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ENTROPY SOLUTIONS FOR STRONGLY NONLINEAR PARABOLIC PROBLEMS WITH LOWER ORDER TERMS IN MUSIELAK-ORLICZ SPACES

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Abstract. We give the existence of entropy solutions to a strongly nonlinear parabolic problem having two lowers order terms. We assume that the nonlinear term is an integrable function on $\mathbb{R}$ satisfying the sign condition, while the right-hand side is assumed to be in $L^1(Q)$ and the second order term is Leray-Lions operator defined on the inhomogeneous Musielak-Orlicz space.

1. Introduction

In the present paper, we prove the existence of an entropy solution to the following nonlinear parabolic problem with homogeneous Dirichlet boundary value conditions

$$(1.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} + A(u) - \operatorname{div}\big(\Phi(u)\big) + g(u)\varphi(x,|\nabla u|) = f & \text{in} \quad Q, \\ u = 0 & \text{on} \quad \partial Q, \\ u(x,0) = u_0 & \text{in} \quad \Omega, \end{cases}$$

where Ω is a bounded subset of $\mathbb{R}^N$, T is a positive real number, $Q = \Omega \times (0,T)$. The operator $A(u) = -\operatorname{div}\big(a(x,t,u,\nabla u)\big)$ is a LerayLions operator defined on a subset of $W_0^1 L_\varphi(Q)$, where φ is a Musielak-Orlicz function, the right-hand side $f \in L^1(Q)$. We assume that g is an integrable function in $\mathbb{R}$ satisfying the sign condition, while the function Φ is a continuous function on $\mathbb{R}$.

When Problem (1.1) is investigated, a difficulty is due to the facts that the datum f only belongs to L^1 and the function Φ is not restricted by any growth condition at infinity, so that proving existence of a weak solution (i.e., in the distribution sense) seems to be an arduous task. Loosely speaking, it would require an $L^1_{\mathrm{loc}}(Q)$ a priori estimate on $\Phi(u)$ to be able to define the nonlinear term $\operatorname{div}(\Phi(u))$ as a distribution on Q. In order to define the solution of (1.1), we use the notion of renormalized solutions introduced by R.-J. DiPerna and P.-L. Lions

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([**14**]) for the study of the Boltzmann equation. It was adapted to the study of some nonlinear elliptic or parabolic problems and evolution problems in fluid mechanics ([**7, 9, 12**]). Let us mention that in a joint work with F. Murat (see D. Blanchard, F. Murat [**7**]), the first author obtained an existence and uniqueness result for Problem (1.1) in the case $\Phi \equiv 0$. The existence and uniqueness of renormalized solution of (1.1) proved in [**30, 31**] in the case $g \equiv 0$. When $g \equiv \Phi \equiv 0$ and f is replaced by $f + \operatorname{div}(F)$, the existence and uniqueness of renormalized solution proved in [**8, 29**].

On Orlicz spaces, Elmahi and Meskine [**17**] proved existence of solutions for (1.1), when $\Phi \equiv 0$ where $g(u)\varphi(x, |\nabla u|) \equiv g(x, t, u, \nabla u)$ in [**18**], without assuming any restriction on the N-function M.

In the framework of variable exponent Sobolev spaces in [**2**] Azroul, Benboubker, Redwane, and Yazough, the existence of renormalized solutions for the problem (1.1) without sign condition involving nonstandard growth in the case $g(u)\varphi(x, |\nabla u|) \equiv g(x, t, u, \nabla u)$ and in the elliptic case (see [**3**]).

In the setting of Musielak-Orlicz spaces, Elemine Vall, Ahmed, Touzani, and Benkirane [**15**] proved the existence of solutions for the problem (1.1), where $\Phi \equiv \Phi(x, t, u)$ and $g \equiv 0$. The problem (1.1) recently solved by Talha, Benkirane, and Elemine Vall in [**35**] when the right-hand side is a measure data, $\Phi \equiv 0$ and $g(u)\varphi(x, |\nabla u|) \equiv g(x, t, u, \nabla u)$. A large number of papers devoted to study the existence of solutions to elliptic and parabolic problems under various assumptions and in different contexts a review on classical results, see [**16, 19, 24, 32, 34, 35**].

The study of nonlinear partial differential equations in this type of spaces is strongly motivated by numerous phenomena of physics, namely the problems related to non-Newtonian fluids of strongly inhomogeneous behavior with a high ability of increasing their viscosity under a different stimuli, like the shear rate, magnetic or electric field. The generalized Orlicz (Musielak-Orlicz) spaces are of interest not only as the natural generalization of these important examples, but also in their own right. They appeared in many problems in PDEs and the calculus of variations [**1, 20**] and have applications to image processing [**11, 25**] and fluid dynamics [**23, 27**].

In this paper, our purpose is to prove the existence of entropy solutions to a strongly nonlinear parabolic equation with minimal restrictions for the Musielak-Orlicz functions and $\Phi(u) \neq 0$, while the right-hand side is an L^1-datum. This result can be applied, for example, for finding an entropy solution to the following equation

$$\frac{\partial u}{\partial t} - \operatorname{div}\Big(\frac{m(x, |\nabla u|)}{|\nabla u|}\nabla u + u|u|^{\sigma}\Big) + \frac{\operatorname{sign(u)}}{1+u^2}\varphi(x, |\nabla u|) = f \in L^1(Q),$$

where m is the partial derivative of $\varphi(x, t)$ with respect to t. A particular case is $\varphi(x, t) = \frac{1}{p(x)} t^{p(x)}$.

The paper is organized as follows: In Section 2, we introduce some basic definitions and properties in inhomogeneous Musielak-Orlicz-Sobolev spaces as well as an abstract theorem. In Section 3, we prepare some auxiliary results, to prove main result. Finally, in Section 4, we give basic assumptions on a, Φ, g, f, and we state the main result and proofs.

2. Preliminaries

In this section we list briefly some definitions and facts about Musielak-Orlicz-Sobolev spaces. Standard reference is [**4, 28**].

2.1. Musielak-Orlicz function

Let Ω be an open set in $\mathbb{R}^N$ and let φ be a real-valued function defined in $\Omega \times \mathbb{R}_+$, satisfying the following conditions:

(a) $\varphi(x, .)$ is an N-function for almost all $x \in \Omega$ (i.e., convex, strictly increasing, continuous, $\varphi(x, 0) = 0$, $\varphi(x, t) > 0$ for all $t > 0$, $\lim\limits_{t \to 0} \operatorname*{ess\,sup}\limits_{x \in \Omega} \frac{\varphi(x,t)}{t} = 0$ and $\lim\limits_{t \to \infty} \operatorname*{ess\,inf}\limits_{x \in \Omega} \frac{\varphi(x,t)}{t} = \infty$),

(b) $\varphi(\cdot, t)$ is a measurable function for all $t > 0$.

The function φ is called a Musielak-Orlicz function.

For a Musielak-Orlicz function φ, we put $\varphi_x(t) = \varphi(x, t)$ and associate its nonnegative reciprocal function φ_x^{-1}, with respect to t, that is

$$\varphi_x^{-1}(\varphi(x, t)) = \varphi(x, \varphi_x^{-1}(t)) = t.$$

The Musielak-Orlicz function φ is said to satisfy the Δ_2-condition if for some $k > 0$ and a non negative function h, integrable in Ω, we have

$$\varphi(x, 2t) \leq k\, \varphi(x, t) + h(x) \qquad \text{for almost all } x \in \Omega \text{ and } t \geq 0. \tag{2.1}$$

When (2.1) holds only for $t \geq t_0$, for some $t_0 > 0$, then φ is said to satisfy the Δ_2-condition near infinity.

Let φ and γ be two Musielak-Orlicz functions, we say that φ dominates γ and we write $\gamma \prec \varphi$, near infinity (resp. globally) if there exist two positive constants c and t_0 such that for almost all $x \in \Omega$

$$\gamma(x, t) \leq \varphi(x, ct) \text{ for all } t \geq t_0, \quad (\text{resp. for all } t \geq 0, \text{ i.e. } t_0 = 0).$$

We say that γ grows essentially less rapidly than φ at 0 (resp., near infinity) and we write $\gamma \prec\prec \varphi$ if for every positive constant c, we have

$$\lim_{t \to 0} \Big(\operatorname*{ess\,sup}_{x \in \Omega} \frac{\gamma(x, ct)}{\varphi(x, t)} \Big) = 0 \quad \Big(\text{resp., } \lim_{t \to \infty} \Big(\operatorname*{ess\,sup}_{x \in \Omega} \frac{\gamma(x, ct)}{\varphi(x, t)} \Big) = 0\Big).$$

Remark 1 ([**6**]). If $\gamma \prec\prec \varphi$ near infinity, then for all $\varepsilon > 0$ there exists $t_0 > 0$ such that for almost all $x \in \Omega$, we have

$$\gamma(x, t) \leq \varphi(x, \varepsilon t) \quad \text{for all } t \geq t_0. \tag{2.2}$$

2.2. Musielak-Orlicz space:

For a Musielak-Orlicz function φ and a measurable function $u\colon \Omega \to \mathbb{R}$, we define the functional

$$\rho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x, |u(x)|)\, \mathrm{d}x$$

The set $K_\varphi(\Omega) = \{u\colon \Omega \to \mathbb{R} \text{ measurable} \mid \rho_{\varphi,\Omega}(u) < \infty\}$ is called the Musielak-Orlicz class (or generalized Orlicz class). The Musielak-Orlicz space (the generalized Orlicz space) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently,

$$L_\varphi(\Omega) = \Big\{u\colon \Omega \to \mathbb{R} \text{ measurable } \Big| \rho_{\varphi,\Omega}\Big(\frac{u}{\lambda}\Big) < \infty \text{ for some } \lambda > 0\Big\}.$$

For a Musielak-Orlicz function φ, we put: $\psi(x,s) = \sup_{t\geq 0}\{st - \varphi(x,t)\}$, ψ is the Musielak-Orlicz function complementary to φ (or conjugate of φ) in the sense of Young with respect to the variable s.

In the space $L_\varphi(\Omega)$, we define the following two norms:

$$\|u\|_{\varphi,\Omega} = \inf\Big\{\lambda > 0 / \int_\Omega \varphi\Big(x, \frac{|u(x)|}{\lambda}\Big)\, \mathrm{d}x \leq 1\Big\}.$$

1. the Luxemburg norm 2. the so–called Orlicz norm

$$|||u|||_{\varphi,\Omega} = \sup_{\|v\|_{\psi,\Omega}\leq 1} \int_\Omega |u(x)v(x)|\, \mathrm{d}x,$$

where ψ is the Musielak-Orlicz function complementary to φ. These two norms are equivalent [**28**].

The closure in $L_\varphi(\Omega)$ of the bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$.

The Musielak function φ is called locally integrable on Ω if $\rho_\varphi(t\chi_E) < \infty$ for all $t > 0$ and all measurable $E \subset \Omega$ with $\mathrm{meas}(E) < \infty$.

Let φ be a locally integrable Musielak function. Then $E_\varphi(\Omega)$ is separable [**13**].

