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ON THE RADICAL OF THE ANNIHILATORS OF LOCAL COHOMOLOGY MODULES

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Abstract. Let R be a noetherian ring, M a non-zero finitely generated R-module of dimension d. Let $\mathfrak{a}$ be an ideal of R and the cohomological dimension of M with respect to $\mathfrak{a}$ equal to d. In this paper, we calculate the radical of the annihilator of the top local cohomology module $\mathrm{H}^d_{\mathfrak{a}}(M)$. In fact, we prove that there exists a submodule $S_R(\mathfrak{a}, M)$ of M such that the radical of the annihilator of $\mathrm{H}^d_{\mathfrak{a}}(M)$ equals the annihilator of the quotient $M/S_R(\mathfrak{a}, M)$. By using this result, for a complete local ring $(R, \mathfrak{m})$, we determine the set of attached prime ideals of $\mathrm{H}^d_{\mathfrak{a}}(M)$.

1. Introduction

Throughout this paper, R is a commutative noetherian ring with identity, $\mathfrak{a}$ is an ideal of R and M is an R-module. Recall that the i-th local cohomology module of M with respect to $\mathfrak{a}$ is defined as

$$\mathrm{H}^i_{\mathfrak{a}}(M) := \varinjlim_{n\geq 1} \mathrm{Ext}^i_R(R/\mathfrak{a}^n, M).$$

One of the important problems in local cohomology is to find the annihilator of the local cohomology module $\mathrm{H}^i_{\mathfrak{a}}(M)$. This problem has been studied by several authors, see for example, [**1**], [**3**], [**4**], and [**8**]. In [**4**], the authors proved that if $(R, \mathfrak{m})$ is a complete noetherian local ring and M is a finitely generated R-module, then $\mathrm{Ann}_R(\mathrm{H}^{\dim M}_{\mathfrak{m}}(M)) = \mathrm{Ann}_R(M/T_R(M))$, where $T_R(M)$ is the largest submodule of M such that $\dim T_R(M) < \dim M$. This result was later extended to non-complete noetherian local rings by Bahmanpour in [**3**]. Also, for an ideal $\mathfrak{a}$ (not necessarily $\mathfrak{a} = \mathfrak{m}$) in an arbitrary noetherian ring R (not necessarily local), in [**1**], Atazadeh et al. proved that $\mathrm{Ann}_R(\mathrm{H}^{\dim M}_{\mathfrak{a}}(M)) = \mathrm{Ann}_R(M/T_R(\mathfrak{a}, M))$ where $T_R(\mathfrak{a}, M)$ is the largest submodule of M such that $\mathrm{cd}(\mathfrak{a}, T_R(\mathfrak{a}, M)) < \mathrm{cd}(\mathfrak{a}, M)$.

This is natural to ask about the radical of the annihilator of the top local cohomology modules. In this paper, we define a new notation $S_R(\mathfrak{a}, M)$ and determine the radical of the annihilator of the top local cohomology module $\mathrm{H}^{\dim M}_{\mathfrak{a}}(M)$ according to $S_R(\mathfrak{a}, M)$.

For an R-module M and an ideal $\mathfrak{a}$, the cohomological dimension of M with respect to $\mathfrak{a}$ is defined as $\mathrm{cd}(\mathfrak{a}, M) := \max\{i \in \mathbb{Z} : \mathrm{H}^i_{\mathfrak{a}}(M) \neq 0\}$. For more details,

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see [**7**]. For any ideal $\mathfrak{a}$ of R, the radical of $\mathfrak{a}$, denoted by $\sqrt{\mathfrak{a}}$, is defined to be the set $\{x \in R : x^n \in \mathfrak{a} \text{ for some } n \in \mathbb{N}\}$.

The following is the main result of this paper.

