ON THE RADICAL OF THE ANNIHILATORS OF LOCAL COHOMOLOGY MODULES

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Abstract. Let $R$ be a noetherian ring, $M$ a non-zero finitely generated $R$-module of dimension $d$. Let $a$ be an ideal of $R$ and the cohomological dimension of $M$ with respect to $a$ equal to $d$. In this paper, we calculate the radical of the annihilator of the top local cohomology module $H^d_a(M)$. In fact, we prove that there exists a submodule $S_R(a, M)$ of $M$ such that the radical of the annihilator of $H^d_a(M)$ equals the annihilator of the quotient $M/S_R(a, M)$. By using this result, for a complete local ring $(R, \mathfrak{m})$, we determine the set of attached prime ideals of $H^d_a(M)$.

1. Introduction

Throughout this paper, $R$ is a commutative noetherian ring with identity, $a$ is an ideal of $R$ and $M$ is an $R$-module. Recall that the $i$-th local cohomology module of $M$ with respect to $a$ is defined as

$$H^i_a(M) := \lim_{\to} \text{Ext}^i_R(R/a^n, M).$$

One of the important problems in local cohomology is to find the annihilator of the local cohomology module $H^d_a(M)$. This problem has been studied by several authors, see for example, [1], [3], [4], and [8]. In [4], the authors proved that if $(R, \mathfrak{m})$ is a complete noetherian local ring and $M$ is a finitely generated $R$-module, then $\text{Ann}_R(H^d_{\mathfrak{m}}(M)) = \text{Ann}_R(M/T_R(M))$, where $T_R(M)$ is the largest submodule of $M$ such that $\dim T_R(M) < \dim M$. This result was later extended to non-complete noetherian local rings by Bahmanpour in [3]. Also, for an ideal $a$ (not necessarily $a = \mathfrak{m}$) in an arbitrary noetherian ring $R$ (not necessarily local), in [1], Atazadeh et al. proved that $\text{Ann}_R(H^d_{\mathfrak{a}}(M)) = \text{Ann}_R(M/T_R(a, M))$ where $T_R(a, M)$ is the largest submodule of $M$ such that $\text{cd}(a, T_R(a, M)) < \text{cd}(a, M)$.

This is natural to ask about the radical of the annihilator of the top local cohomology modules. In this paper, we define a new notation $S_R(a, M)$ and determine the radical of the annihilator of the top local cohomology module $H^d_a(M)$ according to $S_R(a, M)$.

For an $R$-module $M$ and an ideal $a$, the cohomological dimension of $M$ with respect to $a$ is defined as $\text{cd}(a, M) := \max\{i \in \mathbb{Z} : H^i_a(M) \neq 0\}$. For more details,
In this section, we define a radical of the annihilator of the top local cohomology module $H^d_R$. Let $R$ be a noetherian ring and $M$ be a finitely generated $R$-module of finite dimension $d$ such that $\text{cd}(a, M) = d$. Then $\sqrt{\text{Ann}_R H^d_a(M)} = \text{Ann}_R(M/S_R(a, M))$, where $S_R(a, M)$ is the maximal element of the set $\Theta := \{N : N \leq M \text{ and } \{p \in \text{Ass}_R M \mid \text{cd}(a, R/p) = d\} \subseteq \text{Ass}_R(M/N)\}$.

A non-zero $R$-module $M$ is called secondary if its multiplication map by any element $a$ of $R$, is either surjective or nilpotent. A prime ideal $p$ of $R$ is said to be an attached prime of $M$ if $p = (N :_R M)$ for some submodule $N$ of $M$. If $M$ admits a reduced secondary representation, $M = S_1 + S_2 + \cdots + S_n$, then the set of attached primes $\text{Att}_R(M)$ of $M$ is equal to $\{\sqrt{0 :_R S_i} : i = 1, \ldots, n\}$, (see [9]).

We have the following well known result about the attached primes of local cohomology modules.

