

OSCILLATION OF SECOND-ORDER NONLINEAR FORCED DYNAMIC EQUATIONS WITH DAMPING ON TIME SCALES

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ABSTRACT. In this paper, we use Riccati transformation technique to establish some new oscillation criteria for the second-order nonlinear forced dynamic equation with damping on a time scale \mathbb{T}

$$(r(t)g(x^\Delta(t)))^\Delta + p(t)g(x^\Delta(t)) + q(t)f(x^\sigma(t)) = G(t, x^\sigma(t)),$$

where $r(t)$, $p(t)$ and $q(t)$ are real-valued right-dense continuous functions on \mathbb{T} and no sign conditions are imposed on these functions. The function $f: \mathbb{T} \rightarrow \mathbb{T}$ is continuously differentiable and nondecreasing such that $xf(x) > 0$ for $x \neq 0$. Our results not only generalize and extend some existing results, but also can be applied to the oscillation problems that are not covered in literature. Finally, we give some examples to illustrate our main results.

1. INTRODUCTION

The theory of time scales was introduced by Hilger [7] in order to unify, extend and generalize ideas from discrete calculus, quantum calculus and continuous calculus to arbitrary time scale calculus. A time scale \mathbb{T} is an arbitrary closed subset of the real numbers \mathbb{R} . When time scale equals to the reals or to the integers, it represents the classical theories of differential and difference equations. Many other interesting time scales exist, e.g., $\mathbb{T} = q^{\mathbb{N}_0} := \{q^t : t \in \mathbb{N}_0 \text{ for } q > 1\}$ (which has important applications in quantum theory), $\mathbb{T} = h\mathbb{N}$ with $h > 0$, $\mathbb{T} = \mathbb{N}^2$ and $\mathbb{T} = \mathbb{T}^n$ (the space of the harmonic numbers). For an introduction to time scale calculus and dynamic equations, see Bohner and Peterson books [1, 2].

In what follows, we use σ and ρ to denote the forward jump operator and the backward jump operator defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

A point $t \in \mathbb{T}$, $t > \inf \mathbb{T}$ is said to be left-dense if $\rho(t) = t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$.

A function $f: \mathbb{T} \rightarrow \mathbb{R}$ is called right-dense continuous function (rd-continuous) provided it is continuous at right-dense points in \mathbb{T} and its left-sided limits exist

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(finite) at left-dense points in \mathbb{T} . The set of all rd-continuous functions is denoted by $C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.

For $f: \mathbb{T} \rightarrow \mathbb{R}$ we define the function $f^\sigma: \mathbb{T} \rightarrow \mathbb{R}$ by

$$f^\sigma(t) := f(\sigma(t)) \quad \text{for all } t \in \mathbb{T}.$$

Let \mathbb{T}^k be the set defined as $\mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T})$ if $\sup \mathbb{T} < \infty$ and as \mathbb{T} if $\sup \mathbb{T} = \infty$.

For $f: \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}^k$ we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U.$$

We call $f^\Delta(t)$ the delta derivative of f at t and say that f is Δ -differentiable on \mathbb{T}^k provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^k$.

In particular, if f is continuous at $t \in \mathbb{T}^k$ and t is right-scattered, then f is Δ -differentiable at t with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

and if t is right-dense, then f is Δ -differentiable at t if and only if the limit

$$\lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case

$$f^\Delta(t) = \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s},$$

The set of all functions $f: \mathbb{T} \rightarrow \mathbb{R}$ that are differentiable and whose first derivative is rd-continuous function is denoted by $C_{rd}^1(\mathbb{T}) = C_{rd}^1(\mathbb{T}, \mathbb{R})$.

For any Δ -differentiable functions f and g we define

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g^\sigma(t)$$

and

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g^\sigma(t)}, \quad \text{if } gg^\sigma \neq 0.$$

For $a, b \in \mathbb{T}$, we define the Cauchy integral of f by

$$\int_a^b f(t)\Delta t = F(b) - F(a),$$

where $F(t)$ is the Δ -antiderivative of f , i.e., $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^k$.