We say that a sequence of functions $(u_n) \subset L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $\lambda > 0$ such that

$$\lim_{n\to\infty} \rho_{\varphi,\Omega}\Big(\frac{u_n - u}{\lambda}\Big) = 0.$$

For any fixed nonnegative integer m, we define

$$W^m L_\varphi(\Omega) = \{u \in L_\varphi(\Omega) : \text{for all } |\alpha| \leq m,\ D^\alpha u \in L_\varphi(\Omega)\}$$

and

$$W^m E_\varphi(\Omega) = \{u \in E_\varphi(\Omega) : \text{for all } |\alpha| \leq m,\ D^\alpha u \in E_\varphi(\Omega)\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ with nonnegative integers α_i, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $D^\alpha u$ denote the distributional derivatives. The space $W^m L_\varphi(\Omega)$ is called the Musielak-Orlicz Sobolev space.

Let

$$\overline{\rho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\leq m} \rho_{\varphi,\Omega}\big(D^\alpha u\big) \quad \text{and} \quad \|u\|^m_{\varphi,\Omega} = \inf\Big\{\lambda > 0 : \overline{\rho}_{\varphi,\Omega}\big(\frac{u}{\lambda}\big) \leq 1\Big\}$$

for $u \in W^m L_\varphi(\Omega)$, then these functionals are a convex modular and a norm on $W^m L_\varphi(\Omega)$, respectively, and the pair $\big(W^m L_\varphi(\Omega), \|\|\|^m_{\varphi,\Omega}\big)$ is a Banach space if φ satisfies the following condition [**28**]:

$$\text{there exists a constant } c_0 > 0 \text{ such that } \operatorname{ess}\inf_{x\in\Omega} \varphi(x,1) \geq c_0. \tag{2.3}$$

The space $W^m L_\varphi(\Omega)$ is always identified to a subspace of the product $\prod_{|\alpha|\leq m} L_\varphi(\Omega) = \Pi L_\varphi$, this subspace is $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closed.

The space $W_0^m L_\varphi(\Omega)$ is defined as the $\sigma(\Pi L_\varphi,\Pi E_\psi)$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$ and the space $W_0^m E_\varphi(\Omega)$ as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$.

The following spaces of distributions are also used:

$$W^{-m} L_\psi(\Omega) = \Big\{ f \in D'(\Omega);\ f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_\psi(\Omega) \Big\}$$

and

$$W^{-m} E_\psi(\Omega) = \Big\{ f \in D'(\Omega);\ f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in E_\psi(\Omega) \Big\}.$$

We say that a sequence of functions $(u_n) \subset W^m L_\varphi(\Omega)$ is modular convergent to $u \in W^m L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \overline{\rho}_{\varphi,\Omega}\Big(\frac{u_n - u}{k}\Big) = 0.$$

For φ and her complementary function ψ, the following inequality is called the Young inequality [**28**]:

$$ts \leq \varphi(x,t) + \psi(x,s) \quad \text{for all } t,s \geq 0 \text{ and almost all } x \in \Omega. \tag{2.4}$$

This inequality implies that

$$\|\|u\|\|_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) + 1. \tag{2.5}$$

In $L_\varphi(\Omega)$, we have the relation between the norm and the modular

$$\|u\|_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) \qquad \text{if } \|u\|_{\varphi,\Omega} > 1, \tag{2.6}$$

$$\|u\|_{\varphi,\Omega} \geq \rho_{\varphi,\Omega}(u) \qquad \text{if } \|u\|_{\varphi,\Omega} \leq 1. \tag{2.7}$$

For two complementary Musielak-Orlicz functions φ and ψ, let $u \in L_\varphi(\Omega)$ and $v \in L_\psi(\Omega)$, then, we have the Hölder inequality [**28**]

$$\Big| \int_\Omega u(x)v(x)\,\mathrm{d}x \Big| \leq \|u\|_{\varphi,\Omega} \|\|v\|\|_{\psi,\Omega}. \tag{2.8}$$

2.3. Inhomogeneous Musielak-Orlicz-Sobolev spaces

Let Ω be a bounded open subset of $\mathbb{R}^N$ and let $Q = \Omega\times]0,T[$ with some given $T > 0$. Let φ be a Musielak function, denote a real-valued function defined in $Q \times \mathbb{R}_+$. For each a $\alpha \in \mathbb{N}^N$, denote by D_x^α a the distributional derivative on Q

of order α with respect to the variable $x \in \mathbb{R}^N$. The inhomogeneous Musielak-Orlicz-Sobolev space of order 1 is defined as follows

$$W^{1,x}L_\varphi(Q) = \{u \in L_\varphi(Q) : \text{for all } |\alpha| \leq 1 \; D_x^\alpha u \in L_\varphi(Q)\}$$

and

$$W^{1,x}E_\varphi(Q) = \{u \in E_\varphi(Q) : \text{for all } |\alpha| \leq 1 \; D_x^\alpha u \in E_\varphi(Q)\}.$$

The last space is a subspace of the first one and both are Banach spaces under the norm

$$\|u\| = \sum_{|\alpha| \leq m} \|D_x^\alpha u\|_{\varphi,Q}.$$

We can easily show that they form a complementary system when Ω is a Lipschitz domain [**6**]. These spaces are considered as subspaces of the product space $\Pi L_\varphi(Q)$ which has $(N+1)$ copies. We also consider the weak topologies $\sigma(\Pi L_\varphi, \Pi E_\psi)$ and $\sigma(\Pi L_\varphi, \Pi L_\psi)$. If $u \in W^{1,x}L_\varphi(Q)$, then the function $t \mapsto u(t) = u(t,.)$ is defined on $[0,T]$ with values in $W^1L_\varphi(\Omega)$. Further, if $u \in W^{1,x}E_\varphi(Q)$, then this function is $W^1E_\varphi(\Omega)$ valued and strongly measurable. Furthermore, the following imbedding holds: $W^{1,x}E_\varphi(Q) \subset L^1(0,T;W^1E_\varphi(\Omega))$. In general, $W^{1,x}L_\varphi(Q)$ is not a separable space $u \in W^{1,x}L_\varphi(Q)$, we can not conclude that the function $u(t)$ is measurable on $[0,T]$. However, the scalar function $t \mapsto \|u(t)\|_{\varphi,\Omega}$ is in $L^1(0,T)$. The space $W_0^{1,x}E_\varphi(Q)$ is defined as the (norm) closure in $W^{1,x}E_\varphi(Q)$ of $\mathcal{D}(Q)$. We can easily show as in [**6**] that when Ω a is Lipschitz domain, then each element u of the closure of $\mathcal{D}(Q)$ with respect of the weak-* topology $\sigma(\Pi L_\varphi, \Pi E_\psi)$ is limit of some sequence $(u_i) \subset \mathcal{D}(Q)$ in $W^{1,x}L_\varphi(Q)$, for the modular convergence, i.e., there exists $\lambda > 0$ such that for all $|\alpha| \leq 1$,

$$\int_Q \varphi(x, \Big(\frac{D_x^\alpha u_i - D_x^\alpha u}{\lambda}\Big))\,\mathrm{d}x\,\mathrm{d}t \to 0 \qquad \text{as } i \to \infty,$$

this implies that (u_i) converges to u for the weak topology $\sigma(\Pi L_M, \Pi L_\psi)$ in $W^{1,x}L_\varphi(Q)$. Consequently

$$\overline{\mathcal{D}(Q)}^{\sigma(\Pi L_\varphi, \Pi E_\psi)} = \overline{\mathcal{D}(Q)}^{\sigma(\Pi L_\varphi, \Pi L_\psi)}.$$

This space is denoted by $W_0^{1,x}L_\varphi(Q)$. Furthermore, $W_0^{1,x}E_\varphi(Q) = W_0^{1,x}L_\varphi(Q) \cap \Pi E_\varphi$.

We have the following complementary system

$$\begin{pmatrix} W_0^{1,x}L_\varphi(Q) & F \\ W_0^{1,x}E_\varphi(Q) & F_0 \end{pmatrix},$$

F being the dual space of $W_0^{1,x}E_\varphi(Q)$. Except for an isomorphism, it is also the quotient of ΠL_ψ by the polar set $W_0^{1,x}E_\varphi(Q)^\perp$, denoted by $F = W^{-1,x}L_\psi(Q)$ where

$$W^{-1,x}L_\psi(Q) = \Big\{f = \sum_{|\alpha| \leq 1} D_x^\alpha f_\alpha : f_\alpha \in L_\psi(Q)\Big\}.$$

This space is equipped with the usual quotient norm

$$\|f\| = \inf \sum_{|\alpha|\leq 1} \|f_\alpha\|_{\psi,Q},$$

where the infimum is taken on all possible decompositions

$$f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha, \qquad f_\alpha \in L_\psi(Q).$$

The space F_0 is then given by

$$F_0 = \Big\{ f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha : f_\alpha \in E_\psi(Q) \Big\},$$

and is denoted by $F_0 = W^{-1,x} E_\psi(Q)$.

3. Auxiliary results

In this section, we give some preliminaries lemmas.

Lemma 3.1 ([**6**]). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$, and let φ and ψ be two complementary Musielak-Orlicz functions which satisfy the following conditions:*

i) *There exists a constant $c_0 > 0$ such that $\operatorname{ess\,inf}_{x\in\Omega} \varphi(x,1) \geq c_0$,*

ii) *There exists a constant $A > 0$ such that for almost all $x, y \in \Omega$ with $|x-y| \leq \frac{1}{2}$, we have*

$$\frac{\varphi(x,t)}{\varphi(y,t)} \leq t^{\left(\frac{A}{log(\frac{1}{|x-y|})}\right)} \qquad \text{for all } t \geq 1. \tag{3.1}$$

iii)

$$\int_\Omega \varphi(x,\lambda)\,\mathrm{d}x < \infty \qquad \text{for all } \lambda > 0. \tag{3.2}$$

iv) *There exists a constant $c_2 > 0$ such that $\psi(x,1) \leq c_2$, a.e in Ω.*

Under these assumptions, $D(\Omega)$ is dense in $L_\varphi(\Omega)$ with respect to the modular topology, $D(\Omega)$ is dense in $W_0^1 L_\varphi(\Omega)$ for the modular convergence and $D(\overline{\Omega})$ is dense in $W^1 L_\varphi(\Omega)$ for the modular convergence.

Consequently, the action of a distribution S in $W^{-1} L_\psi(\Omega)$ on an element u of $W_0^1 L_\varphi(\Omega)$ is well defined. It is denoted by $\langle S, u\rangle$.

