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HERMITE-HADAMARD TYPE INEQUALITIES OBTAINED VIA RIEMANN-LIOUVILLE FRACTIONAL CALCULUS

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Abstract. We extend some inequalities obtained by M. A. Latif to the framework of Riemann-Liouville fractional calculus.

1. Introduction

The Hermite-Hadamard inequality asserts that for every convex function $f\colon [a,b] \to \mathbb{R}$, one has

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\ \mathrm{d}x \leq \frac{f(a)+f(b)}{2},$$

where $a, b \in I$ with $a < b$. One can easily prove that the left term is closer to the integral mean value than the right one. Therefore,

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leq \frac{1}{2}\left(\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right). \tag{1}$$

See [**5**, p. 52].

A remarkable variety of refinements and generalizations of Hermite-Hadamard inequality have been found; see, for example, [**1**], [**3**], [**5**] and the references cited therein.

Our aim is to establish some new inequalities related to (1), using the Riemann-Liouville fractional integration. We deal with functions whose derivatives in absolute value are convex.

Let $f \in L^1[a,b]$, where $a \geq 0$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\mathrm{d}t \qquad \text{for } x > a,$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\mathrm{d}t \qquad \text{for } x < b,$$

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respectively. Here, $\Gamma(\alpha) = \int_0^\infty \mathrm{e}^{-t}\, t^{\alpha-1}\mathrm{d}t$ is the Gamma function. We also make the convention

$$J^0_{a+}f(x) = J^0_{b-}f(x) = f(x).$$

More details about the Riemann-Liouville fractional integrals may be found in [**2**].

2. Main Results

We assume throughout the present paper that $[a, b]$ is a subinterval of $[0, \infty)$ and $f\colon [a, b] \to \mathbb{R}$ is a function differentiable on (a, b) such that $f' \in L^1[a, b]$. Throughout this section we define the *Hermite-Hadamard $\alpha-$gap* by

$$\begin{aligned}\mathcal{H}_\alpha(x) := {} & \frac{((x-a)^\alpha + (b-x)^\alpha)\, f(x)}{b-a} + \frac{(x-a)^\alpha f(a) + (b-x)^\alpha f(b)}{b-a} \\ & - \frac{2^\alpha \Gamma(\alpha+1)}{b-a}\left[J^\alpha_{x-} f\left(\frac{x+a}{2}\right) + J^\alpha_{a+} f\left(\frac{x+a}{2}\right) \right. \\ & \left. + J^\alpha_{b-} f\left(\frac{x+b}{2}\right) + J^\alpha_{x+} f\left(\frac{x+b}{2}\right)\right].\end{aligned}$$

In the particular case where $\alpha = 1$, this reduces to

$$\mathcal{H}(x) = f(x) + \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{2}{b-a}\int_a^b f(t)\mathrm{d}t.$$

Thus

$$\mathcal{H}\!\left(\frac{a+b}{2}\right) = f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} - \frac{2}{b-a}\int_a^b f(t)\mathrm{d}t.$$

The value of $\mathcal{H}$ was estimated by M. A. Latif [**4**] and it is the purpose of the present paper to generalize some of his results. For this we need a preparation.

Lemma 1. *We have*

$$\begin{aligned}&\mathcal{H}_\alpha(x) \\ &= \frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1 \frac{t^\alpha}{2} f'\left(\frac{1+t}{2}x + \frac{1-t}{2}a\right)\mathrm{d}t - \int_0^1 \frac{t^\alpha}{2} f'\left(\frac{1-t}{2}x + \frac{1+t}{2}a\right)\mathrm{d}t\right) \\ &\quad - \frac{(b-x)^{\alpha+1}}{b-a}\left(\int_0^1 \frac{t^\alpha}{2} f'\left(\frac{1+t}{2}x + \frac{1-t}{2}b\right)\mathrm{d}t + \int_0^1 \frac{t^\alpha}{2} f'\left(\frac{1-t}{2}x + \frac{1+t}{2}b\right)\mathrm{d}t\right),\end{aligned}$$

for all $x \in [a, b]$.