An integration by parts formula reads

$$\int_a^b f(t)g^\Delta(t)\Delta t = [f(t)g(t)]_a^b - \int_a^b f^\Delta(t)g^\sigma(t)\Delta t,$$

or

$$\int_a^b f^\sigma(t)g^\Delta(t)\Delta t = [f(t)g(t)]_a^b - \int_a^b f^\Delta(t)g(t)\Delta t.$$

Finally, the improper integral of the first kind is defined as

$$\int_a^\infty f(t)\Delta t = \lim_{b \rightarrow \infty} \int_a^b f(t)\Delta t.$$

In recent years, there has been much research activity concerning the oscillation and nonoscillation of solution of various equations on time scales. We refer the reader to paper [3]–[6], [8]–[19] and references cited therein.

Some special cases of the equation studied her were investigated in [3]–[4], [6], [10]–[13], [15]–[19].

Here, we are concerned with the oscillation behavior of all solutions of the second-order nonlinear forced dynamic equation with damping on a time scale \mathbb{T} which is unbounded above

$$(1.1) \quad (r(t)g(x^\Delta(t)))^\Delta + p(t)g(x^\Delta(t)) + q(t)f(x^\sigma(t)) = G(t, x^\sigma(t)),$$

where $t \in \mathbb{T}$, $t \geq t_0$.

The equation will be studied under the following assumptions:

- (H₁) $r(t)$, $p(t)$ and $q(t)$ are real-valued right-dense continuous functions on \mathbb{T} and no sign conditions are imposed on $r(t)$, $p(t)$ and $q(t)$,
- (H₂) $f: \mathbb{T} \rightarrow \mathbb{T}$ is continuously differentiable and nondecreasing function such that

$$uf(u) > 0 \quad \text{for } u \neq 0,$$

- (H₃) $G: \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$ is a function such that

$$uG(t, u) > 0 \quad \text{for } u \neq 0,$$

- (H₄) $g \in C(\mathbb{R}, \mathbb{R})$ is continuous and increasing function with $ug(u) > 0$, $u \neq 0$, we have

$$g^{-1}(u\nu) \leq Lg^{-1}(u)g^{-1}(\nu), \quad u, \nu \neq 0, \quad L > 0,$$

- (H₅) $f(x)$ satisfies the superlinear condition

$$(1.2) \quad 0 < \int_\varepsilon^\infty \frac{dx}{f(x)}, \quad \int_{-\infty}^{-\varepsilon} \frac{dx}{f(x)} < \infty, \quad \text{for all } \varepsilon > 0,$$

- (H₅) $f(x)$ satisfies the sublinear condition

$$(1.3) \quad 0 < \int_0^\varepsilon \frac{dx}{f(x)}, \quad \int_{-\varepsilon}^0 \frac{dx}{f(x)} < \infty, \quad \text{for all } \varepsilon > 0,$$

Our results in this paper extend and improve some results established in [3]–[5], [7]–[19]. According to our knowledge, the oscillatory behavior of Eq. (1.1) without strong sign restrictions imposed on $r(t)$, $p(t)$ and $q(t)$ is not discussed before.

By a solution of (1.1), we mean that a nontrivial real valued function x satisfies (1.1) for $t \in \mathbb{T}$. A solution x of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory. Eq. (1.1) is said to be oscillatory if all of its solutions are oscillatory.

2. MAIN RESULTS

Before starting our studies, we begin with the following lemma (Second Mean Value Theorem) which will play an important role in the proof of our main results.