Lemma 3.2. *Let $F\colon \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian with $F(0) = 0$. Let φ be a Musielak-Orlicz function and let $u \in W_0^1 L_\varphi(\Omega)$. Then $F(u) \in W_0^1 L_\varphi(\Omega)$. Moreover, if the set D of discontinuity points of F' is finite, we have*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u) \dfrac{\partial u}{\partial x_i}, & a.e. \text{ in } \{x \in \Omega : u(x) \notin D\}, \\ 0 & a.e. \text{ in } \{x \in \Omega : u(x) \in D\}. \end{cases}$$

Lemma 3.3 (Poincaré's inequality [**34**]). *Let φ a Musielak-Orlicz function which satisfies the assumptions of* Lemma 3.1. *Suppose that $\varphi(x,t)$ decreases with respect to one coordinate of x. Then, there exists a constant $c>0$ which depends only on Ω such that*

$$\int_\Omega \varphi(x,|u(x)|)\mathrm{d}x \leq \int_\Omega \varphi(x,c|\nabla u(x)|)\mathrm{d}x \qquad \text{for all } u \in W_0^1 L_\varphi(\Omega). \tag{3.3}$$

Lemma 3.4 (The Nemytskii Operator). *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure and let φ and ψ be two Musielak-Orlicz functions. Let $f\colon \Omega\times\mathbb{R}^p \to \mathbb{R}^q$ be a Carathéodory function such that for a.e. $x\in\Omega$ and all $s\in\mathbb{R}^p$,*

$$|f(x,s)| \leq c(x) + k_1\psi_x^{-1}\varphi(x,k_2|s|) \tag{3.4}$$

where k_1 and k_2 are real positives constants, and $c(.)\in E_\psi(\Omega)$. Then the Nemytskii Operator N_f defined by $N_f(u)(x) = f(x,u(x))$ is strongly continuous from $\Big(\mathcal{P}\big(E_\varphi(\Omega),\frac{1}{k_2}\big)\Big)^p = \prod\Big\{u\in L_\varphi(\Omega) : d(u,E_\varphi(\Omega)) < \frac{1}{k_2}\Big\}$ into $(L_\psi(\Omega))^q$ for the modular convergence. Furthermore, if $c(\cdot)\in E_\gamma(\Omega)$ and $\gamma \prec\prec \psi$, then N_f is strongly continuous from $\big(\mathcal{P}(E_\varphi(\Omega),\frac{1}{k_2})\big)^p$ to $(E_\gamma(\Omega))^q$.

Lemma 3.5 ([**35**]). *Let $a<b\in\mathbb{R}$ and let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. Then*

$$\Big\{u\in W_0^{1,x}L_\varphi(\Omega\times]a,b[) \;:\; \frac{\partial u}{\partial t}\in W^{-1,x}L_\psi(\Omega\times]a,b[) + L^1(\Omega\times]a,b[)\Big\}$$

is a subset of $\mathcal{C}([a,b];L^1(\Omega))$.

Proposition 3.6 ([**35**]). *Let φ be a Musielak function and let (u_n) be a sequence of $W^{1,x}L_\varphi(Q)$ such that*

$$u_n \rightharpoonup u \text{ weakly in } W^{1,x}L_\varphi(Q) \text{ for } \sigma(\Pi L_\varphi,\Pi L_\psi),$$

and

$$\frac{\partial u_n}{\partial t} = h_n + k_n \text{ in } \mathcal{D}'(Q),$$

with (h_n) bounded in $W^{-1,x}L_\psi(Q)$ and (k_n) bounded in the space $\mathcal{M}(Q)$ of measures on Q. Then $u_n\to u$ strongly in $L^1_{loc}(Q)$. Further, if $(u_n)\subset W_0^{1,x}L_\varphi(Q)$ then $u_n\to u$ strongly in $L^1(Q)$.

4. Assumptions and main results

Let Q be the cylinder $\Omega\times(0,T)$, $+\infty > T > 0$, Ω be a bounded domain of $\mathbb{R}^N$ with the segment property, and let be φ and ψ two complementary Musielak-Orlicz functions. We assume that $\varphi(x,t)$ decreases with respect to one coordinate of x. Let $A\colon D(A)\subset W_0^{1,x}L_\varphi(Q)\to W^{-1,x}L_\psi(Q)$ be a mapping given by

$$A(u) = -\mathrm{div}(a(x,t,u,\nabla u)),$$

where $a(x,t,s,\xi)\colon \Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ is a Carathéodory function. There exist two Musielak-Orlicz functions φ and γ such that $\gamma\prec\prec\varphi$, a positive function

$c(x,t) \in E_\psi(Q)$, and two positive constants ν, β such that for a.e. $(x,t) \in Q$ and for all $s \in \mathbb{R}$, ξ, $\xi' \in \mathbb{R}^N$, $\xi \neq \xi'$,

$$|a(x,t,s,\xi)| \leq \beta\Big(c(x,t) + \psi_x^{-1}\gamma(x,\nu|s|) + \psi_x^{-1}\varphi(x,\nu|\xi|)\Big), \tag{4.1}$$

$$\Big(a(x,t,s,\xi) - a(x,t,s,\xi')\Big)(\xi - \xi') > 0, \tag{4.2}$$

$$a(x,t,s,\xi)\xi \geq \alpha\varphi(x,|\xi|), \tag{4.3}$$

$$\Phi\colon \mathbb{R} \to \mathbb{R}^N \text{ is a continuous function,} \tag{4.4}$$

$$g\colon \mathbb{R} \to \mathbb{R} \text{ is an integrable function on } \mathbb{R} \text{ and } g(u)u \geq 0, \tag{4.5}$$

$$f \in L^1(Q), \tag{4.6}$$

$$u_0 \text{ is an element of } L^1(\Omega). \tag{4.7}$$

We consider the following boundary value problem:

$$(\mathcal{P})\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}\big(a(x,t,u,\nabla u) + \Phi(u)\big) + g(u)\varphi(x,|\nabla u|) = f & \text{in} \quad Q,\\ u = 0 & \text{on} \quad \partial Q,\\ u(x,0) = u_0 & \text{in} \quad \Omega. \end{cases}$$

Remark 2. As already mentioned in the introduction, problem $(\mathcal{P})$ does not admit a weak solution under assumptions (4.1)–(4.7) since the growths of $a(x,t,u,\nabla u)$ and $\Phi(u)$ are not controlled with respect to u (so that these fields are not in general defined as distributions, even when u belongs to $W^{1,x}L_\varphi(Q)$.

Throughout this paper, $\langle , \rangle$ means either the pairing between $W_0^{1,x}L_\varphi(Q) \cap L^\infty(Q)$ and $W^{-1,x}L_\psi(Q) + L^1(Q)$, or between $W_0^{1,x}L_\varphi(Q)$ and $W^{-1,x}L_\psi(Q)$. We recall that for $k > 1$ and s in $\mathbb{R}$, the truncation is defined as

$$T_\ell(s) = \begin{cases} s & \text{if } |s| \leq \ell,\\ \ell\frac{s}{|s|} & \text{if } |s| > \ell. \end{cases}$$

Our main result is collected in the following theorem.

Theorem 4.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$, φ and ψ be two complementary Musielak-Orlicz functions satisfying the assumptions of* Lemma 3.1, *and $\varphi(x,t)$ decreases with respect to one coordinate of x. We assume also that* (4.1)–(4.6) *and* (4.7) *hold true. Then, the problem $(\mathcal{P})$ has at least one entropy solution in the following sense*

(4.8)

$$T_\ell(u) \in W_0^{1,x}L_\varphi(Q) \qquad \textit{for all } \ell > 0,$$

$$\Big\langle \frac{\partial u}{\partial t}, T_\ell(u-v)\Big\rangle + \int_Q a(x,t,u,\nabla u)\cdot\nabla T_\ell(u-v)\,\mathrm{d}x\,\mathrm{d}t + \int_Q \Phi(u)\cdot\nabla T_\ell(u-v)\,\mathrm{d}x\,\mathrm{d}t$$

$$+ \int_Q g(u)\varphi(x,|\nabla u|)T_\ell(u-v)\,\mathrm{d}x\,\mathrm{d}t \leq \int_Q fT_\ell(u-v)\,\mathrm{d}x\,\mathrm{d}t$$

$$u(x,0) = u_0(x) \qquad \textit{for a.e. } x \in \Omega,$$

for all $v \in W_0^{1,x}L_\varphi(Q) \cap L^\infty(Q)$ *such that* $\frac{\partial v}{\partial t} \in W^{-1,x}L_\psi(Q) + L^1(Q)$.

The following remarks are concerned with a few comments on Theorem 4.1.

Remark 3. Equation (4.8) is formally obtained through pointwise multiplication of the problem $(\mathcal{P})$ by $T_\ell(u-v)$. Note that each term in (4.8) has a meaning since $T_\ell(u-v) \in W_0^{1,x}L_\varphi(Q) \cap L^\infty(Q)$. In addition, by Lemma 3.5, we have $v \in C([0,T], L^1(\Omega))$, and then the first and last terms of Eq. (4.8) are well defined.

The proof of this theorem is done in six steps.

Step 1: Approximate problem

Let us introduce the following regularization of the data:

$$\Phi_n(x,t,r) = \Phi(x,t,T_n(r)) \qquad \text{a.e, } (x,t) \in Q, \text{ for all } r \in \mathbb{R}, \tag{4.9}$$

$$f_n \in C_0^\infty(Q): \quad \|f_n\|_{L^1} \le \|f\|_{L^1} \text{ and } f_n \to f \text{ in } L^1(Q) \text{ as } n \text{ tends to } +\infty, \tag{4.10}$$

$$u_{0n} \in C_0^\infty(\Omega): \quad \|u_{0n}\|_{L^1} \le \|u\|_{L^1} \text{ and } u_{0n} \to u_0 \text{ in } L^1(\Omega) \text{ as } n \text{ tends to } +\infty. \tag{4.11}$$

Let consider the following approximate problem:

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - \operatorname{div}\big(a(x,t,u_n,\nabla u_n) + \Phi_n(u_n)\big), & \\ +g(u_n)\varphi(x, \mid \nabla u_n \mid) = f_n & f_n \text{ in } Q, \\ u_n(x,0) = u_{0n}(x) & u_{0n}(x) \text{ in } \Omega, \end{cases} \tag{4.12}$$

where Φ_n is a Lipschitz continuous bounded function from $\mathbb{R}$ into $\mathbb{R}^N$, $(f_n) \subset \mathcal{D}(Q)$ such that $f_n \to f$ strongly in $L^1(Q)$, and $(u_{0n}) \subset \mathcal{D}(\Omega)$ such that $u_{0n} \to u_0$ strongly in $L^1(\Omega)$ ($||u_{n0}||_{L^1(\Omega)} \le \|u_0\|_{L^1(\Omega)}$). As a consequence, proving existence of a weak solution $u_n \in W_0^{1,x}L_\varphi(Q)$ of (4.12) is an easy task (see, e.g, [**26**]).

Step 2: A priori estimates

The estimates derived in this step rely on usual techniques for problems of the type (4.12).

Proposition 4.2. *Assume that* (4.1)–(4.7) *are satisfied and let* u_n *be a solution of the approximate problem* (4.12). *Then for all* $\ell, n > 0$, *we have*

i) $\|T_\ell(u_n)\|_{W_0^{1,x}L_\varphi(Q)} \le C\ell$,
ii) $\lim\limits_{\ell\to\infty} \operatorname{meas}\{(x,t) \in Q : |u_n| > \ell\} = 0$ *uniformly with respect to* n.
iii) $\displaystyle\int_Q g(u_n)\varphi(x,|\nabla u_n|)\,\mathrm{d}x\,\mathrm{d}t \le C_g$, *where* C_g *is a positive constant not depending on* n.