**Theorem 1.1.** Let $R$ be a noetherian ring and $a$ an ideal of $R$. Let $M$ be a finitely generated $R$-module of dimension $d$ such that $\text{cd}(a, M) = d$. Then $\sqrt{\text{Ann}_R H^d_a(M)} = \text{Ann}_R(M/S_R(a, M))$, where $S_R(a, M)$ is the maximal element of the set $\Theta := \{N : N \leq M \text{ and } \{p \in \text{Ass}_R M \mid \text{cd}(a, R/p) = d\} \subseteq \text{Ass}_R(M/N)\}$.

Let $R$ be a noetherian ring and $a$ an ideal of $R$. Let $M$ be a finitely generated $R$-module of finite dimension $d$ such that $\text{cd}(a, M) = d$. Then, by using the above main result, we will show that:

1. If $r \in \sqrt{\text{Ann}_R H^d_a(M)}$ if and only if $\text{Att}_R H^d_a(M) \subseteq \text{Ass}_R M/(rM + S_R(a, M))$, for every non-zero element $r$ of $R$.
2. $\bigcap_{p \in \text{Att}_R H^d_a(M)} p = \text{Ann}_R(M/S_R(a, M))$
3. If $R$ is a complete noetherian local ring, then $\text{Att}_R H^d_a(M) = \text{Min Supp}_R(M/S_R(a, M))$.

2. **Annihilators of Local Cohomology Modules**

Let $R$ be a noetherian ring, $a$ an ideal of $R$ and $M$ be a non-zero finitely generated $R$-module of dimension $d$. Let $\text{cd}(a, M) = d$. In this section we calculate the radical of the annihilator of the top local cohomology module $H^d_a(M)$.

Recall that for an $R$-module $M$ of finite dimension, the submodule $T_R(M)$ of $M$ is defined as follows:

$$T_R(M) := \bigcup \{N : N \leq M \text{ and } \dim N < \dim M\}.$$

In [1], we see the following notation which is a generalization of $T_R(M)$ for an arbitrary ideal $a$ and non-zero finitely generated $R$-module $M$.

$$T_R(a, M) := \bigcup \{N : N \leq M \text{ and } \text{cd}(a, N) < \text{cd}(a, M)\}.$$
In fact, \( T_R(a, M) \) is the largest submodule of \( M \) such that \( \text{cd}(a, T_R(a, M)) < \text{cd}(a, M) \). Clearly, for a local ring \((R, \mathfrak{m})\), we have \( T_R(\mathfrak{m}, M) = T_R(M) \).

It is known that \( T_R(a, M) \) has the following properties. For the details, see [2, Lemma 3.1].

**Proposition 2.1.** Let \( R \) be a noetherian ring and \( a \) an ideal of \( R \). Let \( M \) be a non-zero finitely generated \( R \)-module such that \( \text{cd}(a, M) = \dim M \). Then:

i) \( \text{cd}(a, M/T_R(a, M)) = \dim M \).

ii) \( H_a^{\dim M}(M) \cong H_a^{\dim M}(M/T_R(a, M)) \).

iii) \( \{ p \in \text{Ass}_R M \mid \text{cd}(a, R/p) = \dim M \} = \text{Ass}_R M/T_R(a, M) = \text{Att}_R H_a^{\dim M}(M) \).

The following theorem is main result in [1], on the annihilators of top local cohomology modules.

**Theorem 2.2.** [1, Theorem 2.3] Let \( R \) be a noetherian ring and \( a \) an ideal of \( R \). Let \( M \) be a non-zero finitely generated \( R \)-module such that \( \text{cd}(a, M) = \dim M \). Then \( \text{Ann}_R H_a^{\dim M}(M) = \text{Ann}_R(M/T_R(a, M)) \).

In the following, we give a new notation \( S_R(a, M) \) which plays an essential role in our proofs.

**Definition 2.3.** Let \( a \) be an ideal of \( R \) and \( M \) be a non-zero finitely generated \( R \)-module of dimension \( d \). By \( S_R(a, M) \), we denote the maximal element of the set

\[ \Theta := \{ N : N \subseteq M \text{ and } \{ p \in \text{Ass}_R M \mid \text{cd}(a, R/p) = d \} \subseteq \text{Ass}_R(M/N) \} \]

Note that by Proposition 2.1(iii), \( T_R(a, M) \in \Theta \), and so \( \Theta \) is non-empty. Since \( M \) is a noetherian \( R \)-module, it follows that \( \Theta \) has a maximal element \( S_R(a, M) \) with respect to inclusion.