Theorem 1.1. *Let R be a noetherian ring and $\mathfrak{a}$ an ideal of R. Let M be a finitely generated R-module of finite dimension d such that $\mathrm{cd}(\mathfrak{a}, M) = d$. Then $\sqrt{\mathrm{Ann}_R \mathrm{H}^d_{\mathfrak{a}}(M)} = \mathrm{Ann}_R(M/S_R(\mathfrak{a}, M))$, where $S_R(\mathfrak{a}, M)$ is the maximal element of the set*

$$\Theta := \{N : N \leqslant M \text{ and } \{\mathfrak{p} \in \mathrm{Ass}_R M \mid \mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = d\} \subseteq \mathrm{Ass}_R(M/N)\}.$$

A non-zero R-module M is called secondary if its multiplication map by any element a of R, is either surjective or nilpotent. A prime ideal $\mathfrak{p}$ of R is said to be an attached prime of M if $\mathfrak{p} = (N :_R M)$ for some submodule N of M. If M admits a reduced secondary representation, $M = S_1 + S_2 + \cdots + S_n$, then the set of attached primes $\mathrm{Att}_R(M)$ of M is equal to $\{\sqrt{0 :_R S_i} : i = 1, \dots, n\}$, (see [**9**]).

We have the following well known result about the attached primes of local cohomology modules.

Theorem 1.2 ([**6**, Theorem A])**.** *Let R be a noetherian ring and $\mathfrak{a}$ an ideal of R. Let M be a finitely generated R-module. Then*

$$\mathrm{Att}_R(\mathrm{H}^{\dim M}_{\mathfrak{a}}(M)) = \{\mathfrak{p} \in \mathrm{Ass}_R M \mid \mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = \dim M\}.$$

Let R be a noetherian ring and $\mathfrak{a}$ an ideal of R. Let M be a finitely generated R-module of finite dimension d such that $\mathrm{cd}(\mathfrak{a}, M) = d$. Then, by using the above main result, we will show that:

i) $r \in \sqrt{\mathrm{Ann}_R \mathrm{H}^d_{\mathfrak{a}}(M)}$ if and only if $\mathrm{Att}_R \mathrm{H}^d_{\mathfrak{a}}(M) \subseteq \mathrm{Ass}_R M/(rM + S_R(\mathfrak{a}, M))$, for every non-zero element r of R.

ii) $\bigcap_{\mathfrak{p} \in \mathrm{Att}_R(\mathrm{H}^d_{\mathfrak{a}}(M))} \mathfrak{p} = \mathrm{Ann}_R M/S_R(\mathfrak{a}, M)$

iii) If R is a complete noetherian local ring, then

$$\mathrm{Att}_R \mathrm{H}^{\dim M}_{\mathfrak{a}}(M) = \mathrm{Min}\,\mathrm{Supp}_R(M/S_R(\mathfrak{a}, M)).$$

2. Annihilators of local cohomology modules

Let R be a noetherian ring, $\mathfrak{a}$ an ideal of R and M be a non-zero finitely generated R-module of dimension d. Let $\mathrm{cd}(\mathfrak{a}, M) = d$. In this section we calculate the radical of the annihilator of the top local cohomology module $\mathrm{H}^d_{\mathfrak{a}}(M)$.

Recall that for an R-module M of finite dimension, the submodule $T_R(M)$ of M is defined as follows:

$$T_R(M) := \bigcup \{N : N \leqslant M \text{ and } \dim N < \dim M\}.$$

In [**1**], we see the following notation which is a generalization of $T_R(M)$ for an arbitrary ideal $\mathfrak{a}$ and non-zero finitely generated R-module M

$$T_R(\mathfrak{a}, M) := \bigcup \{N : N \leqslant M \text{ and } \mathrm{cd}(\mathfrak{a}, N) < \mathrm{cd}(\mathfrak{a}, M)\}.$$

In fact, $T_R(\mathfrak{a}, M)$ is the largest submodule of M such that $\operatorname{cd}(\mathfrak{a}, T_R(\mathfrak{a}, M)) < \operatorname{cd}(\mathfrak{a}, M)$. Clearly, for a local ring $(R, \mathfrak{m})$, we have $T_R(\mathfrak{m}, M) = T_R(M)$.