Proof. We use the integration by parts and appropriate substitutions (such as $u = \frac{1+t}{2}x + \frac{1-t}{2}a$, $v = \frac{1-t}{2}x + \frac{1+t}{2}a$ etc.) to show that

$$\int_0^1 \frac{t^\alpha}{2} f'\left(\frac{1+t}{2}x + \frac{1-t}{2}a\right)\mathrm{d}t = \frac{f(x)}{x-a} - \frac{2^\alpha\Gamma(\alpha+1)}{(x-a)^{\alpha+1}} J^\alpha_{x-} f\left(\frac{x+a}{2}\right),$$

$$\int_0^1 \frac{t^\alpha}{2} f'\left(\frac{1-t}{2}x + \frac{1+t}{2}a\right)\mathrm{d}t = -\frac{f(a)}{x-a} + \frac{2^\alpha\Gamma(\alpha+1)}{(x-a)^{\alpha+1}} J^\alpha_{a+} f\left(\frac{x+a}{2}\right),$$

$$\int_0^1 \frac{t^\alpha}{2} f'\left(\frac{1+t}{2}x + \frac{1-t}{2}b\right) \mathrm{d}t = -\frac{f(x)}{b-x} + \frac{2^\alpha \Gamma(\alpha+1)}{(b-x)^{\alpha+1}} J^\alpha_{x+} f\left(\frac{x+b}{2}\right),$$
$$\int_0^1 \frac{t^\alpha}{2} f'\left(\frac{1-t}{2}x + \frac{1+t}{2}b\right) \mathrm{d}t = \frac{f(b)}{b-x} - \frac{2^\alpha \Gamma(\alpha+1)}{(b-x)^{\alpha+1}} J^\alpha_{b-} f\left(\frac{x+b}{2}\right).$$

The proof is complete. □

We are now in a position to state and prove the following theorems.

Theorem 1. *Assume $|f'|$ is convex on $[a,b]$. Then*

$$|\mathcal{H}_\alpha(x)| \le \frac{(x-a)^{\alpha+1}}{b-a} \cdot \frac{|f'(x)| + |f'(a)|}{2(\alpha+1)} + \frac{(b-x)^{\alpha+1}}{b-a} \cdot \frac{|f'(x)| + |f'(b)|}{2(\alpha+1)}.$$

Proof. Using Lemma 1 and taking modulus, we infer from the convexity of $|f'|$ that

$$\begin{aligned}
|\mathcal{H}_\alpha(x)| \le{}& \frac{(x-a)^{\alpha+1}}{b-a} \int_0^1 \frac{t^\alpha}{2}\left[\frac{1+t}{2}|f'(x)| + \frac{1-t}{2}|f'(a)|\right] \mathrm{d}t \\
&+ \frac{(x-a)^{\alpha+1}}{b-a} \int_0^1 \frac{t^\alpha}{2}\left[\frac{1-t}{2}|f'(x)| + \frac{1+t}{2}|f'(a)|\right] \mathrm{d}t \\
&+ \frac{(b-x)^{\alpha+1}}{b-a} \int_0^1 \frac{t^\alpha}{2}\left[\frac{1+t}{2}|f'(x)| + \frac{1-t}{2}|f'(b)|\right] \mathrm{d}t \\
&+ \frac{(b-x)^{\alpha+1}}{b-a} \int_0^1 \frac{t^\alpha}{2}\left[\frac{1-t}{2}|f'(x)| + \frac{1+t}{2}|f'(b)|\right] \mathrm{d}t.
\end{aligned}$$

The result follows after a straightforward computation in the right hand side term. This ends the proof. □

