(u, \tau^1(X_\sigma))e_i \\
&\quad + \frac{1}{2}R(\tau^1(X_\sigma), \nabla_{e_i}X_\sigma)e_i + \frac{1}{2}R(\tau^2(Y_\sigma), \nabla_{e_i}Y_\sigma)e_i \\
&\quad + \frac{1}{4}R(u, \tau^1(X_\sigma)R(u, \nabla_{e_i}X_\sigma)e_i + \frac{1}{4}R(u, \tau^2(Y_\sigma)R(w, \nabla_{e_i}Y_\sigma)e_i \\
&\quad - \frac{1}{2}(\nabla_{e_i}R(u, \tau^1(X_\sigma)e_i - \frac{1}{2}(\nabla_{e_i}R(w, \tau^2(Y_\sigma)e_i \Big\}^0.
\end{aligned}$$

Considering the formula (8), we deduce

$$\begin{aligned}
J_\sigma(\overline{\tau}^V(\sigma)) &= \Big\{ \operatorname{trace}_g \nabla^2(\tau^V(\sigma)) \Big\}^V + \Big\{ \operatorname{trace}_g(R(u, \nabla_*\tau^1(X_\sigma)* \\
&\quad + R(w, \nabla_*\tau^2(Y_\sigma)) * + R(\tau^V(\sigma), \nabla_*\sigma)* \\
&\quad + \frac{1}{2}R(u, \tau^1(X_\sigma))R(u, \nabla_*X_\sigma) * + \frac{1}{2}R(w, \tau^2(Y_\sigma))R(w, \nabla_*Y_\sigma)*) \Big\}^0.
\end{aligned}$$

□

Lemma 4.3. *Let (M, g) be a Riemannian manifold and (T^2M, g^D) be its tangent bundle of order two equipped with the diagonal metric. If $\sigma \in \Gamma(T^2M)$ is a* *smooth section, then the Jacobi tensor* $J_\sigma(\tau^0(\sigma))$ *is given by*

$$\begin{aligned}
&J_\sigma(\overline{\tau}^0(\sigma))_p\\
&= \mathrm{trace}_g\Big\{2R(\tau^0(X_\sigma),*)\nabla_*X_\sigma - R(*,\nabla_*\tau^0(X_\sigma))u + \frac{1}{2}R(R(u,\nabla_*X_\sigma)*,\tau^0(X_\sigma))u\Big\}^1\\
&\quad+\mathrm{trace}_g\Big\{2R(\tau^0(Y_\sigma),*)\nabla_*Y_\sigma - R(*,\nabla_*\tau^0(Y_\sigma))w + \frac{1}{2}R(R(w,\nabla_*Y_\sigma)*,\tau^0(Y_\sigma))w\Big\}^2\\
&\quad+\mathrm{trace}_g\Big\{\nabla_*\nabla_*\tau^0(\sigma) + R(u,\nabla_*X_\sigma)\nabla_*\tau^0(X_\sigma) + R(w,\nabla_*Y_\sigma)\nabla_*\tau^0(Y_\sigma)\\
&\quad+\frac{1}{2}R(u,\nabla_*\nabla_*X_\sigma)\tau^0(X_\sigma) + \frac{1}{2}R(w,\nabla_*\nabla_*Y_\sigma)\tau^0(Y_\sigma) + R(u,R(\tau^0(X_\sigma),*)u)*\\
&\quad+R(w,R(\tau^0(Y_\sigma),*)w)* + R(\tau^0(\sigma),*)* + (\nabla_{\tau^0(X_\sigma)}R)(u,\nabla_*X_\sigma)*\\
&\quad+(\nabla_{\tau^0(Y_\sigma)}R)(w,\nabla_*Y_\sigma)*\Big\}^0_p
\end{aligned}$$

for all $p=(x,u,w)\in T^2M$.

Proof. Let $p=(x,u,w)\in T^2M$ and $\{e_i\}_{i=1}^m$ be a local orthonormal frame on M such that $(\nabla_{e_i}e_i)_x=0$, denoted by

$$F_i = F_{iX}+F_{iY} = \frac{1}{2}R(e_i,\tau^0(X_\sigma))* + \frac{1}{2}R(e_i,\tau^0(Y_\sigma))* \tag{40}$$

$$G = G_X+G_Y = \frac{1}{2}R(*,\nabla_*X_\sigma)\tau^0(X_\sigma) + \frac{1}{2}R(*,\nabla_*Y_\sigma)\tau^0(Y_\sigma). \tag{41}$$