It is known that $T_R(\mathfrak{a}, M)$ has the following properties. For the details, see [**2**, Lemma 3.1].

Proposition 2.1. *Let R be a noetherian ring and $\mathfrak{a}$ an ideal of R. Let M be a non-zero finitely generated R-module such that $\operatorname{cd}(\mathfrak{a}, M) = \dim M$. Then:*

i) $\operatorname{cd}(\mathfrak{a}, M/T_R(\mathfrak{a}, M)) = \dim M$.

ii) $\mathrm{H}^{\dim M}_{\mathfrak{a}}(M) \simeq \mathrm{H}^{\dim M}_{\mathfrak{a}}(M/T_R(\mathfrak{a}, M))$.

iii) $\{\mathfrak{p} \in \operatorname{Ass}_R M \mid \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = \dim M\} = \operatorname{Ass}_R M/T_R(\mathfrak{a}, M) = \operatorname{Att}_R \mathrm{H}^{\dim M}_{\mathfrak{a}}(M)$.

The following theorem is main result in [**1**], on the annihilators of top local cohomology modules.

Theorem 2.2. [**1**, Theorem 2.3] *Let R be a noetherian ring and $\mathfrak{a}$ an ideal of R. Let M be a non-zero finitely generated R-module such that $\operatorname{cd}(\mathfrak{a}, M) = \dim M$. Then $\operatorname{Ann}_R \mathrm{H}^{\dim M}_{\mathfrak{a}}(M) = \operatorname{Ann}_R(M/T_R(\mathfrak{a}, M))$.*

In the following, we give a new notation $S_R(\mathfrak{a}, M)$ which plays an essential role in our proofs.

Definition 2.3. Let $\mathfrak{a}$ be an ideal of R and M be a non-zero finitely generated R-module of dimension d. By $S_R(\mathfrak{a}, M)$, we denote the maximal element of the set

$$\Theta := \{N : N \leqslant M \text{ and } \{\mathfrak{p} \in \operatorname{Ass}_R M \mid \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = d\} \subseteq \operatorname{Ass}_R(M/N)\}.$$

Note that by Proposition 2.1(iii), $T_R(\mathfrak{a}, M) \in \Theta$, and so Θ is non-empty. Since M is a noetherian R-module, it follows that Θ has a maximal element $S_R(\mathfrak{a}, M)$ with respect to inclusion.

Lemma 2.4. *Let R be a noetherian ring and $\mathfrak{a}$ an ideal of R. Let M be a finitely generated R-module of finite dimension such that $\operatorname{cd}(\mathfrak{a}, M) = \dim M$. Then $\dim(M/S_R(\mathfrak{a}, M)) = \operatorname{cd}(\mathfrak{a}, M/S_R(\mathfrak{a}, M)) = \dim M$.*

Proof. Since $\operatorname{cd}(\mathfrak{a}, M) = \dim M$, thus $\mathrm{H}^{\dim M}_{\mathfrak{a}}(M) \neq 0$, and so $\operatorname{Att}_R(\mathrm{H}^{\dim M}_{\mathfrak{a}}(M)) = \{\mathfrak{p} \in \operatorname{Ass}_R M \mid \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = \dim M\} \neq \emptyset$ by Theorem 1.2.