Our next result reads as

Theorem 2. *Assume $|f'|^q$ is convex on $[a,b]$ for some fixed $q>1$. Then*

$$\begin{aligned}
|\mathcal{H}_\alpha(x)| \le{}& \left(\frac{1}{2}\right)^{1+2/q} \times \left(\frac{1}{\alpha p+1}\right)^{1/p} \times \\
&\times \left\{ \frac{(x-a)^{\alpha+1}}{b-a}\left[\left(3|f'(x)|^q + |f'(a)|^q\right)^{1/q} + \left(|f'(x)|^q + 3|f'(a)|^q\right)^{1/q}\right]\right. \\
&\left. + \frac{(b-x)^{\alpha+1}}{b-a}\left[\left(3|f'(x)|^q + |f'(b)|^q\right)^{1/q} + \left(|f'(x)|^q + 3|f'(b)|^q\right)^{1/q}\right]\right\}
\end{aligned}$$

for all $x \in [a,b]$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. According to Lemma 1 and Hölder's inequality, we have

$$\begin{aligned}
|\mathcal{H}_\alpha(x)| \le{}& \frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1 \left(\frac{t^\alpha}{2}\right)^p \mathrm{d}t\right)^{1/p}\left((I_1)^{\frac{1}{q}} + (I_2)^{1/q}\right) \\
&+ \frac{(b-x)^{\alpha+1}}{b-a}\left(\int_0^1 \left(\frac{t^\alpha}{2}\right)^p \mathrm{d}t\right)^{1/p}\left((I_3)^{\frac{1}{q}} + (I_4)^{1/q}\right)
\end{aligned}$$

for all $x \in [a,b]$. Here

$$\begin{aligned} I_1 &= \int_0^1 \left| f'\left(\frac{1+t}{2}x + \frac{1-t}{2}a\right)\right|^q \mathrm{d}t \\ &\le \int_0^1 \left(\frac{1+t}{2}|f'(x)|^q + \frac{1-t}{2}|f'(a)|^q\right) \mathrm{d}t = \frac{3|f'(x)|^q + |f'(a)|^q}{4}, \\ I_2 &= \int_0^1 \left| f'\left(\frac{1-t}{2}x + \frac{1+t}{2}a\right)\right|^q \mathrm{d}t \le \frac{|f'(x)|^q + 3|f'(a)|^q}{4}, \\ I_3 &= \int_0^1 \left| f'\left(\frac{1+t}{2}x + \frac{1-t}{2}b\right)\right|^q \mathrm{d}t \le \frac{3|f'(x)|^q + |f'(b)|^q}{4}, \end{aligned}$$

and

$$I_4 = \int_0^1 \left| f'\left(\frac{1-t}{2}x + \frac{1+t}{2}b\right)\right|^q \mathrm{d}t \le \frac{|f'(x)|^q + 3|f'(b)|^q}{4}.$$

These last inequalities hold due to the convexity of $|f'|^q$ on $[a,b]$. The proof is complete. □

Theorem 3. *Assume $|f'|^q$ is convex on $[a,b]$ for some fixed $q \ge 1$. Then the following inequality*

$$\begin{aligned} |\mathcal{H}_\alpha(x)| \le & \left[\frac{1}{2(\alpha+1)}\right]^{1-1/q} \cdot \left[\frac{1}{4(\alpha+1)(\alpha+2)}\right]^{1/q} \\ & \times \left\{ \frac{(x-a)^{\alpha+1}}{b-a} \left[\left((2\alpha+3)|f'(x)|^q + |f'(a)|^q\right)^{1/q} \right.\right. \\ & \qquad \left. + \left(|f'(x)|^q + (2\alpha+3)|f'(a)|^q\right)^{1/q}\right] \\ & + \frac{(b-x)^{\alpha+1}}{b-a} \left[\left((2\alpha+3)|f'(x)|^q + |f'(b)|^q\right)^{1/q} \right. \\ & \qquad \left.\left. + \left(|f'(x)|^q + (2\alpha+3)|f'(b)|^q\right)^{1/q}\right] \right\} \end{aligned}$$

holds for all $x \in [a,b]$.