First, using Lemma 3.4 and Proposition 2.17, we calculate

$$\begin{aligned}
\mathrm{trace}_g\,\widetilde{\nabla}^2(\overline{\tau}^0(\sigma))_p &= \sum_{i=1}^m\Big\{\widetilde{\nabla}^\sigma_{e_i}\widetilde{\nabla}^\sigma_{e_i}(\tau^0(\sigma))^0\Big\}\\
&= \sum_{i=1}^m\Big\{(\widetilde{\nabla}^\sigma_{e_i^0+(\nabla_{e_i}X_\sigma)^1+(\nabla_{e_i}Y_\sigma)^2}(\nabla_{e_i}\tau(\sigma))^0 - F^1_{iX} - F^2_{iY} + G^0_i)\Big\}_p.
\end{aligned}$$

From Proposition 2.17, we have

$$\begin{aligned}
\mathrm{trace}_g\,\widetilde{\nabla}^2(\overline{\tau}^0(\sigma))_p = &\sum_{i=1}^m\Big\{(\nabla_{e_i}\nabla_{e_i}\tau^0(\sigma))^0 + (\frac{1}{2}R(u,\nabla_{e_i}X_\sigma)\nabla_{e_i}\tau^0(X_\sigma)\\
&+\frac{1}{2}R(w,\nabla_{e_i}Y_\sigma)\nabla_{e_i}\tau^0(Y_\sigma))^0 - (\nabla_{e_i}F_{iX})^1 - (\nabla_{e_i}F_{iY})^2\\
&-(\frac{1}{2}R(e_i,\nabla_{e_i}\tau^0(X_\sigma))u)^1 - (\frac{1}{2}R(e_i,\nabla_{e_i}\tau^0(Y_\sigma))w)^2\\
&-\frac{1}{2}(R(u,F_{iX}(u))e_i)^0 - \frac{1}{2}(R(w,F_{iY}(w))e_i)^0\\
&-(F_{iX}(\nabla_{e_i})X_\sigma)^1 - (F_{iY}(\nabla_{e_i})Y_\sigma)^2 + (\nabla_{e_i}G)^0\\
&-\frac{1}{2}(R(e_i,G_X(u))u)^1 - \frac{1}{2}(R(e_i,G_Y(w))w)^2\\
&+(G_X(\nabla_{e_i}X_\sigma))^0 + (G_Y(\nabla_{e_i}Y_\sigma))^0\\
&+\frac{1}{2}(R(u,\nabla_{e_i}X_\sigma)G_X(u))^0 + \frac{1}{2}(R(w,\nabla_{e_i}Y_\sigma)G_Y(w))^0\Big\}_p.
\end{aligned} \tag{42}$$

Substituting (40) and (41) in (42), we arrive at

$$
\begin{aligned}
&\operatorname{trace}_g \widetilde{\nabla}^2(\overline{\tau}^0(\sigma))_p \\
&= \sum_{i=1}^{m} \Big\{ (\nabla_{e_i}\nabla_{e_i}\tau^0(\sigma)) + R(u,\nabla_{e_i}X_\sigma)\nabla_{e_i}\tau^0(X_\sigma) \\
&+ R(w,\nabla_{e_i}Y_\sigma)\nabla_{e_i}\tau^0(Y_\sigma) + \frac{1}{2}R(u,\nabla_{e_i}\nabla_{e_i}X_\sigma)\tau^0(X_\sigma) \\
&+ \frac{1}{2}R(w,\nabla_{e_i}\nabla_{e_i}Y_\sigma)\tau^0(Y_\sigma) + \frac{1}{2}(\nabla_{e_i}R)(u,\nabla_{e_i}X_\sigma)\tau^0(X_\sigma) \\
&+ \frac{1}{2}(\nabla_{e_i}R)(w,\nabla_{e_i}Y_\sigma)\tau^0(Y_\sigma) + \frac{1}{4}R(u,\nabla_{e_i}X_\sigma)R(u,\nabla_{e_i}X_\sigma)\tau^0(X_\sigma) \\
&+ \frac{1}{4}R(w,\nabla_{e_i}Y_\sigma)R(w,\nabla_{e_i}Y_\sigma)\tau^0(Y_\sigma) - \frac{1}{4}R(u,R(e_i,\tau^0(X_\sigma))u)e_i \\
&- \frac{1}{4}R(w,R(e_i,\tau^0(Y_\sigma))w)e_i \Big\}_p^0 - \sum_{i=1}^{m}\Big\{\frac{1}{2}R(e_i,\tau^0(X_\sigma))\nabla_{e_i}X_\sigma \\
&+ R(e_i,\nabla_{e_i}\tau^0(X_\sigma))u + \frac{1}{2}(\nabla_{e_i}R)(e_i,\tau^0(X_\sigma))u \\
&+ \frac{1}{4}R(e_i,R(u,\nabla_{e_i}X_\sigma)\tau^0(X_\sigma))u\Big\}_p^1 \\
&- \sum_{i=1}^{m}\Big\{\frac{1}{2}R(e_i,\tau^0(Y_\sigma))\nabla_{e_i}Y_\sigma + R(e_i,\nabla_{e_i}\tau^0(Y_\sigma))w \\
&+ \frac{1}{2}(\nabla_{e_i}R)(e_i,\tau^0(Y_\sigma))w + \frac{1}{4}R(e_i,R(w,\nabla_{e_i}Y_\sigma)\tau^0(Y_\sigma))w\Big\}_p^2 .
\end{aligned}
\tag{43}
$$