Lemma 2.1. ([2, Theorem 5.45]). *Let h be a bounded function that is integrable on $[a, b]_{\mathbb{T}}$. Let m_H and M_H be the infimum and supremum of the function $H(t) := \int_a^t h(s)\Delta s$ on $[a, b]_{\mathbb{T}}$ respectively. Suppose that g is a nonnegative and non-increasing function on $[a, b]_{\mathbb{T}}$. Then there is some number Λ with $m_H \leq \Lambda \leq M_H$ such that*

$$\int_a^b h(t)g(t)\Delta t = g(a)\Lambda.$$

Our first result is stated in terms of an auxiliary function $\phi = \phi(t)$.

Theorem 2.1. *Assume that there exist $e(t)$ and a C_{rd}^1 function ϕ such that*

$$(2.1) \quad \phi^\sigma(t)|G(t, u)| \leq e(t)|f(u)|, \quad \text{for } u \neq 0$$

$$(2.2) \quad \phi(t)r(t) > 0, \quad \phi^\Delta(t)r(t) = \phi^\sigma(t)p(t), \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}},$$

$$(2.3) \quad \int_{t_0}^{\infty} g^{-1}\left(\frac{1}{\phi(t)r(t)}\right)\Delta t = \int_{t_0}^{\infty} [\phi^\sigma(t)q(t) - e(t)]\Delta t = \infty,$$

then every solution of Eq. (1.1) is oscillatory.

Proof. Assume that Eq. (1.1) has a nonoscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we assume that $x(t) > 0$ on $[T, \infty)_{\mathbb{T}}$ for some $T \in [t_0, \infty)_{\mathbb{T}}$. Define

$$w(t) := \frac{\phi(t)r(t)g(x^\Delta(t))}{f(x(t))}.$$

Using the delta derivative rules of the product and quotient of two functions and then chain rule (see [1, Theorem 1.90]), we get

$$\begin{aligned} w^\Delta(t) &= \frac{\phi^\sigma(t)}{f(x(\sigma(t)))} (r(t)g(x^\Delta(t)))^\Delta + \left[\frac{\phi(t)}{f(x(t))} \right]^\Delta r(t)g(x^\Delta(t)) \\ &= \frac{\phi^\sigma(t)}{f(x(\sigma(t)))} (r(t)g(x^\Delta(t)))^\Delta \\ &\quad + \left[\frac{\phi^\Delta(t)}{f(x(\sigma(t)))} - \frac{\phi(t) \int_0^1 f'(x_h(t)) dh}{f(x(t))f(x(\sigma(t)))} x^\Delta(t) \right] r(t)g(x^\Delta(t)), \end{aligned}$$

where $x_h(t) := (1-h)x(t) + hx^\sigma(t) > 0$ for $0 \leq h \leq 1$, $t \in [T, \infty)_{\mathbb{T}}$. Therefore, from (1.1), we have

$$(2.4) \quad \begin{aligned} w^\Delta(t) &= -\phi^\sigma(t)q(t) + \frac{\phi^\sigma(t)G(t, x^\sigma(t))}{f(x(\sigma(t)))} - \frac{\phi^\sigma(t)p(t)}{f(x(\sigma(t)))} g(x^\Delta(t)) \\ &\quad + \frac{\phi^\Delta(t)r(t)}{f(x(\sigma(t)))} g(x^\Delta(t)) - \frac{\phi(t)r(t) \int_0^1 f'(x_h(t)) dh}{f(x(t))f(x(\sigma(t)))} x^\Delta(t) g(x^\Delta(t)). \end{aligned}$$

From (H₂), (H₄), (2.1) and (2.2), we get

$$(2.5) \quad w^\Delta(t) \leq -[\phi^\sigma(t)q(t) - e(t)].$$

Integrating the above inequality from T to t , we get

$$\frac{\phi(t)r(t)g(x^\Delta(t))}{f(x(t))} \leq \frac{\phi(T)r(T)g(x^\Delta(T))}{f(x(T))} - \int_T^t [\phi^\sigma(s)q(s) - e(s)]\Delta s.$$