Proof. Using Lemma 1, the convexity of $|f'|^q$ on $[a,b]$ and the power-mean inequality, we have

$$\begin{aligned} |\mathcal{H}_\alpha(x)| \le & \frac{(x-a)^{\alpha+1}}{b-a} \left(\int_0^1 \frac{t^\alpha}{2} \mathrm{d}t\right)^{1-1/q} \left((J_1)^{1/q} + (J_2)^{1/q}\right) \\ & + \frac{(b-x)^{\alpha+1}}{b-a} \left(\int_0^1 \frac{t^\alpha}{2} \mathrm{d}t\right)^{1-1/q} \left((J_3)^{1/q} + (J_4)^{1/q}\right), \end{aligned}$$

where

$$J_1 = \int_0^1 \frac{t^\alpha}{2} \left| f'\left(\frac{1+t}{2}x + \frac{1-t}{2}a\right)\right|^q \mathrm{d}t \le \frac{(2\alpha+3)|f'(x)|^q + |f'(a)|^q}{4(\alpha+1)(\alpha+2)},$$

$$J_2 = \int_0^1 \frac{t^\alpha}{2} \left| f'\left(\frac{1-t}{2}x + \frac{1+t}{2}a\right)\right|^q \mathrm{d}t \le \frac{|f'(x)|^q + (2\alpha+3)|f'(a)|^q}{4(\alpha+1)(\alpha+2)},$$

$$J_3 = \int_0^1 \frac{t^\alpha}{2} \left| f'\left(\frac{1+t}{2}x + \frac{1-t}{2}b\right)\right|^q \mathrm{d}t \le \frac{(2\alpha+3)|f'(x)|^q + |f'(b)|^q}{4(\alpha+1)(\alpha+2)},$$

and

$$J_4 = \int_0^1 \frac{t^\alpha}{2} \left| f'\left(\frac{1-t}{2}x + \frac{1+t}{2}b\right)\right|^q \mathrm{d}t \le \frac{|f'(x)|^q + (2\alpha+3)|f'(b)|^q}{4(\alpha+1)(\alpha+2)}.$$

Hence the proof of the theorem is complete. □

Theorem 4. *Assume $|f'|^q$ is concave on $[a,b]$ for some fixed $q>1$. Then*

$$|\mathcal{H}_\alpha(x)| \le \frac{1}{2}\left(\frac{1}{\alpha p+1}\right)^{1/p} \left\{ \frac{(x-a)^{\alpha+1}}{b-a}\left[\left|f'\left(\frac{3x+a}{4}\right)\right| + \left|f'\left(\frac{x+3a}{4}\right)\right|\right] \right.$$
$$\left. + \frac{(b-x)^{\alpha+1}}{b-a}\left[\left|f'\left(\frac{3x+b}{4}\right)\right| + \left|f'\left(\frac{x+3b}{4}\right)\right|\right]\right\}$$

for all $x \in [a,b]$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1 and Hölder's integral inequality for $q>1$ and $p=\frac{q}{q-1}$, we have

$$|\mathcal{H}_\alpha(x)| \le \frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1 \left(\frac{t^\alpha}{2}\right)^p \mathrm{d}t\right)^{1/p} (K_1)^{1/q}$$
$$+ \frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1 \left(\frac{t^\alpha}{2}\right)^p \mathrm{d}t\right)^{1/p} (K_2)^{1/q}$$
$$+ \frac{(b-x)^{\alpha+1}}{b-a}\left(\int_0^1 \left(\frac{t^\alpha}{2}\right)^p \mathrm{d}t\right)^{1/p} (K_3)^{1/q}$$
$$+ \frac{(b-x)^{\alpha+1}}{b-a}\left(\int_0^1 \left(\frac{t^\alpha}{2}\right)^p \mathrm{d}t\right)^{1/p} (K_4)^{1/q}$$

for all $x \in [a,b]$. Here,

$$K_1 = \int_0^1 \left| f'\left(\frac{1+t}{2}x + \frac{1-t}{2}a\right)\right|^q \mathrm{d}t$$
$$\leq \left| f'\left(\int_0^1 \left(\frac{1+t}{2}x + \frac{1-t}{2}a\right)\mathrm{d}t\right)\right|^q = \left| f'\left(\frac{3x+a}{4}\right)\right|^q.$$

and similarly,

$$K_2 = \int_0^1 \left| f'\left(\frac{1-t}{2}x + \frac{1+t}{2}a\right)\right|^q \mathrm{d}t \leq \left| f'\left(\frac{x+3a}{4}\right)\right|^q,$$
$$K_3 = \int_0^1 \left| f'\left(\frac{1+t}{2}x + \frac{1-t}{2}b\right)\right|^q \mathrm{d}t \leq \left| f'\left(\frac{3x+b}{4}\right)\right|^q,$$
$$K_4 = \int_0^1 \left| f'\left(\frac{1-t}{2}x + \frac{1+t}{2}b\right)\right|^q \mathrm{d}t \leq \left| f'\left(\frac{x+3b}{4}\right)\right|^q.$$

We used the concavity of $|f'|^q$ on $[a,b]$ and Jensen's integral inequality in order to obtain the last four inequalities. This completes the proof of the theorem. □

Our final result is the following theorem.