If $\mathfrak{p} \in \{\mathfrak{p} \in \operatorname{Ass}_R M \mid \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = \dim M\}$, then $\mathfrak{p} \in \operatorname{Ass}_R(M/S_R(\mathfrak{a}, M))$ by definition of $S_R(\mathfrak{a}, M)$. Thus $\operatorname{Supp}_R(R/\mathfrak{p}) \subseteq \operatorname{Supp}_R(M/S_R(\mathfrak{a}, M))$. Now by [**7**, Theorem 2.2], we have $\dim M = \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) \leq \operatorname{cd}(\mathfrak{a}, M/S_R(\mathfrak{a}, M))$. But, $\operatorname{cd}(\mathfrak{a}, M/S_R(\mathfrak{a}, M)) \leq \dim(M/S_R(\mathfrak{a}, M)) \leq \dim M$. Therefore,

$$\dim M = \operatorname{cd}(\mathfrak{a}, M/S_R(\mathfrak{a}, M)) = \dim(M/S_R(\mathfrak{a}, M)),$$

as required. □

Lemma 2.5. *Let R be a noetherian ring and $\mathfrak{a}$ an ideal of R. Let M be a finitely generated R-module of finite dimension d such that $\operatorname{cd}(\mathfrak{a}, M) = d$. Then $\operatorname{Att}_R \mathrm{H}^d_{\mathfrak{a}}(M) = \operatorname{Att}_R \mathrm{H}^d_{\mathfrak{a}}(M/S_R(\mathfrak{a}, M))$, and so*

$$\sqrt{\operatorname{Ann}_R(\mathrm{H}^d_{\mathfrak{a}}(M))} = \sqrt{\operatorname{Ann}_R(\mathrm{H}^d_{\mathfrak{a}}(M/S_R(\mathfrak{a}, M)))}.$$

Proof. Let $S := S_R(\mathfrak{a}, M)$. The short exact sequence

$$0 \to S \to M \to M/S \to 0$$

induces an exact sequence

$$\cdots \to \mathrm{H}^d_{\mathfrak{a}}(S) \to \mathrm{H}^d_{\mathfrak{a}}(M) \to \mathrm{H}^d_{\mathfrak{a}}(M/S) \to 0.$$

Thus, $\mathrm{Att}_R(\mathrm{H}^d_{\mathfrak{a}}(M/S)) \subseteq \mathrm{Att}_R(\mathrm{H}^d_{\mathfrak{a}}(M))$. On the other hand, by definition of S

$$\mathrm{Att}_R(\mathrm{H}^d_{\mathfrak{a}}(M)) = \{\mathfrak{p} \in \mathrm{Ass}_R M \mid \mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = d\} \subseteq \mathrm{Ass}_R(M/S).$$

Hence, in view of Lemma 2.4 and Theorem 1.2, we have

$$\mathrm{Att}_R \mathrm{H}^d_{\mathfrak{a}}(M) \subseteq \{\mathfrak{p} \in \mathrm{Ass}_R(M/S) \mid \mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = d\} = \mathrm{Att}_R \mathrm{H}^d_{\mathfrak{a}}(M/S).$$

Therefore, $\mathrm{Att}_R \mathrm{H}^d_{\mathfrak{a}}(M) = \mathrm{Att}_R \mathrm{H}^d_{\mathfrak{a}}(M/S)$ and by [**5**, 7.2.11], we have

$$\sqrt{\mathrm{Ann}_R(\mathrm{H}^d_{\mathfrak{a}}(M))} = \sqrt{\mathrm{Ann}_R(\mathrm{H}^d_{\mathfrak{a}}(M/S))},$$

the proof is complete. □

Proposition 2.6. *Let R be a noetherian ring and $\mathfrak{a}$ an ideal of R. Let M be a finitely generated R-module of finite dimension d such that $\mathrm{cd}(\mathfrak{a}, M) = d$. Then $\mathrm{Att}_R \mathrm{H}^d_{\mathfrak{a}}(M) \subseteq \mathrm{Ass}_R M/U$ for every submodule U of $S_R(\mathfrak{a}, M)$.*