**Lemma 2.4.** Let \( R \) be a noetherian ring and \( a \) an ideal of \( R \). Let \( M \) be a finitely generated \( R \)-module of finite dimension such that \( \text{cd}(a, M) = \dim M \). Then \( \dim(M/S_R(a, M)) = \text{cd}(a, M/S_R(a, M)) = \dim M \).

**Proof.** Since \( \text{cd}(a, M) = \dim M \), thus \( H_a^{\dim M}(M) \neq 0 \), and so \( \text{Att}_R(H_a^{\dim M}(M)) = \{ p \in \text{Ass}_R M \mid \text{cd}(a, R/p) = \dim M \} \neq \emptyset \) by Theorem 1.2.

If \( p \in \{ p \in \text{Ass}_R M \mid \text{cd}(a, R/p) = \dim M \} \), then \( p \in \text{Ass}_R(M/S_R(a, M)) \) by definition of \( S_R(a, M) \). Thus \( \text{Supp}_R(R/p) \subseteq \text{Supp}_R(M/S_R(a, M)) \). Now by [7, Theorem 2.2], we have \( \dim M = \text{cd}(a, R/p) \leq \text{cd}(a, M/S_R(a, M)) \). But, \( \text{cd}(a, M/S_R(a, M)) \leq \dim(M/S_R(a, M)) \leq \dim M \). Therefore,

\[ \dim M = \text{cd}(a, M/S_R(a, M)) = \dim(M/S_R(a, M)) \]

as required. \( \square \)

**Lemma 2.5.** Let \( R \) be a noetherian ring and \( a \) an ideal of \( R \). Let \( M \) be a finitely generated \( R \)-module of finite dimension \( d \) such that \( \text{cd}(a, M) = d \). Then \( \text{Att}_R H_a^d(M) = \text{Att}_R H_a^d(M/S_R(a, M)) \), and so

\[ \sqrt{\text{Ann}_R(H_a^d(M))} = \sqrt{\text{Ann}_R(H_a^d(M/S_R(a, M)))} \]
Thus \( \text{cd}(R) \leq \text{dim}(M) \leq \text{dim}(M/U) \leq \dim M \), and so

\[ \text{cd}(M/U) = \dim(M/U) = \dim M. \]

Now, the short exact sequence

\[ 0 \to S/U \to M/U \to M/S \to 0 \]

implies that \( \text{Att}_R(H^d_a(M)) \subseteq \text{Att}_R(H^d_a(M/U)). \) By Lemma 2.5, we conclude

\[ \text{Att}_R(H^d_a(M)) \subseteq \text{Att}_R(H^d_a(M/U)) \subseteq \text{Ass}_R(M/U), \]

as required. \( \square \)

**Lemma 2.7.** Let \( R \) be a noetherian ring and \( a \) an ideal of \( R \). Let \( M \) be a finitely generated \( R \)-module of finite dimension \( d \) such that \( \text{cd}(a,M) = d \). Then \( \sqrt{\text{Ann}_R(M/S_R(a,M))} = \text{Ann}_R(M/S_R(a,M)). \)

**Proof.** Let \( S := S_R(a,M). \) Let \( x \in \sqrt{\text{Ann}_R(M/S)}. \) There exists an integer \( n \) such that \( x^n M \subseteq S. \) Thus Lemma 2.5 implies that

\[ \text{Att}_R(H^d_a(M)) = \text{Att}_R(H^d_a(M/S)) = \text{Att}_R(H^d_a(M/x^n M + S)). \]

Since \( \text{Supp}_R(M/x^n M + S) = \text{Supp}_R(M/x^M + S), \) [6, Corollary 3] implies that

\[ \text{Att}_R(H^d_a(M/x^n M + S)) = \text{Att}_R(H^d_a(M/x^M + S)). \]