Theorem 5. *Suppose $|f'|$ is concave on $[a,b]$. Then*

$$|\mathcal{H}_\alpha(x)| \leq \frac{(x-a)^{\alpha+1}}{2(\alpha+1)(b-a)}\left[\left|f'\left(\frac{(2\alpha+3)x+a}{2(\alpha+2)}\right)\right| + \left|f'\left(\frac{x+(2\alpha+3)a}{2(\alpha+2)}\right)\right|\right]$$
$$+ \frac{(b-x)^{\alpha+1}}{2(\alpha+1)(b-a)}\left[\left|f'\left(\frac{(2\alpha+3)x+b}{2(\alpha+2)}\right)\right| + \left|f'\left(\frac{x+(2\alpha+3)b}{2(\alpha+2)}\right)\right|\right]$$

for all $x \in [a,b]$.

Proof. Using Lemma 1 and taking modulus, we infer from the concavity of $|f'|$ that

$$|\mathcal{H}_\alpha(x)| \leq \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 \left|f'\left(\frac{1+t}{2}x + \frac{1-t}{2}a\right)\right|\frac{t^\alpha}{2}\mathrm{d}t$$
$$+ \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 \left|f'\left(\frac{1-t}{2}x + \frac{1+t}{2}a\right)\right|\frac{t^\alpha}{2}\mathrm{d}t$$
$$+ \frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 \left|f'\left(\frac{1+t}{2}x + \frac{1-t}{2}b\right)\right|\frac{t^\alpha}{2}\mathrm{d}t$$
$$+ \frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 \left|f'\left(\frac{1-t}{2}x + \frac{1+t}{2}b\right)\right|\frac{t^\alpha}{2}\mathrm{d}t$$

$$\leq \frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1 \frac{t^\alpha}{2}\mathrm{d}t\right)\left|f'\left(\frac{\int_0^1\left(\frac{1+t}{2}x+\frac{1-t}{2}a\right)\frac{t^\alpha}{2}\mathrm{d}t}{\int_0^1\frac{t^\alpha}{2}\mathrm{d}t}\right)\right|$$
$$+\frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1 \frac{t^\alpha}{2}\mathrm{d}t\right)\left|f'\left(\frac{\int_0^1\frac{t^\alpha}{2}\left(\frac{1-t}{2}x+\frac{1+t}{2}a\right)\mathrm{d}t}{\int_0^1\frac{t^\alpha}{2}\mathrm{d}t}\right)\right|$$
$$+\frac{(b-x)^{\alpha+1}}{b-a}\left(\int_0^1 \frac{t^\alpha}{2}\mathrm{d}t\right)\left|f'\left(\frac{\int_0^1\frac{t^\alpha}{2}\left(\frac{1+t}{2}x+\frac{1-t}{2}b\right)\mathrm{d}t}{\int_0^1\frac{t^\alpha}{2}\mathrm{d}t}\right)\right|$$
$$+\frac{(b-x)^{\alpha+1}}{b-a}\left(\int_0^1 \frac{t^\alpha}{2}\mathrm{d}t\right)\left|f'\left(\frac{\int_0^1\frac{t^\alpha}{2}\left(\frac{1-t}{2}x+\frac{1+t}{2}b\right)\mathrm{d}t}{\int_0^1\frac{t^\alpha}{2}\mathrm{d}t}\right)\right|$$

for all $x \in [a, b]$, which is equivalent to the inequality in the statement of Theorem 5. □

The case where $\alpha = 1$ in our Theorems 2–5 was previously noted by M. A. Latif [**4**].

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