Proof. Let $S := S_R(\mathfrak{a}, M)$. First note that by Lemma 2.4, we have

$$\dim M = \dim(M/S) \leq \dim(M/U) \leq \dim M,$$
$$\dim M = \mathrm{cd}(\mathfrak{a}, M/S) \leq \mathrm{cd}(\mathfrak{a}, M/U) \leq \dim M,$$

Thus $\mathrm{cd}(M/U) = \dim(M/U) = \dim M$. Now, the short exact sequence

$$0 \to S/U \to M/U \to M/S \to 0$$

implies that $\mathrm{Att}_R(\mathrm{H}^d_{\mathfrak{a}}(M/S)) \subseteq \mathrm{Att}_R(\mathrm{H}^d_{\mathfrak{a}}(M/U))$. By Lemma 2.5, we conclude that $\mathrm{Att}_R(\mathrm{H}^d_{\mathfrak{a}}(M)) \subseteq \mathrm{Att}_R(\mathrm{H}^d_{\mathfrak{a}}(M/U)) \subseteq \mathrm{Ass}_R M/U$, as required □

Lemma 2.7. *Let R be a noetherian ring and $\mathfrak{a}$ an ideal of R. Let M be a finitely generated R-module of finite dimension d such that $\mathrm{cd}(\mathfrak{a}, M) = d$. Then $\sqrt{\mathrm{Ann}_R(M/S_R(\mathfrak{a}, M))} = \mathrm{Ann}_R(M/S_R(\mathfrak{a}, M))$.*

Proof. Let $S := S_R(\mathfrak{a}, M)$. Let $x \in \sqrt{\mathrm{Ann}_R(M/S)}$. There exists an integer n such that $x^n M \subseteq S$. Thus Lemma 2.5 implies that

$$\mathrm{Att}_R(\mathrm{H}^d_{\mathfrak{a}}(M)) = \mathrm{Att}_R(\mathrm{H}^d_{\mathfrak{a}}(M/S)) = \mathrm{Att}_R(\mathrm{H}^d_{\mathfrak{a}}(M/x^n M + S)).$$

Since $\mathrm{Supp}_R(M/x^n M + S) = \mathrm{Supp}_R(M/xM + S)$, [**6**, Corollary 3] implies that

$$\mathrm{Att}_R(\mathrm{H}^d_{\mathfrak{a}}(M/x^n M + S)) = \mathrm{Att}_R(\mathrm{H}^d_{\mathfrak{a}}(M/xM + S)).$$

Hence, $\mathrm{Att}_R(\mathrm{H}^d_{\mathfrak{a}}(M)) = \mathrm{Att}_R(\mathrm{H}^d_{\mathfrak{a}}(M/xM + S))$. Thus

$$\mathrm{Att}_R(\mathrm{H}^d_{\mathfrak{a}}(M)) = \{\mathfrak{p} \in \mathrm{Ass}_R M \mid \mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = d\} \subseteq \mathrm{Ass}_R(M/xM + S).$$

By maximality of S, we conclude that $xM + S \subseteq S$. Therefore, $xM \subseteq S$, and so $x \in \mathrm{Ann}_R(M/S)$, the proof is complete. □

The following theorem is the main result of this paper.

Theorem 2.8. *Let R be a noetherian ring and $\mathfrak{a}$ an ideal of R. Let M be a finitely generated R-module of finite dimension d such that* $\operatorname{cd}(\mathfrak{a}, M) = d$. *Then* $\sqrt{\operatorname{Ann}_R \mathrm{H}^d_{\mathfrak{a}}(M)} = \operatorname{Ann}_R(M/S_R(\mathfrak{a}, M))$.