On the other hand, we have

$$
\begin{aligned}
&\operatorname{trace}_g\Big\{\widetilde{R}(\overline{\tau}^0(\sigma),d\sigma)d\sigma\Big\}_p \\
&= \sum_{i=1}^{m}\Big\{R(\tau^0(\sigma),e_i)e_i + \frac{3}{4}R(u,R(\tau^0(X_\sigma),e_i)u)e_i \\
&+ \frac{3}{4}R(w,R(\tau^0(Y_\sigma),e_i)w)e_i + \nabla_{\tau^0}(X_\sigma R)(u,\nabla_{e_i}X_\sigma)e_i \\
&+ \nabla_{\tau^0}(Y_\sigma R)(w,\nabla_{e_i}Y_\sigma)e_i - \frac{1}{2}(\nabla_{e_i}R)(u,\nabla_{e_i}X_\sigma)\tau^0(X_\sigma) \\
&- \frac{1}{2}(\nabla_{e_i}R)(w,\nabla_{e_i}Y_\sigma)\tau^0(Y_\sigma) - \frac{1}{4}R(u,\nabla_{e_i}X_\sigma)R(u,\nabla_{e_iX_\sigma})\tau^0(X_\sigma) \\
&- \frac{1}{4}R(w,\nabla_{e_i}Y_\sigma)R(w,\nabla_{e_iY_\sigma})\tau^0(Y_\sigma)\Big\}_p^0 \\
&+ \sum_{i=1}^{m}\Big\{\frac{1}{2}(\nabla_{e_i}R)\tau^0(X_\sigma,e_i)u + \frac{1}{2}R(R(u,\nabla_{e_i}X_\sigma)e_i,\tau^0(X_\sigma))u \\
&+ \frac{3}{2}R(\tau^0(X_\sigma),e_i)\nabla_{e_i}X_\sigma - \frac{1}{4}R(R(u,\nabla_{e_i}X_\sigma)\tau^0(X_\sigma),e_i)u\Big\}^1 \\
&+ \sum_{i=1}^{m}\Big\{\frac{1}{2}(\nabla_{e_i}R)\tau^0(Y_\sigma,e_i)w + \frac{1}{2}R(R(w,\nabla_{e_i}Y_\sigma)e_i,\tau^0(Y_\sigma))w \\
&+ \frac{3}{2}R(\tau^0(Y_\sigma),e_i)\nabla_{e_i}Y_\sigma - \frac{1}{4}R(R(w,\nabla_{e_i}Y_\sigma)\tau^0(Y_\sigma),e_i)w\Big\}^2 .
\end{aligned}
\tag{44}
$$

By summing (43) and (44), the proof of Lemma 4.3 is completed. □

From Lemma 4.2 and 4.3, we deduce the following theorems

Theorem 4.4. *Let (M,g) be a Riemannian manifold and (T^2M, g^D) be its tangent bundle of order two equipped with the diagonal metric. If $\sigma\colon M \to T^2M$ is a smooth section, then the bitension field of σ is given by*