Hence, \( \text{Att}_R(H^d_a(M)) = \text{Att}_R(H^d_a(M/x^M + S)). \) Thus

\[ \text{Att}_R(H^d_a(M)) = \{ p \in \text{Ass}_R M \mid \text{cd}(a,R/p) = d \} \subseteq \text{Ass}_R(M/x^M + S). \]

By maximality of \( S \), we conclude that \( xM + S \subseteq S \). Therefore, \( xM \subseteq S \), and so \( x \in \text{Ann}_R(M/S) \), the proof is complete. \( \square \)

The following theorem is the main result of this paper.
Theorem 2.8. Let $R$ be a noetherian ring and $a$ an ideal of $R$. Let $M$ be a finitely generated $R$-module of finite dimension $d$ such that $\text{cd}(a, M) = d$. Then
\[
\sqrt{\text{Ann}_R H_a^d(M)} = \text{Ann}_R (M/S_R(a, M)).
\]

Proof. Let $S := S_R(a, M)$ and $T := T_R(a, M)$. At first, we show that $T_R(a, M/S) = 0$. It suffices to show that for any non-zero submodule $L/S$ of $M/S$ we have $\text{cd}(a, L/S) = \text{cd}(a, M/S)$. It is easy to see that
\[
\{p \in \text{Ass} M : \text{cd}(a, R/p) = d\} \subseteq \text{Ass}_R (M/S) \subseteq \text{Ass}_R L/S \cup \text{Ass}_R M/L.
\]
If $\{p \in \text{Ass} M : \text{cd}(a, R/p) = d\} \subseteq \text{Ass}_R (M/L)$ then since $S \subseteq L$ from the maximality of $S$, we get a contradiction. Thus there exists a prime ideal $p$ such that $p \in \text{Ass}_R L/S$. Hence $d = \text{cd}(a, R/p) \leq \text{cd}(a, L/S) \leq \text{cd}(a, M/S) \leq \text{cd}(a, M) = d$. It follows that $\text{cd}(a, L/S) = \text{cd}(a, M/S)$ and so $T_R(a, M/S) = 0$. On the other hand, by Lemma 2.5 and Theorem 2.2, we have
\[
\sqrt{\text{Ann}_R H_a^d(M)} = \sqrt{\text{Ann}_R (M/S)} = \sqrt{(\text{Ann}_R (M/S)/T_R(a, M/S))}.
\]
Since $T_R(a, M/S) = 0$, by Lemma 2.7, we get
\[
\sqrt{\text{Ann}_R H_a^d(M)} = \sqrt{\text{Ann}_R (M/S)} = \text{Ann}_R (M/S),
\]
as required. \hfill \Box

Corollary 2.9. Let $R$ be a noetherian ring and $a$ an ideal of $R$. Let $M$ be a finitely generated $R$-module of finite dimension $d$ such that $\text{cd}(a, M) = d$. Then
\[
\sqrt{\text{Ann}_R H_a^d(M)} = 0 \text{ if and only if } \text{Att}_R H_a^d(M) \subseteq \text{Ass}_R M/(rM + S_R(a, M)) \text{ for every non-zero element } r \text{ of } R.
\]

Proof. Set $S := S_R(a, M)$ and let $0 \neq r \in R$. If $r \in \sqrt{\text{Ann}_R H_a^d(M)}$, then $r \in \text{Ann}_R M/S$ by Theorem 2.8. Thus $rM \subseteq S$, and so $\text{Att}_R H_a^d(M) \subseteq \text{Ass}_R M/(rM + S)$ by Proposition 2.6. Conversely, if $\text{Att}_R H_a^d(M) \subseteq \text{Ass}_R M/(rM + S)$, then $rM \subseteq S$ by maximality of $S$. Thus $r \in \text{Ann}_R M/S = \sqrt{\text{Ann}_R H_a^d(M)}$ and the proof is complete. \hfill \Box

Corollary 2.10. Let $R$ be a noetherian ring and $M$ a finitely generated $R$-module of dimension $d$ such that $\text{cd}(a, M) = d$. Then
\[
\bigcap_{p \in \text{Att}_R (H_a^d(M))} p = \text{Ann}_R M/S_R(a, M).
\]