Proof. Let $S := S_R(\mathfrak{a}, M)$ and $T := T_R(\mathfrak{a}, M)$. At first, we show that $T_R(\mathfrak{a}, M/S) = 0$. It suffices to show that for any non-zero submodule L/S of M/S we have $\operatorname{cd}(\mathfrak{a}, L/S) = \operatorname{cd}(\mathfrak{a}, M/S)$. It is easy to see that

$$\{\mathfrak{p} \in \operatorname{Ass} M : \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = d\} \subseteq \operatorname{Ass}_R(M/S) \subseteq \operatorname{Ass}_R L/S \cup \operatorname{Ass}_R M/L.$$

If $\{\mathfrak{p} \in \operatorname{Ass} M : \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = d\} \subseteq \operatorname{Ass}_R(M/L)$ then since $S \subsetneq L$ from the maximality of S, we get a contradiction. Thus there exists a prime ideal $\mathfrak{p} \in \{\mathfrak{p} \in \operatorname{Ass} M : \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = d\}$ such that $\mathfrak{p} \in \operatorname{Ass}_R L/S$. Hence $d = \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) \leq \operatorname{cd}(\mathfrak{a}, L/S) \leq \operatorname{cd}(\mathfrak{a}, M/S) \leq \operatorname{cd}(\mathfrak{a}, M) = d$. It follows that $\operatorname{cd}(\mathfrak{a}, L/S) = \operatorname{cd}(\mathfrak{a}, M/S)$ and so $T_R(\mathfrak{a}, M/S) = 0$. On the other hand, by Lemma 2.5 and Theorem 2.2, we have

$$\sqrt{\operatorname{Ann}_R(\mathrm{H}^d_{\mathfrak{a}}(M))} = \sqrt{\operatorname{Ann}_R(\mathrm{H}^d_{\mathfrak{a}}(M/S))} = \sqrt{(\operatorname{Ann}_R(M/S)/T_R(\mathfrak{a}, M/S))}.$$

Since $T_R(\mathfrak{a}, M/S) = 0$, by Lemma 2.7, we get

$$\sqrt{\operatorname{Ann}_R(\mathrm{H}^d_{\mathfrak{a}}(M))} = \sqrt{\operatorname{Ann}_R(M/S)} = \operatorname{Ann}_R(M/S),$$

as required. □

Corollary 2.9. *Let R be a noetherian ring and $\mathfrak{a}$ an ideal of R. Let M be a finitely generated R-module of finite dimension d such that* $\operatorname{cd}(\mathfrak{a}, M) = d$. *Then* $\sqrt{\operatorname{Ann}_R \mathrm{H}^d_{\mathfrak{a}}(M)} = 0$ *if and only if* $\operatorname{Att}_R \mathrm{H}^d_{\mathfrak{a}}(M) \nsubseteq \operatorname{Ass}_R M/(rM + S_R(\mathfrak{a}, M))$ *for every non-zero element r of R.*

Proof. Set $S := S_R(\mathfrak{a}, M)$ and let $0 \neq r \in R$. If $r \in \sqrt{\operatorname{Ann}_R \mathrm{H}^d_{\mathfrak{a}}(M)}$, then $r \in \operatorname{Ann}_R M/S$ by Theorem 2.8. Thus $rM \subseteq S$, and so $\operatorname{Att}_R \mathrm{H}^d_{\mathfrak{a}}(M) \subseteq \operatorname{Ass}_R M/(rM + S)$ by Proposition 2.6. Conversely, if $\operatorname{Att}_R \mathrm{H}^d_{\mathfrak{a}}(M) \subseteq \operatorname{Ass}_R M/(rM + S)$, then $rM \subseteq S$ by maximality of S. Thus $r \in \operatorname{Ann}_R M/S = \sqrt{\operatorname{Ann}_R \mathrm{H}^d_{\mathfrak{a}}(M)}$ and the proof is complete. □

Corollary 2.10. *Let R be a noetherian ring and M a finitely generated R-module of dimension d such that* $\operatorname{cd}(\mathfrak{a}, M) = d$. *Then*

$$\bigcap_{\mathfrak{p} \in \operatorname{Att}_R(\mathrm{H}^d_{\mathfrak{a}}(M))} \mathfrak{p} = \operatorname{Ann}_R M/S_R(\mathfrak{a}, M).$$