Proof. We take $T_\ell(u_n)\chi_{(0,\tau)}$ as a test function in (4.12). For every $\tau \in (0,T)$, we get

$$\begin{aligned} &\Big\langle \frac{\partial u_n}{\partial t}, T_\ell(u_n)\chi_{(0,\tau)}\Big\rangle + \int_{Q_\tau} a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\nabla T_\ell(u_n)\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_{Q_\tau} \Phi_n(u_n)\nabla T_\ell(u_n)\mathrm{d}x\,\mathrm{d}t + \int_{Q_\tau} g(u_n)\varphi(x,|\nabla u_n|)T_\ell(u_n)\mathrm{d}x\,\mathrm{d}t \\ &= \int_{Q_\tau} f_n T_\ell(u_n)\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{4.13}$$

which implies that

$$\begin{aligned} &\int_\Omega S_\ell(u_n(\tau))\,\mathrm{d}x + \int_{Q_\tau} a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\nabla T_\ell(u_n)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_{Q_\tau} \Phi_n(u_n)\nabla T_\ell(u_n)\mathrm{d}x\,\mathrm{d}t \\ &= \int_{Q_\tau} f_n T_\ell(u_n)\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_\tau} g(u_n)\varphi(x,|\nabla u_n|)T_\ell(u_n)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega S_\ell(u_{0n})\,\mathrm{d}x, \end{aligned}$$

where

$$S_\ell(r) = \int_0^r T_\ell(\sigma)\,\mathrm{d}\,\sigma = \begin{cases} \dfrac{r^2}{2} & \text{if } |r| \le \ell, \\ \ell|r| - \dfrac{r^2}{2} & \text{if } |r| > \ell. \end{cases} \tag{4.14}$$

The Lipschitz character of Φ_n, and Stokes formula together with the boundary condition $u_n = 0$ on $(0,T)\times\partial\Omega$, make it possible to obtain

$$\int_{Q_\tau} \Phi_n(u_n)\nabla T_\ell(u_n)\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{4.15}$$

Due to the definition of S_ℓ and (4.11), we have

$$0 \le \int_\Omega S_\ell(u_{0n})\,\mathrm{d}x \le \ell\int_\Omega |u_{0n}|\,\mathrm{d}x \le \ell\|u_0\|_{L^1(\Omega)}. \tag{4.16}$$

For $\theta, \epsilon > 0$, now consider a function $\varrho_\theta^\epsilon \in C^1(\mathbb{R})$ such that

$$\varrho_\theta^\epsilon(s) = \begin{cases} 0 & \text{if } |s| \le \theta, \\ \operatorname{sign}(s) & \text{if } |s| > \theta + \epsilon, \end{cases} \tag{4.17}$$

and

$$(\varrho_\theta^\epsilon)'(s) \ge 0 \qquad \text{for all } s \in \mathbb{R}.$$

Then, by using $\varrho_\theta^\epsilon(u_n)$ as a test function in (4.12) and following [**33**], we can see that

$$\int_{\{|u_n|>\theta\}} |g(u_n)\varphi(x,|\nabla u_n|)|\,\mathrm{d}x\,\mathrm{d}t \le \int_{\{|u_n|>\theta\}} |f_n|\,\mathrm{d}x\,\mathrm{d}t + \int_{\{|u_n|>\theta\}} |u_{0n}|\,\mathrm{d}x\,\mathrm{d}t, \tag{4.18}$$

and so by letting $\theta \to 0$ and using Fatou's lemma, we deduce that $g(u_n)\varphi(x, |\nabla u_n|)$ is a bounded sequence in $L^1(Q_\tau)$, then, we obtain iii). By using (4.15), (4.16), iii), and (4.5), it yields

(4.19)
$$\begin{aligned}
&\int_\Omega S_\ell(u_n(\tau))\,\mathrm{d}x + \int_{Q_\tau} a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\nabla T_\ell(u_n)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{Q_\tau} f_n T_\ell(u_n)\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_\tau} g(u_n)\varphi(x,|\nabla u_n|)T_\ell(u_n)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega S_\ell(u_{0n})\,\mathrm{d}x \\
&= \int_{Q_\tau} f_n T_\ell(u_n)\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_\tau} |g(u_n)\varphi(x,|\nabla u_n|)T_\ell(u_n)|\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega S_\ell(u_{0n})\,\mathrm{d}x \\
&\le \ell||f_n||_{L^1(Q_\tau)} + \ell C_g + \ell||u_0||_{L^1(\Omega)} \\
&\le \ell(||f_n||_{L^1(Q_\tau)} + \ell C_g + ||u_0||_{L^1(\Omega)}) \\
&\le \ell C_0,
\end{aligned}$$

where here and below C_i denote positive constants not depending on n and ℓ. Using (4.19) and the fact that $S_\ell(u_n) \ge 0$ allows us to deduce that

$$\int_{Q_\tau} a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\nabla T_\ell(u_n)\,\mathrm{d}x\,\mathrm{d}t \le \ell C_0, \tag{4.20}$$

which implies by virtue of (4.3), that

$$\int_{Q_\tau} \varphi(x,|\nabla T_\ell(u_n)|)\,\mathrm{d}x\,\mathrm{d}t \le \ell C_1. \tag{4.21}$$

From the that above inequality (4.19), we deduce that

$$\int_\Omega S_\ell(u_n(\tau))\,\mathrm{d}x \le \ell C_0, \text{ for any } \tau \text{ in } [0,T]. \tag{4.22}$$

On the other hand, thanks to Lemma 3.3, there exists a constant $\lambda > 0$ depending only on Ω such that

$$\int_{Q_\tau} \varphi(x,|v(x)|)\,\mathrm{d}x\,\mathrm{d}t \le \int_{Q_\tau} \varphi(x,\lambda|\nabla v(x)|)\,\mathrm{d}x\,\mathrm{d}t, \quad \forall v \in W_0^1 L_\varphi(\Omega). \tag{4.23}$$

Taking $v = \frac{T_\ell(u_n)}{\lambda}$ in (4.23) and using (4.21), one has

$$\int_{Q_\tau} \varphi(x,\frac{|T_\ell(u_n)|}{\lambda})\,\mathrm{d}x\,\mathrm{d}t \le \ell C_1. \tag{4.24}$$

On the other hand, one has

$$\begin{aligned}\operatorname{meas}\{|u_n|>\ell\} &\le \frac{1}{\operatorname*{ess\,inf}_{x\in\Omega}\varphi(x,\frac{\ell}{\lambda})}\int_{\{|u_n|>\ell\}}\varphi\Big(x,\frac{\ell}{\lambda}\Big)\,\mathrm{d}x\,\mathrm{d}t\\ &\le \frac{1}{\operatorname*{ess\,inf}_{x\in\Omega}\varphi(x,\frac{\ell}{\lambda})}\int_{\Omega}\varphi(x,\frac{1}{\lambda}|T_\ell(u_n)|)\,\mathrm{d}x\,\mathrm{d}t\\ &\le \frac{C_1\ell}{\operatorname*{ess\,inf}_{x\in\Omega}\varphi(x,\frac{\ell}{\lambda})}\qquad \text{for all } n, \text{ and } \ell\ge 0.\end{aligned}\tag{4.25}$$

For any $\beta>0$, we have

$$\begin{aligned}\operatorname{meas}\{|u_n-u_m|>\beta\}\le \operatorname{meas}\{|u_n|>\ell\}+\operatorname{meas}\{|u_m|>\ell\}\\ +\operatorname{meas}\{|T_\ell(u_n)-T_\ell(u_m)|>\beta\},\end{aligned}$$

and so that

$$\operatorname{meas}\{|u_n-u_m|>\beta\}\le \frac{2C_1\ell}{\operatorname*{ess\,inf}_{x\in\Omega}\varphi(x,\frac{\ell}{\lambda})}+\operatorname{meas}\{|T_\ell(u_n)-T_\ell(u_m)|>\beta\}.\tag{4.26}$$

By using (4.24) and Poincaré's inequality in Musielak-Orlicz spaces, we deduce that $(T_\ell(u_n))_n$ is bounded in $W_0^{1,x}L_\varphi(Q)$, and then there exists $\omega_\ell\in W_0^{1,x}L_\varphi(Q)$ such that $T_\ell(u_n)\rightharpoonup\omega_\ell$ weakly in $W_0^{1,x}L_\varphi(Q)$ for $\sigma(\Pi L_\varphi,\Pi E_\psi)$, strongly in $L^1(Q)$ and a.e. in Q.

Consequently, we can assume that $(T_\ell(u_n))_n$ is a Cauchy sequence in measure in Q.

Let $\varepsilon>0$, then by (4.26) and the fact that $\frac{2C_1\ell}{\operatorname*{ess\,inf}_{x\in\Omega}\varphi(x,\frac{\ell}{\lambda})}\to 0$ as $\ell\to+\infty$, there exists some $\ell=\ell(\varepsilon)>0$ such that

$$\operatorname{meas}\{|u_n-u_m|>\lambda\}<\varepsilon\qquad\text{for all } n,m\ge h_0(\ell(\varepsilon),\lambda).$$

This proves that (u_n) is a Cauchy sequence in measure, thus, (u_n) converges almost everywhere to some measurable function u. □

Step 3: Boundedness of $a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))$ in $(L_\psi(Q))^N$

Proposition 4.3. *Let u_n be a solution of the approximate problem* (4.12)*, then we have the following properties:*

$$u_n\to u\ \textit{a.e. in } Q.\tag{4.27}$$

$$a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\rightharpoonup\phi_\ell\ \textit{weakly in } (L_\psi(Q))^N\ \textit{for } \sigma(\Pi L_\varphi,\Pi E_\psi),\tag{4.28}$$

for some $\phi_\ell\in(L_\psi(Q))^N$.

Proof. From (4.21), we have that $(T_\ell(u_n))$ is bounded in $W_0^{1,x}L_\varphi(Q)$ for every $\ell>0$. Consider now $C^2(\mathbb{R})$ nondecreasing function $\zeta_\ell(s)=s$ for $|s|\le\frac{\ell}{2}$ and $\zeta_\ell(s)=\ell$ sign (s) for $|s|\ge 0$. Multiplying the approximating equation by $\zeta_\ell'(u_n)$, we obtain

$$
\begin{aligned}
\frac{\partial(\zeta_\ell(u_n))}{\partial t} &= \operatorname{div}\big(a(x,t,u_n,\nabla u_n)\zeta_\ell'(u_n)\big) - a(x,t,u_n,\nabla u_n)\zeta_\ell''(u_n)\nabla u_n \\
&\quad + \operatorname{div}\big(\Phi_n(u_n)\zeta_\ell'(u_n)\big) - \Phi_n(u_n)\zeta_\ell''(u_n)\nabla u_n \\
&\quad - g(u_n)\varphi(x,|\nabla u_n|)\zeta_\ell'(u_n) + f_n\zeta_\ell'(u_n)
\end{aligned}
\tag{4.29}
$$

in the sense of distributions. This implies, thanks to (4.24) and the fact that ζ_ℓ' has compact support, that $\zeta_\ell'(u_n)$ is bounded in $W_0^{1,x}L_\varphi(Q)$, while its time derivative $\frac{\partial(\zeta_\ell(u_n))}{\partial t}$ is bounded in $W_0^{-1,x}L_\varphi(Q) + L^1(Q)$, hence Proposition 3.6 allows us to conclude that $\zeta_\ell(u_n)$ is compact in $L^1(Q)$. Due to the choice of ζ_ℓ, we conclude that for each ℓ, the sequence $(T_\ell(u_n))$ converges almost everywhere in Q, which implies that (u_n) converges almost everywhere to some measurable function u in Q. Therefore, following [**7**], we can see that there exists a measurable function $u \in L^\infty(0,T;L^1(\Omega))$ such that for every $\ell > 0$ and a subsequence, not relabeled,

$$u_n \to u \text{ a.e. in } Q,$$

and

$$
\begin{aligned}
T_\ell(u_n) \rightharpoonup T_\ell(u) \quad & \text{weakly in } W_0^{1,x}L_\varphi(Q) \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi), \\
& \text{strongly in } L^1(Q) \text{ and a.e. in } Q.
\end{aligned}
\tag{4.30}
$$