Proof. By [5, 7.2.11] and Theorem 2.8, we have
\[
\bigcap_{p \in \text{Att}_R (H_a^d(M))} p = \sqrt{\text{Ann}_R H_a^d(M)} = \text{Ann}_R (M/S_R(a, M)),
\]
as required. \hfill \Box
Corollary 2.11. Let $R$ be a noetherian ring and $a$ an ideal of $R$. Let $M$ be a finitely generated $R$-module of finite dimension $d$ such that $\text{cd}(a, M) = d$. Then
\[ \sqrt{\text{Ann}_R(M/T_R(a, M))} = \text{Ann}_R(M/S_R(a, M)). \]

Proof. It follows from Theorems 2.2 and 2.8. □

In the next result, we determine $V(\text{Ann}_R H^d_a(M))$, the set of prime ideals of $R$ containing ideal $\text{Ann}_R H^d_a(M)$.

Corollary 2.12. Let $R$ be a noetherian ring and $a$ an ideal of $R$. Let $M$ be a finitely generated $R$-module of finite dimension $d$ such that $\text{cd}(a, M) = d$. Then
\[ V(\text{Ann}_R H^d_a(M)) = \text{Supp}_R(M/T_R(a, M)) = \text{Supp}_R(M/S_R(a, M)). \]

Proof. By Theorem 2.2, we have
\[ V(\sqrt{\text{Ann}_R(M/T_R(a, M))}) = V(\text{Ann}_R(M/S_R(a, M))). \]
Thus $\text{Supp}_R(M/T_R(a, M)) = \text{Supp}_R(M/S_R(a, M))$, the proof is complete. □

In the following Corollary, for a complete noetherian local ring $(R, m)$, we show that the set $\text{Att}_R H^d_a(M)$ is equal to the minimal elements of the set $\text{Supp}_R(M/S_R(a, M))$.

Corollary 2.13. Let $(R, m)$ be a complete noetherian local ring and $a$ an ideal of $R$. Let $M$ be a finitely generated $R$-module of finite dimension $d$ such that $\text{cd}(a, M) = d$. Then
\[ \text{Att}_R H^d_a(M) = \text{Min } \text{Supp}_R(M/S_R(a, M)). \]

Proof. By [10, Theorem 2.11 (i)], we have $\text{Att}_R H^d_a(M) = \text{Min } V(\text{Ann}_R H^d_a(M))$. Now Corollary 2.12 completes the proof. □

Corollary 2.14. Let $R$ be a noetherian ring, $a$ an ideal of $R$, $M$ and $N$ be two finitely generated $R$-modules of dimension $d$ with $\text{cd}(a, M) = \text{cd}(a, N) = d$. Then
\[ \text{Att}_R H^d_a(M) = \text{Att}_R H^d_a(N) \]
if and only if
\[ \text{Supp}_R(M/S_R(a, M)) = \text{Supp}_R(N/S_R(a, N)). \]

Proof. If
\[ \text{Att}_R H^d_a(M) = \text{Att}_R H^d_a(N), \]
then
\[ \text{Ann}_R M/S_R(a, M) = \text{Ann}_R N/S_R(a, N) \]
by Corollary 2.10, and so
\[ V(\text{Ann}_R(M/S_R(a, M))) = V(\text{Ann}_R(N/S_R(a, N))). \]
Thus
\[ \text{Supp}_R(M/S_R(a, M)) = \text{Supp}_R(N/S_R(a, N)). \]
Conversely, if
\[ \text{Supp}_R(M/S_R(a, M)) = \text{Supp}_R(N/S_R(a, N)), \]
then [6, Corollary 3] implies that
\[ \text{Att}_R(H^d_a(M/S_R(a, M))) = \text{Att}_R(H^d_a(N/S_R(a, N))). \]
Thus
\[ \text{Att}_R(H^d_a(M)) = \text{Att}_R(H^d_a(N)) \]
by Lemma 2.5, as required.

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