Proof. By [**5**, 7.2.11] and Theorem 2.8, we have

$$\bigcap_{\mathfrak{p} \in \operatorname{Att}_R(\mathrm{H}^d_{\mathfrak{a}}(M))} \mathfrak{p} = \sqrt{\operatorname{Ann}_R \mathrm{H}^d_{\mathfrak{a}}(M)} = \operatorname{Ann}_R(M/S_R(\mathfrak{a}, M)),$$

as required. □

Corollary 2.11. *Let R be a noetherian ring and $\mathfrak{a}$ an ideal of R. Let M be a finitely generated R-module of finite dimension d such that* $\operatorname{cd}(\mathfrak{a}, M) = d$. *Then* $\sqrt{\operatorname{Ann}_R(M/T_R(\mathfrak{a}, M))} = \operatorname{Ann}_R(M/S_R(\mathfrak{a}, M))$.

Proof. It follows from Theorems 2.2 and 2.8. □

In the next result, we determine $\operatorname{V}(\operatorname{Ann}_R \operatorname{H}^d_{\mathfrak{a}}(M))$, the set of prime ideals of R containing ideal $\operatorname{Ann}_R \operatorname{H}^d_{\mathfrak{a}}(M)$.

Corollary 2.12. *Let R be a noetherian ring and $\mathfrak{a}$ an ideal of R. Let M be a finitely generated R-module of finite dimension d such that* $\operatorname{cd}(\mathfrak{a}, M) = d$. *Then*

$$\operatorname{V}(\operatorname{Ann}_R \operatorname{H}^d_{\mathfrak{a}}(M)) = \operatorname{Supp}_R(M/T_R(\mathfrak{a}, M)) = \operatorname{Supp}_R(M/S_R(\mathfrak{a}, M)).$$

Proof. By Theorem 2.2, we have

$$\operatorname{V}(\operatorname{Ann}_R \operatorname{H}^d_{\mathfrak{a}}(M)) = \operatorname{V}(\operatorname{Ann}_R(M/T_R(\mathfrak{a}, M))) = \operatorname{Supp}_R(M/T_R(\mathfrak{a}, M))$$

and by Corollary 2.11,

$$\operatorname{V}(\sqrt{(\operatorname{Ann}_R(M/T_R(\mathfrak{a}, M))}) = \operatorname{V}(\operatorname{Ann}_R(M/S_R(\mathfrak{a}, M))).$$

Thus $\operatorname{Supp}_R(M/T_R(\mathfrak{a}, M)) = \operatorname{Supp}_R(M/S_R(\mathfrak{a}, M))$, the proof is complete. □

In the following Corollary, for a complete noetherian local ring $(R, \mathfrak{m})$, we show that the set $\operatorname{Att}_R \operatorname{H}^d_{\mathfrak{a}}(M)$ is equal to the minimal elements of the set $\operatorname{Supp}_R(M/S_R(\mathfrak{a}, M))$.

Corollary 2.13. *Let $(R, \mathfrak{m})$ be a complete noetherian local ring and $\mathfrak{a}$ an ideal of R. Let M be a finitely generated R-module of finite dimension d such that* $\operatorname{cd}(\mathfrak{a}, M) = d$. *Then* $\operatorname{Att}_R \operatorname{H}^d_{\mathfrak{a}}(M) = \operatorname{Min} \operatorname{Supp}_R(M/S_R(\mathfrak{a}, M))$.