We prove that $a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))$ is bounded sequence in $(L_\psi(Q))^N$. Let $w \in (E_\varphi(Q)^N$ with $\|w\|_{\varphi,Q} \le 1$. By using (4.2), we have

$$\Big(a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) - a(x,t,T_\ell(u_n),\frac{w}{\nu})\Big)\Big(\nabla T_\ell(u_n) - \frac{w}{\nu}\Big) > 0,$$

hence

$$
\begin{aligned}
&\int_Q a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\frac{w}{\nu}\,\mathrm{d}x\,\mathrm{d}t \\
&\le \int_Q a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\nabla T_\ell(u_n)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_Q a\Big(x,t,T_\ell(u_n),\frac{w}{\nu}\Big)\Big(\nabla T_\ell(u_n) - \frac{w}{\nu}\Big)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{4.31}
$$

Thanks to (4.20), we have

$$\int_Q a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\nabla T_\ell(u_n)\,\mathrm{d}x\,\mathrm{d}t \le C_2, \tag{4.32}$$

where C_2 is a positive constant which is independent of n.

On the other hand, by using (4.1) for λ large enough ($\lambda > \beta$), we have

$$
\begin{aligned}
&\int_Q \psi_x\Big(\frac{a(x,t,T_\ell(u_n),\frac{w}{\nu})}{3\lambda}\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\le \int_Q \psi_x\Big(\frac{\beta\big(c(x,t) + \psi_x^{-1}(\gamma(x,|T_\ell(u_n)|)) + \psi_x^{-1}(\varphi(x,|w|))\big)}{3\lambda}\Big)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

$$\begin{aligned}
&\le \frac{\beta}{\lambda}\int_Q \psi_x\Big(\frac{c(x,t)+\psi_x^{-1}(\gamma(x,|T_\ell(u_n)|))+\psi_x^{-1}(\varphi(x,|w|))}{3}\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\le \frac{\beta}{3\lambda}\Big(\int_Q \psi_x(c(x,t))\,\mathrm{d}x\,\mathrm{d}t+\int_Q \gamma(x,|T_\ell(u_n)|)\,\mathrm{d}x\,\mathrm{d}t+\int_Q \varphi(x,|w|)\,\mathrm{d}x\,\mathrm{d}t\Big)\\
&\le \frac{\beta}{3\lambda}\Big(\int_Q \psi_x(c(x,t))\,\mathrm{d}x\,\mathrm{d}t+\int_Q \gamma(x,|T_\ell(u_n)|)\,\mathrm{d}x\,\mathrm{d}t+\int_Q \varphi(x,|w|)\,\mathrm{d}x\,\mathrm{d}t\Big)\\
&\le C_3.
\end{aligned}$$

Now, since γ grows essentially less rapidly than φ near infinity and by using the Remark 1, for all $\varepsilon > 0$ such that $\gamma(x,|T_\ell(u_n)|) \le \varphi(x,\varepsilon|T_\ell(u_n)|)$, we have

$$\begin{aligned}
\int_Q \psi_x\Big(\frac{a(x,t,T_\ell(u_n),\frac{w}{\nu})}{3\lambda}\Big)\,\mathrm{d}x\,\mathrm{d}t \le \frac{\beta}{3\lambda}\Big(&\int_Q \psi_x(c(x,t))\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_Q \varphi(x,\varepsilon|T_\ell(u_n)|)\,\mathrm{d}x\,\mathrm{d}t+\int_Q \varphi(x,|w|)\,\mathrm{d}x\,\mathrm{d}t\Big),
\end{aligned}$$

hence $a(x,t,T_\ell(u_n),\frac{w}{\nu})$ is bounded in $(L_\psi(Q))^N$. Which implies that second term of the right hand side of (4.31) is bounded, consequently, we obtain

$$\int_Q a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))w\,\mathrm{d}x\,\mathrm{d}t \le C_3 \quad \text{for all } w \in (E_\varphi(Q)^N \text{ with } \|w\|_{\varphi,Q} \le 1,$$

where C_3 is a positive constant which is independent of n.

Hence, thanks the Banach-Steinhaus Theorem, the sequence $(a(x,t,T_\ell(u_n), \nabla T_\ell(u_n)))_n$ is a bounded sequence in $(L_\psi(Q))^N$, thus up to a subsequence

(4.33)
$$a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) \rightharpoonup \phi_k \text{ weakly stars in } (L_\psi(Q))^N \text{ for } \sigma(\Pi L_\psi,\Pi E_\varphi)$$

for some $\phi_k \in (L_\psi(Q))^N$. □

Step 4: Almost everywhere convergence of the gradients

Let ρ_m be a truncation defined by

$$\rho_m(s) = \begin{cases} 1 & \text{if} \quad |s| \le m,\\ m+1-|s| & \text{if} \quad |s| \le m+1,\\ 0 & \text{if} \quad |s| \ge m+1 \end{cases} \tag{4.34}$$

where $m > \ell$. We set

$$\begin{aligned}
T_\ell^*(s) &= \Big(\int_0^{T_\ell(s)} \exp\Big(\int_0^t g(s)\mathrm{d}s\Big)\mathrm{d}t\Big)\Big(\exp\Big(-\int_0^\infty g(s)\mathrm{d}s\Big)\Big),\\
R_m(s) &= \Big(\int_0^s \rho_m(t)\exp\Big(\int_0^t g(s)\mathrm{d}s\Big)\mathrm{d}t,\\
\omega_{\mu,j}^i &= T_\ell(v_j)_\mu + \exp(-\mu t)T_\ell(w_i).
\end{aligned}$$

Let $(v_j) \in D(Q)$ be a sequence such that

$$v_j \to T_\ell^*(u) \text{ in } W_0^{1,x}L_\varphi(Q) \text{ for the modular convergence,} \tag{4.35}$$

and let $(\omega_j) \subset \mathcal{D}(Q)$ be a sequence such that $v_j \geq T_\ell^*(\omega_j)$ and ω_j converges strongly to $T_\ell^*(u_0)$ in $L^2(\Omega)$.

Also $T_\ell(v_j)_\mu$ is the mollification with respect to time of $T_\ell(v_j)$, see [**5**]. Note that $\omega^i_{\mu,j}$ is a smooth function having the following properties:

$$\frac{\partial}{\partial t}(\omega^i_{\mu,j}) = \mu(T_\ell(v_j) - \omega^i_{\mu,j}), \omega^i_{\mu,j}(0) = T_\ell(\omega_i), |\omega^i_{\mu,j}| \leq \ell, \tag{4.36}$$

$$\omega^i_{\mu,j} \to T_\ell^*(u)_\mu + \exp(-\mu t)T_\ell(w_i) \text{ in } W_0^{1,x}L_\varphi(Q) \tag{4.37}$$

for the modular convergence as $j \to \infty$,

$$T_\ell^*(u)_\mu + \exp(-\mu t)T_\ell(w_i) \to T_\ell^*(u) \text{ in } W_0^{1,x}L_\varphi(Q) \tag{4.38}$$

for the modular convergence as $\mu \to \infty$.
Using the admissible test function $Z^{\mu,m}_{i,j,n} = (T_\ell^*(u_n) - \omega^i_{\mu,j})\rho_m(u_n)\exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)$ as a test function in (4.12), leads to

$$\begin{aligned}
&\Big\langle \frac{\partial u_n}{\partial t}, Z^{\mu,m}_{i,j,n}\Big\rangle \\
&+ \int_Q a(x,t,u_n,\nabla u_n)(\nabla T_\ell^*(u_n) - \nabla\omega^i_{\mu,j})\rho_m(u_n)\exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t && (1)\\
&+ \int_{\{m\leq u_n\leq m+1\}} a(x,t,u_n,\nabla u_n)(T_\ell^*(u_n) - \omega^i_{\mu,j})\nabla u_n\rho'_m(u_n)\exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t && (2)\\
&+ \int_{\{m\leq u_n\leq m+1\}} \Phi_n(u_n)(T_\ell^*(u_n(u_n) - \omega^i_{\mu,j})\nabla u_n\rho'_m(u_n)\exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t && (3)\\
&+ \int_Q \Phi_n(u_n)(\nabla T_\ell^*(u_n) - \nabla\omega^i_{\mu,j})\rho_m(u_n)\exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t && (4)\\
&= \int_Q f_n Z^{\mu,m}_{i,j,n}\,\mathrm{d}x\,\mathrm{d}t - \int_Q g(u_n)\varphi(x,|\nabla u_n|)Z^{\mu,m}_{i,j,n}\,\mathrm{d}x\,\mathrm{d}t = (5) + (6).
\end{aligned}$$

We denote $\epsilon(n,j,\mu,i)$ any quantity such that

$$\lim_{i\to\infty}\lim_{\mu\to\infty}\lim_{j\to\infty}\lim_{n\to\infty}\epsilon(n,j,\mu,i) = 0.$$

Let us recall that for $u_n \in W_0^{1,x}L_\varphi(Q)$, there exists a smooth function $u_{n\sigma}$ such that

$$u_{n\sigma} \to u_n \text{ for the modular convergence in } W_0^{1,x}L_\varphi(Q) \cap L^2(Q),$$

$$\frac{\partial u_{n\sigma}}{\partial t} \to \frac{\partial u_n}{\partial t} \text{ for the modular convergence in } W^{-1,x}L\psi(Q) + L^2(Q),$$

$$\begin{aligned}
\Big\langle \frac{\partial u_n}{\partial t}, Z^{\mu,m}_{i,j,n}\Big\rangle &= \lim_{\sigma\to 0^+}\int_Q (u_{n\sigma})'(T^*_\ell(u_{n\sigma}) - \omega^i_{\mu,j})\rho_m(u_n)\exp\Big(\int_0^{u_{n\sigma}} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&= \lim_{\sigma\to 0^+}\Big(\int_Q (R_m(u_{n\sigma}) - T^*_\ell(u_{n\sigma}))'(T^*_\ell(u_{n\sigma}) - \omega^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_Q T^*_\ell(u_{n\sigma})'(T^*_\ell(u_{n\sigma}) - \omega^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t\Big)\\
&= \lim_{\sigma\to 0^+}\int_\Omega \Big[(R_m(u_{n\sigma}) - T^*_\ell(u_{n\sigma}))(T^*_\ell(u_{n\sigma}) - \omega^i_{\mu,j})\mathrm{d}x\Big]_0^T\\
&\quad - \int_Q (R_m(u_{n\sigma}) - T^*_\ell(u_{n\sigma}))(T^*_\ell(u_{n\sigma}) - \omega^i_{\mu,j})'\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_Q T^*_\ell(u_{n\sigma})'(T^*_\ell(u_{n\sigma}) - \omega^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t = I_1 + I_2 + I_3.
\end{aligned}$$