$$\begin{aligned}\tau_2(\sigma)_p = \; & \operatorname{trace}_g \Big\{\nabla^2\tau^1(X_\sigma) + 2R(\tau^0(X_\sigma),*)\nabla_* X_\sigma - R(*,\nabla_*\tau^0(X_\sigma))u \\ & + \frac{1}{2}R(R(u,\nabla_*)*,\tau^0(X_\sigma))u\Big\}^1 \\ & + \operatorname{trace}_g \Big\{\nabla^2\tau^2(Y_\sigma) + 2R(\tau^0(Y_\sigma),*)\nabla_* Y_\sigma \\ & - R(*,\nabla_*\tau^0(Y_\sigma))w + \frac{1}{2}R(R(w,\nabla_*)*,\tau^0(Y_\sigma))w\Big\}^2 \\ & + \operatorname{trace}_g \Big\{R(u,\nabla_*\tau^1(X_\sigma))* + R(w,\nabla_*\tau^2(Y_\sigma))* \\ & + R(\tau^1(X_\sigma),\nabla_* X_\sigma)* \\ & + R(\tau^2(Y_\sigma),\nabla_* Y_\sigma)* + \frac{1}{2}R(u,\tau^1(X_\sigma))R(u,\nabla_* X_\sigma)* \\ & + \frac{1}{2}R(w,\tau^2(Y_\sigma))R(w,\nabla_* Y_\sigma)* + \nabla_*\nabla_*\tau^0(\sigma) \\ & + R(u,\nabla_* X_\sigma)\nabla_*\tau^0(X_\sigma) \\ & + R(w,\nabla_* Y_\sigma)\nabla_*\tau^0(Y_\sigma) + R(\tau^0(\sigma),*)* \\ & + \frac{1}{2}R(u,\nabla_*\nabla_* X_\sigma)\tau^0(X_\sigma) \\ & + \frac{1}{2}R(w,\nabla_*\nabla_* Y_\sigma)\tau^0(Y_\sigma) + R(u,R(\tau^0(X_\sigma),*)u)* \\ & + R(w,R(\tau^0(Y_\sigma),*)w)* + (\nabla_{\tau^0(X_\sigma)}R)(u,\nabla_* X_\sigma)* \\ & + (\nabla_{\tau^0(Y_\sigma)}R)(w,\nabla_* Y_\sigma)*\Big\}^0_p\end{aligned}$$

for all $p \in T^2M$.

Theorem 4.5. *Let (M,g) be a Riemannian manifold and (T^2M, g^D) be its tangent bundle of order two equipped with the diagonal metric. A section $\sigma\colon M \to T^2M$ is biharmonic if and only if the following conditions are verified:*

1) $$\begin{aligned}0 = \operatorname{trace}_g & \Big\{\nabla^2\tau^1(X_\sigma) + 2R(\tau^0(X_\sigma),*)\nabla_* X_\sigma - R(*,\nabla_*\tau^0(X_\sigma))u \\ & + \frac{1}{2}R(R(u,\nabla_*)*,\tau^0(X_\sigma))u\Big\}_p,\end{aligned}$$

2) $$\begin{aligned}0 = \operatorname{trace}_g & \Big\{\nabla^2\tau^2(Y_\sigma) + 2R(\tau^0(Y_\sigma),*)\nabla_* Y_\sigma - R(*,\nabla_*\tau^0(Y_\sigma))w \\ & + \tfrac{1}{2}R(R(w,\nabla_*)*,\tau^0(Y_\sigma))w\Big\}_p,\end{aligned}$$

$$
\begin{aligned}
3)\qquad 0 = \operatorname{trace}_g \Big\{ & R(u,\nabla_*\tau^1(X_\sigma)) * + R(w,\nabla_*\tau^2(Y_\sigma)) * \\
& + R(\tau^1(X_\sigma),\nabla_* X_\sigma) * + R(\tau^2(Y_\sigma),\nabla_* Y_\sigma) * \\
& + \frac{1}{2} R(u,\tau^1(X_\sigma)) R(u,\nabla_* X_\sigma) * + \tfrac{1}{2} R(w,\tau^2(Y_\sigma)) R(w,\nabla_* Y_\sigma) * \\
& + \nabla_*\nabla_*\tau^0(\sigma) + R(u,\nabla_* X_\sigma)\nabla_*\tau^0(X_\sigma) \\
& + R(w,\nabla_* Y_\sigma)\nabla_*\tau^0(Y_\sigma) \\
& + \frac{1}{2} R(u,\nabla_*\nabla_* X_\sigma)\tau^0(X_\sigma) + \tfrac{1}{2} R(w,\nabla_*\nabla_* Y_\sigma)\tau^0(Y_\sigma) \\
& + R(u, R(\tau^0(X_\sigma),*)u) * + R(w, R(\tau^0(Y_\sigma),*)w) * \\
& + R(\tau^0(\sigma),*) * + (\nabla_{\tau^0(X_\sigma)} R)(u,\nabla_* X_\sigma) * \\
& + (\nabla_{\tau^0(Y_\sigma)} R)(w,\nabla_* Y_\sigma) * \Big\}_p
\end{aligned}
$$

for all $p = S^{-1}(x,u,w) \in T^2M$.

Corollary 4.6. *Let* (M,g) *be a Riemannian manifold and* (T^2M, g^D) *be its tangent bundle of order two equipped with the diagonal metric. If* $\sigma\colon M \to T^2M$ *is a section such that* X_σ *and* Y_σ *are biharmonic vector fields, then* σ *is biharmonic.*

(For biharmonic vector see [**1**]).

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