Proof. By [**10**, Theorem 2.11 (i)], we have $\operatorname{Att}_R \operatorname{H}^d_{\mathfrak{a}}(M) = \operatorname{Min} \operatorname{V}(\operatorname{Ann}_R \operatorname{H}^d_{\mathfrak{a}}(M))$. Now Corollary 2.12 completes the proof. □

Corollary 2.14. *Let R be a noetherian ring, $\mathfrak{a}$ an ideal of R, M and N be two finitely generated R-modules of dimension d with* $\operatorname{cd}(\mathfrak{a}, M) = \operatorname{cd}(\mathfrak{a}, N) = d$. *Then*

$$\operatorname{Att}_R \operatorname{H}^d_{\mathfrak{a}}(M) = \operatorname{Att}_R \operatorname{H}^d_{\mathfrak{a}}(N)$$

if and only if

$$\operatorname{Supp}_R(M/S_R(\mathfrak{a}, M)) = \operatorname{Supp}_R(N/S_R(\mathfrak{a}, N)).$$

Proof. If

$$\operatorname{Att}_R \operatorname{H}^d_{\mathfrak{a}}(M) = \operatorname{Att}_R \operatorname{H}^d_{\mathfrak{a}}(N),$$

then

$$\operatorname{Ann}_R M/S_R(\mathfrak{a}, M) = \operatorname{Ann}_R N/S_R(\mathfrak{a}, N)$$

by Corollary 2.10, and so

$$\operatorname{V}(\operatorname{Ann}_R(M/S_R(\mathfrak{a}, M))) = \operatorname{V}(\operatorname{Ann}_R(N/S_R(\mathfrak{a}, N))).$$

Thus

$$\operatorname{Supp}_R(M/S_R(\mathfrak{a}, M)) = \operatorname{Supp}_R(N/S_R(\mathfrak{a}, N)).$$

Conversly, if

$$\mathrm{Supp}_R(M/S_R(\mathfrak{a},M)) = \mathrm{Supp}_R(N/S_R(\mathfrak{a},N)),$$

then [**6**, Corollary 3] implies that

$$\mathrm{Att}_R(\mathrm{H}^d_{\mathfrak{a}}(M/S_R(\mathfrak{a},M))) = \mathrm{Att}_R(\mathrm{H}^d_{\mathfrak{a}}(N/S_R(\mathfrak{a},N))).$$

Thus

$$\mathrm{Att}_R(\mathrm{H}^d_{\mathfrak{a}}(M)) = \mathrm{Att}_R(\mathrm{H}^d_{\mathfrak{a}}(N))$$

by Lemma 2.5, as required. □

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References

1. Atazadeh A., Sedghi M. and Naghipour R., *On the annihilators and attached primes of top local cohomology modules*, Arch. Math. **102** (2014) 225–236.

2. Atazadeh A., Sedghi M. and Naghipour R., *Cohomological dimension filtration and annihilators of top local cohomology modules*, Colloq. Math. **139** (2015), 25–35.

3. Bahmanpour K., *Annihilators of local cohomology modules*, Comm. Algebra **43** (2015), 2509–2515.

4. Bahmanpour K., Azami J. and Ghasemi G., *On the annihilators of local cohomology modules*, J. Algebra **363** (2012), 8–13.

5. Brodmann M. and Sharp R. Y., *Local cohomology: an algebraic introduction with geometric applications*, Cambridge Univ. Press 1998.

6. Dibaei M. T. and Yassemi S., *Attached primes of the top local cohomology modules with respect to an ideal*, Arch. Math. **84** (2005), 292–297.

7. Divaani-Aazar K., Naghipour R. and Tousi M., *Cohomological dimension of certain algebraic varieties*, Proc. Amer. Math. Soc. **130** (2002), 3537–3544.

8. Lynch L. R., *Annihilators of top local cohomology*, Comm. Algebra **40** (2012), 542–551.

9. MacDonald I. G., *Secondary representations of modules over a commutative ring*, In Symposia Mat. 11, Istituto Nazionale di alta Matematica, Roma, 1973, 23–43.

10. Rezaei Sh., *Some results on top local cohomolgy and top formal local cohomology modules*, Comm. Algebra **45(2)** (2017), 1935–1940.

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