Remark also that

$$R_m(u_{n\sigma}) \geq T^*_\ell(u_{n\sigma}) \text{ if } u_{n\sigma} < \ell \text{ and } R_m(u_{n\sigma}) > \ell = T^*_\ell(u_{n\sigma}) \geq |\omega^i_{\mu,j})| \text{ if } u_{n\sigma} \geq \ell,$$

$$\begin{aligned}
I_1 &= \int_\Omega (R_m(u_{n\sigma})(T) - T^*_\ell(u_{n\sigma})(T))(T^*_\ell(u_{n\sigma})(T) - \omega^i_{\mu,j}(T))\mathrm{d}x,\\
&\quad - \int_\Omega (R_m(u_{n\sigma})(0) - T^*_\ell(u_{n\sigma})(0))(T^*_\ell(u_{n\sigma})(0) - \omega^i_{\mu,j}(0))\mathrm{d}x = I_1^1 + I_1^2,\\
I_1^1 &\geq \int_{\left\{u_{n\sigma}(T)\leq \ell\right\}} (R_m(u_{n\sigma})(T) - T^*_\ell(u_{n\sigma})(T))(T^*_\ell(u_{n\sigma})(T) - \omega^i_{\mu,j}(T))\mathrm{d}x,
\end{aligned}$$

and it is easy to see that,

$$\limsup_{\sigma\to 0^+} I_1^1 \geq \epsilon(n,j,\mu),$$

$$\begin{aligned}
I_1^2 &= -\int_{\left\{u_{n\sigma}(0)\leq \ell\right\}} (R_m(u_{n\sigma})(0) - T^*_\ell(u_{n\sigma})(0))(T^*_\ell(u_{n\sigma})(0) - T_\ell(\omega_i))\mathrm{d}x\\
&\quad - \int_{\left\{u_{n\sigma}(0)> \ell\right\}} (R_m(u_{n\sigma})(0) - T^*_\ell(u_{n\sigma})(0))(T^*_\ell(u_{n\sigma})(0) - T_\ell(\omega_i))\mathrm{d}x.
\end{aligned}$$

For the first part, it is the same as I_1^1 and for the second part, we have

$$I_1^2 \geq \epsilon(n,j,\mu) - \int_{\left\{u_{n\sigma}(0)\geq \ell\right\}} (R_m(u_{n\sigma})(0) - T^*_\ell(u_{n\sigma})(0))(T^*_\ell(u_{n\sigma})(0) - T_\ell(\omega_i))\mathrm{d}x.$$

$$\limsup_{\sigma\to 0^+} I_1 \geq \epsilon(n,j,\mu) - \int_{\left\{u_{0n}\geq \ell\right\}} (R_m(u_{0n}) - T^*_\ell(u_{0n})(T^*_\ell(u_{0n}) - T_\ell(\omega_i))\mathrm{d}x = J_1.$$

Now letting $n \to \infty$, we have

$$\lim_{n\to+\infty} J_1 = \int_{\left\{u_0\geq \ell\right\}} (R_m(u_0) - T^*_\ell(u_0)(T^*_\ell(u_0) - T_\ell(\omega_i))\mathrm{d}x$$

and by letting $i \to \infty$, we obtain

$$\limsup_{\sigma\to 0^+} I_1 \geq \epsilon(n,j,i,\mu).$$

Concerning I_2, we remark that $T_\ell^*(u_{n\sigma})' = 0$ if $u_{n\sigma} > \ell$, then

$$I_2 = -\int_{\{u_{n\sigma}\le\ell\}} (R_m(u_{n\sigma}) - T_\ell^*(u_{n\sigma}))(T_\ell^*(u_{n\sigma}) - \omega_{\mu,j}^i)' \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_{\{u_{n\sigma}>\ell\}} (R_m(u_{n\sigma}) - T_\ell^*(u_{n\sigma}))(\omega_{\mu,j}^i)' \,\mathrm{d}x\,\mathrm{d}t = I_2^1 + I_2^2.$$

As in $I_1, I_1^2 \ge \epsilon(n,j,\mu)$ and

$$I_2^2 = \int_{\{u_{n\sigma}>\ell\}} (R_m(u_{n\sigma}) - T_\ell^*(u_{n\sigma}))(\omega_{\mu,j}^i)' \,\mathrm{d}x\,\mathrm{d}t$$
$$\ge \mu \int_{\{u_{n\sigma}>\ell\}} (R_m(u_{n\sigma}) - T_\ell^*(u_{n\sigma}))(T_\ell(v_j) - T_\ell^*(u_{n\sigma}))' \,\mathrm{d}x\,\mathrm{d}t$$

by using the fact that

$$(R_m(u_{n\sigma}) - T_\ell^*(u_{n\sigma}))(T_\ell^*(u_{n\sigma} - \omega_{\mu,j}^i)\chi_{\{u_{n\sigma}>\ell\}} \ge 0.$$

So

$$\limsup_{\sigma\to 0^+} I_2^2 \ge \mu \int_{\{u_{n\sigma}>\ell\}} (R_m(u_{n\sigma}) - T_\ell^*(u_{n\sigma}))(T_\ell(v_j) - T_\ell^*(u_{n\sigma}))' \,\mathrm{d}x\,\mathrm{d}t$$
$$= \epsilon(n,j).$$

About I_3,

$$I_3 = \int_Q T_\ell^*(u_{n\sigma})'(T_\ell^*(u_{n\sigma}) - \omega_{\mu,j}^i) \,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_Q (T_\ell^*(u_{n\sigma}) - \omega_{\mu,j}^i)'(T_\ell^*(u_{n\sigma}) - \omega_{\mu,j}^i) \,\mathrm{d}x\,\mathrm{d}t + \int_Q (\omega_{\mu,j}^i)'(T_\ell^*(u_{n\sigma}) - \omega_{\mu,j}^i) \,\mathrm{d}x\,\mathrm{d}t.$$

Set $\phi(r) = \frac{r^2}{2}, \qquad \phi \ge 0$, then

$$I_3 = \Big[\int_\Omega \phi(T_\ell^*(u_{n\sigma}) - \omega_{\mu,j}^i)\mathrm{d}x\Big]_0^T + \mu\int_Q (T_\ell(v_j) - \omega_{\mu,j}^i)(T_\ell^*(u_{n\sigma}) - \omega_{\mu,j}^i) \,\mathrm{d}x\,\mathrm{d}t.$$
$$\ge \epsilon(n,j,\mu) - \int_\Omega \phi(T_\ell^*(u_{n\sigma})(0) - T_\ell(\omega_i))\mathrm{d}x$$
$$+ \mu\int_Q (T_\ell(v_j) - \omega_{\mu,j}^i)(T_\ell^*(u_{n\sigma}) - \omega_{\mu,j}^i) \,\mathrm{d}x\,\mathrm{d}t (\text{ as in } I_2).$$

So,

$$\limsup_{\sigma\to 0^+} I_3 \ge \epsilon(n,j,\mu) - \int_\Omega \phi(T_\ell^*(u_{0n}) - T_\ell(\omega_i))\mathrm{d}x$$
$$+ \mu\int_Q (T_\ell(v_j) - T_\ell^*(u_n))(T_\ell^*(u_{n\sigma}) - \omega_{\mu,j}^i) \,\mathrm{d}x\,\mathrm{d}t$$
$$= -\int_\Omega \phi(T_\ell^*(u_{0n}) - \omega_i)\mathrm{d}x + \mu\int_Q (T_\ell(v_j) - \omega_{\mu,j}^i)(T_\ell^*(u_{n\sigma}) - \omega_{\mu,j}^i) \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \epsilon(n,j,\mu)$$

and, we easily deduce

$$\limsup_{\sigma\to 0^+} I_3 \ge \epsilon(n,j,i,\mu).$$

Finally, we conclude that

$$\Big\langle \frac{\partial u_n}{\partial t}, (T_\ell^*(u_n) - \omega_{\mu,j}^i)\rho_m(u_n) \exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\Big\rangle \geq \epsilon(n,j,i,\mu). \tag{4.39}$$

We are interested now in the terms of (1), (2), (4), and (5). Let us remark that

$$\begin{aligned} \nabla T_\ell^*(u) &= \Big(\exp\Big(-\int_0^\infty g(s)\mathrm{d}s\Big)\Big)\exp\Big(\int_0^{T_\ell(u_n)} g(s)\mathrm{d}s\Big)\nabla T_\ell(u) \\ &=: \lambda(u)\nabla T_\ell(u). \end{aligned} \tag{4.40}$$

About (1),

$$\begin{aligned} &\int_Q a(x,t,u_n,\nabla u_n)(\nabla T_\ell^*(u_n) - \nabla\omega_{\mu,j}^i)\rho_m(u_n)\exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad= \int_{\{u_n\leq \ell\}} a(x,t,u_n,\nabla u_n)(\nabla T_\ell^*(u_n) - \nabla\omega_{\mu,j}^i)\rho_m(u_n)\exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad\quad+ \int_{\{u_n> \ell\}} a(x,t,u_n,\nabla u_n)(\nabla T_\ell^*(u_n) - \nabla\omega_{\mu,j}^i)\rho_m(u_n)\exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad= \int_Q a(x,t,T_\ell(u_n),\nabla u_n)(\nabla T_\ell^*(u_n) - \nabla\omega_{\mu,j}^i)\exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad\quad+ \int_{\{u_n> \ell\}} a(x,t,u_n,\nabla u_n)(\nabla T_\ell^*(u_n) - \nabla\omega_{\mu,j}^i)\rho_m(u_n)\exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big),\,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

recall that $\rho_m(u_n) = 1$ on $|u_n| \leq \ell$.

Let $s > 0$, $Q_s = \{(x,t) \in Q : |\nabla T_\ell(u)| \leq s\}$, $Q_s^j = \{(x,t) \in Q : |\nabla T_\ell(v_j)| \leq s\}$.

$$\begin{aligned} &\int_Q a(x,t,u_n,\nabla u_n)(\nabla T_\ell^*(u_n) - \nabla\omega_{\mu,j}^i)\rho_m(u_n)\exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad= \int_Q (a(x,t,T_\ell(u_n),\nabla u_n) - a(x,t,T_\ell(u_n),\nabla T_\ell(v_j)\chi_s^j)(\nabla T_\ell^*(u_n) - \nabla T_\ell(v_j)\chi_s^j) \\ &\quad\quad\times\exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_Q a(x,t,T_\ell(u_n),\nabla T_\ell(v_j)\chi_s^j)(\nabla T_\ell^*(u_n) - \nabla T_\ell(v_j)\chi_s^j) \\ &\quad\quad\times\exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad\quad+ \int_Q a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\nabla T_\ell(v_j)\chi_s^j\exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad\quad- \int_Q a(x,t,u_n,\nabla u_n)\nabla\omega_{\mu,j}^i\rho_m(u_n)\exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad= J_1 + J_2 + J_3 + J_4. \end{aligned}$$

We go to the limit as n, j, m, and $s \to \infty$ in the last three integrals of the last side.