From condition (2.3), we conclude that there exists a sufficiently large $T_1 \geq T$ such that

$$(2.6) \quad g(x^\Delta(t)) < 0, \text{ for } t \in [T_1, \infty)_{\mathbb{T}},$$

and consequently

$$x^\Delta(t) < 0, \quad \text{for } t \in [T_1, \infty)_{\mathbb{T}}.$$

Also, from (2.3), there exists $T_2 \geq T_1$ such that

$$(2.7) \quad \int_{T_2}^t [\phi^\sigma(s)q(s) - e(s)]\Delta s \geq 0, \quad \text{for all } t \geq T_2.$$

Multiplying both sides of Eq. (1.1) by $\phi^\sigma(t)$ and integrating it from T_2 to t , we get

$$(2.8) \quad \begin{aligned} & \int_{T_2}^t \phi^\sigma(s)(r(s)g(x^\Delta(s)))^\Delta \Delta s + \int_{T_2}^t \phi^\sigma(s)p(s)g(x^\Delta(s))\Delta s \\ & + \int_{T_2}^t \phi^\sigma(s)q(s)f(x(\sigma(s)))\Delta s \\ & = \int_{T_2}^t \phi^\sigma(s)G(s, x^\sigma(s))\Delta s. \end{aligned}$$

Integrating by parts and using (2.2), we have

$$(2.9) \quad \begin{aligned} & \int_{T_2}^t \phi^\sigma(s)(r(s)g(x^\Delta(s)))^\Delta \Delta s + \int_{T_2}^t \phi^\sigma(s)p(s)g(x^\Delta(s))\Delta s \\ & = \phi(t)r(t)g(x^\Delta(t)) - \phi(T_2)r(T_2)g(x^\Delta(T_2)) \\ & \quad - \int_{T_2}^t \phi^\Delta(s)r(s)g(x^\Delta(s))\Delta s + \int_{T_2}^t \phi^\sigma(s)p(s)g(x^\Delta(s))\Delta s \\ & = \phi(t)r(t)g(x^\Delta(t)) - \phi(T_2)r(T_2)g(x^\Delta(T_2)). \end{aligned}$$

Also, integrating by parts, applying chain rule and then using (H₂), (2.1) and (2.7), we get

$$\begin{aligned}
 & \int_{T_2}^t \phi^\sigma(s)q(s)f(x(\sigma(s)))\Delta s - \int_{T_2}^t \phi^\sigma(s)G(s, x^\sigma(s))\Delta s \\
 & \geq \int_{T_2}^t \phi^\sigma(s)q(s)f(x(\sigma(s)))\Delta s - \int_{T_2}^t f(x(\sigma(s)))e(s)\Delta s \\
 & \geq \int_{T_2}^t f(x(\sigma(s)))[\phi^\sigma(s)q(s) - e(s)]\Delta s \\
 (2.10) \quad & = f(x(t)) \int_{T_2}^t [\phi^\sigma(\theta)q(\theta) - e(\theta)]\Delta\theta \\
 & \quad - \int_{T_2}^t \left[\int_0^1 f'(x_h(s))dh \right] x^\Delta(s) \int_{T_2}^s [\phi^\sigma(\theta)q(\theta) - e(\theta)]\Delta\theta\Delta s \\
 & \geq f(x(t)) \int_{T_2}^t [\phi^\sigma(s)q(s) - e(s)]\Delta s \geq 0.
 \end{aligned}$$

Using (H₄), (2.9) and (2.10) in (2.8), we get

$$\phi(t)r(t)g(x^\Delta(t)) \leq \phi(T_2)r(T_2)g(x^\Delta(T_2)),$$

and consequently

$$x^\Delta(t) \leq L_1L_2g^{-1}\left(\phi(T_2)r(T_2)\right)x^\Delta(T_2) \int_{T_2}^t g^{-1}\left(\frac{1}{\phi(s)r(s)}\right)\Delta s, \quad L_1, L_2 > 0.$$