As for the inequality (4.21), we can prove that

$$\begin{aligned} &T_\ell^*(u_n) \rightharpoonup T_\ell^*(u) \text{ in } W_0^{1,x}L_\varphi(Q) \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi), \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{strongly in } L^1(Q), \text{ and a.e. in } Q. \end{aligned} \tag{4.41}$$

Now, starting with J_2, by letting $n\to\infty$, we have

$$J_2=\int_Q a(x,t,T_\ell(u),\nabla T_\ell(v_j)\chi_s^j)(\nabla T_\ell^*(u) - \nabla T_\ell(v_j)\chi_s^j)\exp\Big(\int_0^u g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t+\epsilon(n).$$

Since

$$a(x,t,T_\ell(u_n),\nabla T_\ell(v_j)\chi_s^j)\to a(x,t,T_\ell(u),\nabla T_\ell(v_j)\chi_s^j) \text{ strongly in} (E_\psi(Q))^N,$$

$$a(x,t,T_\ell(u_n),\nabla T_\ell(v_j)\chi_s^j)\to a(x,t,T_\ell(u),\nabla T_\ell(u)\chi_s) \text{ strongly in } (E_\psi(Q))^N,$$

and

$$\nabla T_\ell(v_j)\chi_s^j\to \nabla T_\ell^*(u)\chi_s \text{ strongly in } (L_\varphi(Q))^N,$$

then

$$J_2=\epsilon(n,j). \tag{4.42}$$

Following the same way as in J_2 and using (4.14), one has

$$J_3=\int_Q \phi_\ell\nabla T_\ell^*(u)\exp\Big(\int_0^u g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t+\epsilon(n,j,\mu,s). \tag{4.43}$$

Concerning the terms J_4

$$\begin{aligned}J_4&=-\int_Q a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla\omega_{\mu,j}^i\rho_m(u_n)\exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t\\&=-\int_{\{|u_n|\le\ell\}} a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla\omega_{\mu,j}^i\rho_m(u_n)\exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t\\&\quad-\int_{\{\ell<|u_n|\le m+1\}} a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla\omega_{\mu,j}^i\rho_m(u_n)\exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

By letting first n then j and finally μ go to infinity

$$J_4=-\int_Q \phi_\ell\nabla T_\ell^*(u)\exp\Big(\int_0^u g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t+\epsilon(n,j,\mu). \tag{4.44}$$

We conclude then that

$$\begin{aligned}&\int_Q a(x,t,u_n,\nabla u_n)(\nabla T_\ell^*(u_n)-\nabla\omega_{\mu,j}^i)\rho_m(u_n)\exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t\\&=\int_Q (a(x,t,T_\ell(u_n),\nabla u_n)-a(x,t,T_\ell(u_n),\nabla T_\ell(v_j)\chi_s^j)\\&\quad\times(\nabla T_\ell^*(u_n)-\nabla T_\ell(v_j)\chi_s^j)\exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t+\epsilon(n,j,\mu,s).\end{aligned} \tag{4.45}$$

About (2),

$$\begin{aligned}&\Big|\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)(T_\ell^*(u_n)-\omega_{\mu,j}^i)\nabla u_n\rho_m'(u_n)\exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t\Big|\\&\le C(k)\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Then by (4.32), we deduce that

$$\Big|\int_{\{m\leq|u_n|\leq m+1\}} a(x,t,u_n,\nabla u_n)(T_\ell^*(u_n)-\omega_{\mu,j}^i)\nabla u_n\rho_m'(u_n)\exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t\Big|$$
$$\leq \epsilon(n,\mu,m).$$

About (3) and (4),
using Lebesgue's convergence theorem shows that

$$\Phi_n(u_n)\rho_m(u_n)\to\Phi(u)\rho_m(u) \text{ strongly in } (E_\psi(Q)^N) \text{ as } n\to\infty,$$

and

$$\Phi_n(u_n)\chi_{\{m\leq|u_n|\leq m+1\}}(\nabla T_\ell^*(u_n)-\nabla\omega_{i,j}^\mu)\rho_m(u_n)$$
$$\to Phi(u)\chi_{\{m\leq|u_n|\leq m+1\}}(\nabla T_\ell^*(u)-\nabla\omega_{i,j}^\mu)\rho_m(u), \text{ strongly in } (E_\psi(Q)^N) \quad \text{as } n\to\infty$$

Then by virtue of $\nabla T_\ell^*(u_n)\rightharpoonup\nabla T_\ell^*(u)$ weakly in $(L_\varphi(Q)^N)$,
and

$$\nabla u_n\chi_{\{m\leq|u_n|\leq m+1\}}=\nabla T_{m+1}^*(u_n)\chi_{\{m\leq|u_n|\leq m+1\}} \qquad \text{a.e. in } Q,$$

one has

$$\int_Q\Phi_n(u_n)(\nabla T_\ell^*(u_n)-\nabla\omega_{i,j}^\mu)\rho_m(u_n)\,\mathrm{d}x\,\mathrm{d}t\to\int_Q\Phi(u)(\nabla T_\ell^*(u)-\nabla\omega_{i,j}^\mu)\rho_m(u)\,\mathrm{d}x\,\mathrm{d}t$$

as $n\to\infty$, and

$$\int_{\{m\leq|u_n|\leq m+1\}}\Phi_n(u_n)(T_\ell^*(u_n)-\omega_{i,j}^\mu)\nabla u_n\rho_m'(u_n)\,\mathrm{d}x\,\mathrm{d}t$$
$$\to\int_{\{m\leq|u_n|\leq m+1\}}\Phi(u)(T_\ell^*(u)-\omega_{i,j}^\mu)\nabla u\rho_m'(u)\,\mathrm{d}x\,\mathrm{d}t \quad \text{as } n\to+\infty.$$

On the other hand, by using the modular convergence of $(\omega_{i,j}^\mu)$ as $j\to+\infty$ and letting μ tend to infinity, we deduce that

$$\int_Q\Phi_n(u_n)(\nabla T_\ell^*(u_n)-\nabla\omega_{\mu,j}^i)\rho_m(u_n)\exp\Big(\int_0^{u_n}g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t=\epsilon(n,j,\mu) \tag{4.46}$$

and

$$\int_{\{m\leq|u_n|\leq m+1\}}\Phi_n(u_n)(T_\ell^*(u_n(u_n)-\omega_{\mu,j}^i)\nabla u_n\rho_m'(u_n)\exp\Big(\int_0^{u_n}g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t \tag{4.47}$$
$$=\epsilon(n,j,\mu).$$

About (3).
Similarly, by the almost everywhere convergence of u_n, we have $(T_\ell^*(u_n)-\omega_{\mu,j}^i)\rho_m(u_n)\exp(\int_0^{u_n}g(s)\mathrm{d}s$ converges to $(T_\ell^*(u)-\omega_{\mu,j}^i)\rho_m(u)\exp(\int_0^u g(s)\mathrm{d}s$ in $L^1(Q)$ weakly, * and then

$$\int_Q f_n(T_\ell^*(u_n)-\omega_{i,j}^\mu)\rho_m(u_n)\,\mathrm{d}x\,\mathrm{d}t\to\int_Q f_n(T_\ell^*(u)-\omega_{i,j}^\mu)\rho_m(u)\,\mathrm{d}x\,\mathrm{d}t.$$

So,

$$(T^*_\ell(u) - \omega^\mu_{i,j})\rho_m(u) \to (T^*_\ell(u) - T^*_\ell(u)_\mu - \exp(-\mu t)T^*_\ell(w_i)$$

in $L^\infty(Q)$ weakly * as $j \to \infty$, and also

$$(T^*_\ell(u) - T^*_\ell(u))_\mu - \exp(-\mu t)T^*_\ell(w_i) \to 0 \text{ in } L^\infty(Q) \text{ weak * as } \mu \to \infty.$$

Then, we deduce that

$$\int_Q f_n(T^*_\ell(u_n) - \omega^\mu_{i,j})\rho_m(u_n)\,\mathrm{d}x\,\mathrm{d}t = \epsilon(n, j, \mu). \tag{4.48}$$

Now taking into account the estimation of (1), (2), (3), (4), and (5), we obtain

$$\int_Q (a(x, t, T_\ell(u_n), \nabla T_\ell(u_n)) - a(x, t, T_\ell(u_n), \nabla T_\ell(v_j)\chi^j_s)(\nabla T^*_\ell(u_n) - \nabla T_\ell(v_j)\chi^j_s)$$
$$\times \exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t = \epsilon(n, j, \mu, i, s, m).$$

On the other hand, we get

$$\int_Q (a(x, t, T_\ell(u_n), \nabla T_\ell(u_n)) - a(x, t, T_\ell(u_n), \nabla T^*_\ell(u)\chi_s)$$
$$\times (\nabla T^*_\ell(u_n) - \nabla T^*_\ell(u)\chi_s)\exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$- \int_Q (a(x, t, T_\ell(u_n), \nabla T_\ell(u_n)) - a(x, t, T_\ell(u_n), \nabla T_\ell(v_j)\chi^j_s)$$
$$\times (\nabla T^*_\ell(u_n) - \nabla T_\ell(v_j)\chi^j_s)\exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_Q (a(x, t, T_\ell(u_n), \nabla T_\ell(u_n))(\nabla T_\ell(v_j)\chi^j_s - \nabla T^*_\ell(u)\chi_s)\exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$- \int_Q (a(x, t, T_\ell(u_n), \nabla T^*_\ell(u_n)\chi_s)(\nabla T_\ell(v_j)\chi^j_s - \nabla T_\ell{*}(u)\chi_s)\exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_Q (a(x, t, T_\ell(u_n), \nabla T^*_\ell(v_j)\chi^j_s)(\nabla T^*_\ell(u_n) - \nabla T_\ell(v_j)\chi^j_s)\exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t.$$

Each term of the last right hand side is of the form $\epsilon(n, j, s)$, which gives

$$\int_Q (a(x, t, T_\ell(u_n), \nabla T_\ell(u_n)) - a(x, t, T^*_\ell(u_n), \nabla T^*_\ell(u)\chi_s)$$
$$\times (\nabla T^*_\ell(u_n) - \nabla T^*_\ell(u)\chi_s)\exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_Q (a(x, t, T_\ell(u_n), \nabla T_\ell(u_n)) - a(x, t, T_\ell(u_n), \nabla T_\ell(v_j)\chi^j_s)$$
$$\times (\nabla T^*_\ell(u_n) - \nabla T_\ell(v_j)\chi^j_s)\exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t + \epsilon(n, j, s).$$

Following the same technique used in [**17**] for all $r < s$

$$\int_Q (a(x, t, T_\ell(u_n), \nabla T_\ell(u_n)) - a(x, t, T_\ell(u_n), \nabla T^*_\ell(u))(\nabla T^*_\ell(u_n) - \nabla T^*_\ell(u))\,\mathrm{d}x\,\mathrm{d}t \to 0. \tag{4.49}$$