Assuming $M = L_1L_2g^{-1}\left(\phi(T_2)r(T_2)\right)x^\Delta(T_2)$ and $M < 0$, we get

$$x(t) - x(T_2) \leq M \int_{T_2}^t g^{-1}\left(\frac{1}{\phi(s)r(s)}\right)\Delta s.$$

From condition (2.3), $\lim_{t \rightarrow \infty} x(t) = -\infty$, which is a contradiction. This completes the proof. \square

Theorem 2.2. *Assume that there exist $e(t)$ and a C_{rd}^1 function ϕ such that (2.1) and (2.2) hold. If*

$$(2.11) \quad \int_{t_0}^{\infty} g^{-1}\left(\frac{1}{\phi(t)r(t)}\right)\Delta t < \infty,$$

$$(2.12) \quad \int_{t_0}^{\infty} [\phi^\sigma(t)q(t) - e(t)]\Delta t = \infty,$$

and

$$(2.13) \quad \int_{t_0}^{\infty} g^{-1}\left(\frac{1}{\phi(t)r(t)}\left[\int_{t_0}^t [\phi^\sigma(s)q(s) - e(s)]\Delta s\right]\right)\Delta t = \infty,$$

then every solution of Eq. (1.1) is either oscillatory or tends to zero as t tends to infinity.

Proof. Assume that Eq. (1.1) has a nonoscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we assume that $x(t) > 0$ on $[T, \infty)_{\mathbb{T}}$ for some $T \in [t_0, \infty)_{\mathbb{T}}$. Proceeding as in the proof of Theorem 2.1 to get (2.6), i.e., there exists a sufficiently large $T_1 \geq T$ such that

$$g(x^\Delta(t)) < 0, \quad \text{for } t \in [T_1, \infty)_{\mathbb{T}}.$$

and

$$w^\Delta(t) \leq -[\phi^\sigma(t)q(t) - e(t)].$$

Integrating the above inequality from T_1 to t , we get

$$\begin{aligned} \frac{\phi(t)r(t)g(x^\Delta(t))}{f(x(t))} - \frac{\phi(T_1)r(T_1)g(x^\Delta(T_1))}{f(x(T_1))} &\leq - \int_{T_1}^t [\phi^\sigma(s)q(s) - e(s)]\Delta s. \\ \frac{\phi(t)r(t)g(x^\Delta(t))}{f(x(t))} &\leq - \int_{T_1}^t [\phi^\sigma(s)q(s) - e(s)]\Delta s. \end{aligned}$$

$$(2.14) \quad g(x^\Delta(t)) \leq - \left(\frac{1}{\phi(t)r(t)} \int_{T_1}^t [\phi^\sigma(s)q(s) - e(s)]\Delta s \right) f(x(t)).$$

Since $x(t) > 0$ and $x^\Delta(t) < 0$ for all $t \geq T_1 \geq T$, then, $x(t) \rightarrow N$ as $t \rightarrow \infty$.

Assume that $N > 0$, then $x(t) \geq N$ and consequently $f(x(t)) \geq f(N) > 0$ for all $t \geq T_1$.

From (2.14) for $t \geq T_1$, we have

$$g(x^\Delta(t)) \leq -f(N) \left(\frac{1}{\phi(t)r(t)} \int_{T_1}^t [\phi^\sigma(s)q(s) - e(s)]\Delta s \right).$$

and consequently

$$(2.15) \quad \begin{aligned} x^\Delta(t) &\leq -Lg^{-1}(f(N))g^{-1} \\ &\times \left(\frac{1}{\phi(t)r(t)} \int_{T_1}^t [\phi^\sigma(s)q(s) - e(s)]\Delta s \right), \quad L > 0. \end{aligned}$$

Integrating the inequality (2.15) from T_1 to t , we have

$$(2.16) \quad \begin{aligned} x(t) &\leq x(t_1) - Lg^{-1}(f(N)) \int_{T_1}^t g^{-1} \\ &\times \left(\frac{1}{\phi(s)r(s)} \int_{T_1}^s [\phi^\sigma(\theta)q(\theta) - e(\theta)]\Delta\theta \right) \Delta s. \end{aligned}$$

From (2.13), $x(t) \rightarrow -\infty$ as $t \rightarrow \infty$, but this contradicts the fact $x(t) > 0$. Thus $N = 0$ and $x(t) \rightarrow 0$ as $t \rightarrow \infty$. This completes the proof. \square

Remark 2.1. The function f may be linear, superlinear and sublinear, since we do not assume additional conditions like (1.2) or (1.3).