On the other hand, by using (4.40), we get

$$(\lambda(u_n)-\lambda(u))\nabla T_\ell(u)\chi_{\{|\nabla T_\ell(u)|\le r\}} \to 0 \text{ strongly in } (E_\varphi(Q))^N$$

and

$$a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) - a(x,t,T_\ell(u_n),\nabla T_\ell(u)) \rightharpoonup \phi_\ell - a(x,t,T_\ell(u),\nabla T_\ell(u)) \\ \text{weakly in } (L_\psi(Q))^N,$$

which gives

$$\int_Q (a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) \\ - a(x,t,T_\ell(u_n),\nabla T_\ell(u)))\nabla T_\ell(u)((\lambda(u_n)-\lambda(u))\,\mathrm{d}x\,\mathrm{d}t \to 0. \tag{4.50}$$

By using:
- (4.41),
- the monotonicity condition,
- (4.40) and the decomposition

$$\nabla T_\ell^*(u_n) - \nabla T_\ell^*(u) \\ = \lambda(u_n)(\nabla T_\ell(u_n) - \nabla T_\ell(u)) + (\lambda(u_n)-\lambda(u))\nabla T_\ell(u),$$

- (4.49) and (4.50),

we obtain

$$\lim_{n\to\infty}\int_Q (a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) - a(x,t,T_\ell(u_n),\nabla T_\ell(u)) \\ \times (\nabla T_\ell(u_n) - \nabla T_\ell(u))\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Thus, there exists a subsequence also denoted by u_n such that

$$\nabla T_\ell(u_n) \to \nabla T_\ell(u) \text{ a.e. in } Q. \tag{4.51}$$

We deduce then that

$$a(x,t,T_\ell(u_n),\nabla T_\ell(u_n) \rightharpoonup a(x,t,T_\ell(u),\nabla T_\ell(u) \text{ in } (L_\psi(Q))^N \tag{4.52}$$

for $\sigma(\Pi L_\varphi, \Pi E_\psi)$.

Step 5: Modular convergence of the truncations

We have proved that

$$\int_Q (a(x,t,T_\ell(u_n),\nabla T_\ell(u_n) - a(x,t,T_\ell(u_n),\nabla T_\ell(v_j)\chi_s^j)) \\ \times (\nabla T_\ell^*(u_n) - \nabla T_\ell(v_j)\chi_s^j) \times \exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t \le \epsilon(n,j,\mu,,i,s,m). \tag{4.53}$$

And, we can also deduce that

$$\begin{aligned}&\int_Q (a(x,t,T_\ell(u_n), \nabla T_\ell(u_n) - a(x,t,T_\ell(u_n), \nabla T_\ell^*(u)\chi_s))\\ &\quad\times (\nabla T_\ell^*(u_n) - \nabla T_\ell^*(u)\chi_s) \exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_Q (a(x,t,T_\ell(u_n), \nabla T_\ell(u_n) - a(x,t,T_\ell(u_n), \nabla T_\ell(v_j)\chi_s^j))\\ &\quad\times (\nabla T_\ell^*(u_n) - \nabla T_\ell(v_j)\chi_s^j) \exp\Big(\int_0^{u_n} g(s)\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t + \epsilon(n,j,s).\end{aligned}$$

Then

$$\begin{aligned}&\int_Q (a(x,t,T_\ell(u_n), \nabla T_\ell(u_n)\nabla T_\ell^*(u_n)\,\mathrm{d}x\,\mathrm{d}t\\ &\le \int_Q (a(x,t,T_\ell(u_n), \nabla T_\ell(u_n)\nabla T_\ell^*(u)\chi_s\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+ \int_Q (a(x,t,T_\ell(u_n), \nabla T_\ell^*(u)\chi_s)(\nabla T_\ell^*(u_n) - T_\ell(u)\chi_s)\,\mathrm{d}x\,\mathrm{d}t + \epsilon(n,j,\mu,,i,s,m)\end{aligned}$$

and

$$\begin{aligned}&\limsup_n \int_Q (a(x,t,T_\ell(u_n), \nabla T_\ell(u_n)\nabla T_\ell^*(u_n)\,\mathrm{d}x\,\mathrm{d}t\\ &\le \int_Q (a(x,t,T_\ell(u_n), \nabla T_\ell(u_n)\nabla T_\ell^*(u)\chi_s\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+ \lim_n \epsilon(n,j,\mu,,i,s,m).\end{aligned}$$

Then

$$\begin{aligned}&\limsup_n \int_Q (a(x,t,T_\ell(u_n), \nabla T_\ell(u_n)\nabla T_\ell^*(u_n)\,\mathrm{d}x\,\mathrm{d}t\\ &\le \int_Q (a(x,t,T_\ell(u_n), \nabla T_\ell(u_n)\nabla T_\ell^*(u)\,\mathrm{d}x\,\mathrm{d}t\\ &\le \liminf_n \int_Q (a(x,t,T_\ell(u_n), \nabla T_\ell(u_n)\nabla T_\ell^*(u)\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

as $n \to \infty$, we deduce

$$\begin{aligned}&a(x,t,T_\ell(u_n), \nabla T_\ell(u_n)\nabla T_\ell^*(u_n)\\ &\to a(x,t,T_\ell(u), \nabla T_\ell(u)\nabla T_\ell^*(u) \text{ in } L^1(Q).\end{aligned}$$

Using the same argument as above, we obtain

$$\begin{aligned}&a(x,t,T_\ell(u_n), \nabla T_\ell(u_n)\nabla T_\ell(u_n)\\ &\to a(x,t,T_\ell(u), \nabla T_\ell(u)\nabla T_\ell(u) \text{ in } L^1(Q)\end{aligned}$$

and Vitali's theorem, and (4.2) gives

$$\nabla T_\ell(u_n) \to \nabla T_\ell(u) \text{ for the modular convergence in } (L_\varphi(Q))^N.$$

Step 6: Passing to the limit

Let $v \in W_0^{1,x}L_\varphi(Q)$ such that $\frac{\partial v}{\partial t} \in W^{-1,x}L_\psi(Q) + L^1(Q)$. There exists a prolongation $\overline{v}$ of v such that (see the proof of [**35**, Lemma 5.3])

$$\begin{cases} \overline{v} = v \quad \text{on } Q, \\ \overline{v} \in W_0^{1,x}L_\varphi(\Omega \times \mathbb{R}) \cap L^1(\Omega \times \mathbb{R}) \cap L^\infty(\Omega \times \mathbb{R}), \\ \text{and} \quad \dfrac{\partial \overline{v}}{\partial t} \in W^{-1,x}L_\psi(\Omega \times \mathbb{R}) + L^1(\Omega \times \mathbb{R}). \end{cases}$$

By Lemma 3.5, there exists a sequence $(w_j)_j$ in $D(\Omega \times \mathbb{R})$ such that $w_j \to \overline{v}$ in $W_0^{1,x}L_\varphi(\Omega \times \mathbb{R})$ and $\frac{\partial w_j}{\partial t} \to \frac{\partial \overline{v}}{\partial t}$ in $W^{-1,x}L_\psi(\Omega \times \mathbb{R}) + L^1(\Omega \times \mathbb{R})$ for the modular convergence and $\|w_j\|_{\infty,Q} \le (N+2)\|v\|_{\infty,Q}$. Using $T_\ell(u_n - w_j)\chi_{[0,\tau]}$ as a test function in $(\mathcal{P}_n)$, for every $\tau \in [0,T]$, one has

$$\begin{aligned} &\int_{Q_\tau} \frac{\partial u_n}{\partial t} T_\ell(u_n - w_j)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_{Q_\tau} a(x,t,u_n,\nabla u_n) \cdot \nabla T_\ell(u_n - w_j)\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \int_{Q_\tau} \Phi(u_n) \cdot \nabla T_\ell(u_n - w_j)\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad\quad \times \int_{Q_\tau} g(u_n)\varphi(x,|\nabla u_n|) T_\ell(u_n - w_j)\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad\qquad \le \int_{Q_\tau} f_n T_\ell(u_n - w_j)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{4.54}$$

For the first term of (4.54), we get

$$\begin{aligned} \int_{Q_\tau} \frac{\partial u_n}{\partial t} T_\ell(u_n - w_j)\,\mathrm{d}x\,\mathrm{d}t &= \left[\int_\Omega T_\ell(u_n - w_j)\,\mathrm{d}x\right]_0^\tau + \int_{Q_\tau} \frac{\partial w_j}{\partial t} T_\ell(u_n - w_j)\,\mathrm{d}x\,\mathrm{d}t \\ &= \left[\int_\Omega T_\ell(u - w_j)\,\mathrm{d}x\right]_0^\tau + \int_{Q_\tau} \frac{\partial w_j}{\partial t} T_\ell(u - w_j)\,\mathrm{d}x\,\mathrm{d}t + \varepsilon(n) \\ &= \int_{Q_\tau} \frac{\partial u}{\partial t} T_\ell(u - w_j)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

For the second term of (4.54), we have if $|u_n| > \lambda$, then $|u_n - w_j| \ge |u_n| - \|w_j\|_\infty > k$, therefore $\{|u_n - w_j| \le k\} \subseteq \{|u_n| \le k + (N+2)\|v\|_\infty\}$, which implies

$$\begin{aligned} &\liminf_{n\to+\infty} \int_Q a(x,t,u_n,\nabla u_n)\nabla T_k(u_n - w_j)\,\mathrm{d}x\,\mathrm{d}t \\ &\ge \int_Q a(x,t,T_{k+(N+2)\|v\|_\infty}(u), \nabla T_{k+(N+2)\|v\|_\infty}(u)) \\ &\quad \times (\nabla T_{k+(N+2)\|v\|_\infty}(u) - \nabla w_j)\chi_{\{|u-v|\le k\}}\,\mathrm{d}x\,\mathrm{d}t, \\ &= \int_Q a(x,t,u,\nabla u)(\nabla u - \nabla w_j)\chi_{\{|u-w_j|\le k\}}\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_Q a(x,t,u,\nabla u)\nabla T_k(u - w_j)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{4.55}$$

By using 4.9 and the fact that $\nabla T_\ell(u_n - w_j) \rightharpoonup \nabla T_\ell(u - w_j)$ in $L_\varphi(Q)$ as $n \to +\infty$, we can see that

$$\int_{Q_\tau} \Phi(u_n) \cdot \nabla T_\ell(u_n - w_j)\,\mathrm{d}x\,\mathrm{d}t \to \int_{Q_\tau} \Phi(u) \cdot \nabla T_\ell(u - w_j)\,\mathrm{d}x\,\mathrm{d}t.$$

Consequently, by using the strong convergence of $(g(u_n)\varphi(x, |\nabla u_n|))_n$ and $((f_n))_n$, one has

$$\begin{aligned}
&\int_{Q_\tau} \frac{\partial u}{\partial t} T_\ell(u - w_j)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{Q_\tau} a(x, t, u, \nabla u) \cdot \nabla T_\ell(u - w_j)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_{Q_\tau} \Phi(u) \cdot \nabla T_\ell(u - w_j) d\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\quad + \int_{Q_\tau} g(u)\varphi(x, |\nabla u|) T_\ell(u - w_j)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad \leq \int_{Q_\tau} f T_\ell(u - w_j)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{4.56}$$

Thus, by using the modular convergence in j, we achieve this step.

As a conclusion of step 1 to step 6, the proof of Theorem 4.1 is complete.

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