In the following, we state some new results for special cases of Eq. (1.1):

Case I. When $g(x) = x^\gamma$, then Eq. (1.1) becomes

$$(2.17) \quad (r(t)(x^\Delta(t))^\gamma)^\Delta + p(t)(x^\Delta(t))^\gamma + q(t)f(x^\sigma(t)) = G(t, x^\sigma(t)),$$

where $t \in \mathbb{T}$, $t \geq t_0$, and γ is the quotient of odd positive integers. Eq. (2.17) with $G(t, x^\sigma) = 0$ and $r(t) = 1$ was studied in [5]. Our results extend and generalize some results of [5] because we do not require any restriction sign on $r(t)$, $p(t)$ and $q(t)$. In this case, we have the following results:

Corollary 2.1. *Assume that there exist $e(t)$ and a C_{rd}^1 function ϕ such that (2.1) and (2.2) hold. If*

$$(2.18) \quad \int_{t_0}^{\infty} \frac{\Delta t}{(\phi(t)r(t))^{\frac{1}{\gamma}}} = \int_{t_0}^{\infty} [\phi^\sigma(t)q(t) - e(t)]\Delta t = \infty,$$

then every solution of Eq. (2.17) is oscillatory.

Corollary 2.2. *Assume that there exist $e(t)$ and a C_{rd}^1 function ϕ such that (2.1) and (2.2) hold. If*

$$(2.19) \quad \int_{t_0}^{\infty} \frac{\Delta t}{(\phi(t)r(t))^{\frac{1}{\gamma}}} < \infty,$$

$$(2.20) \quad \int_{t_0}^{\infty} [\phi^\sigma(t)q(t) - e(t)]\Delta t = \infty,$$

and

$$(2.21) \quad \int_{t_0}^{\infty} \left(\frac{1}{\phi(t)r(t)} \left[\int_{t_0}^t [\phi^\sigma(s)q(s) - e(s)]\Delta s \right] \right)^{\frac{1}{\gamma}} \Delta t = \infty,$$

then every solution of Eq. (2.17) is either oscillatory or tends to zero as t tends to infinity.

Case II. When $g(x) = x$, then Eq. (1.1) becomes

$$(2.22) \quad (r(t)x^\Delta(t))^\Delta + p(t)x^\Delta(t) + q(t)f(x^\sigma(t)) = G(t, x^\sigma(t)),$$

where $t \in \mathbb{T}$, $t \geq t_0$.

Here, we can obtain new results not necessarily $\phi^\Delta(t)r(t) = \phi^\sigma(t)p(t)$. Eq. (2.22) with $G(t, x^\sigma) = 0$ was studied in [9, 14]. So, our results are not only extension for the results in [9, 14], but also improve the results established by [16] when $\mathbb{T} = \mathbb{R}$ and $G(t, x^\sigma) = e(t)$ and by [18] when $f(x) = x^\alpha$, as our results do not require any restriction sign on $r(t)$, $p(t)$ and $q(t)$. In this case, we have the following results.

Theorem 2.3. *Assume that f satisfies (1.2). Also, assume that there exist $e(t)$ and a C_{rd}^1 function ϕ such that (2.1) holds. If*

$$(2.23) \quad \phi(t)r(t) > 0, \quad P(t) \geq 0, \quad P^\Delta(t) \leq 0, \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}},$$

where $P(t) := \phi^\Delta(t)r(t) - \phi^\sigma(t)p(t)$, and

$$(2.24) \quad \int_{t_0}^{\infty} \frac{\Delta t}{\phi(t)r(t)} = \int_{t_0}^{\infty} [\phi^\sigma(t)q(t) - e(t)]\Delta t = \infty,$$

then every solution of Eq. (2.22) is oscillatory.

Proof. Assume that Eq. (2.22) has a nonoscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we assume that $x(t) > 0$ on $[T, \infty)_{\mathbb{T}}$ for some $T \in [t_0, \infty)_{\mathbb{T}}$. Define

$$w(t) := \frac{\phi(t)r(t)x^\Delta(t)}{f(x(t))}.$$

Proceeding as in the proof of Theorem 2.1, Eq. (2.4) will have the form

$$(2.25) \quad \begin{aligned} w^\Delta(t) = & -\phi^\sigma(t)q(t) + \frac{\phi^\sigma(t)G(t, x^\sigma(t))}{f(x(\sigma(t)))} + \frac{P(t)x^\Delta(t)}{f(x(\sigma(t)))} \\ & - \frac{\phi(t)r(t) \int_0^1 f'(x_h(t))dh (x^\Delta(t))^2}{f(x(t))f(x(\sigma(t)))}. \end{aligned}$$

Using (H_2) , (2.1), (2.23), we get

$$(2.26) \quad w^\Delta(t) \leq -[\phi^\sigma(t)q(t) - e(t)] + \frac{P(t)x^\Delta(t)}{f(x(\sigma(t)))}.$$

Integrating the above inequality from T to $t (\geq T)$, we get

$$(2.27) \quad w(t) \leq w(T) - \int_T^t [\phi^\sigma(s)q(s) - e(s)]\Delta s + \int_T^t \frac{P(s)x^\Delta(s)}{f(x(\sigma(s)))}\Delta s,$$

We claim that $\int_T^t \frac{P(s)x^\Delta(s)}{f(x(\sigma(s)))}\Delta s$ is bounded above for all $t \geq T$. Since $P(t) \geq 0$ and $P^\Delta(s) \leq 0$, we have from Lemma 2.1 that for each $t \in [T, \infty)_{\mathbb{T}}$,

$$(2.28) \quad \int_T^t \frac{P(s)x^\Delta(s)}{f(x(\sigma(s)))}\Delta s = P(T) \int_T^t \frac{x^\Delta(s)}{f(x(\sigma(s)))}\Delta s = P(T)\Lambda(t),$$

where $m(t) \leq \Lambda(t) \leq M(t)$. The numbers $m(t)$ and $M(t)$ denote the infimum and supremum, respectively, of the function $\int_T^s \frac{x^\Delta(\theta)}{f(x(\sigma(\theta)))}\Delta\theta$ on the interval $[T, t]_{\mathbb{T}}$. Define

$$F(x(s)) := \int_{x(T)}^{x(s)} \frac{d\theta}{f(\theta)}.$$

Then

$$(2.29) \quad (F(x(s)))^\Delta = \int_0^1 F'(x_h(s))dh x^\Delta(s) = \int_0^1 \frac{dh}{f(x_h(s))}x^\Delta(s).$$

For a fixed point $s \in [T, \infty)_{\mathbb{T}}$, we have

$$x_h(s) := (1-h)x(s) + hx^\sigma(s) \begin{cases} \geq x^\sigma(s), & \text{if } x^\Delta(s) \leq 0 \\ \leq x^\sigma(s), & \text{if } x^\Delta(s) \geq 0 \end{cases}$$

and so

$$\frac{x^\Delta(s)}{f(x_h(s))} \geq \begin{cases} \frac{x^\Delta(s)}{f(x(\sigma(s)))}, & \text{if } x^\Delta(s) \leq 0 \\ \frac{x^\Delta(s)}{f(x(\sigma(s)))}, & \text{if } x^\Delta(s) \geq 0. \end{cases}$$

Then

$$\frac{x^\Delta(s)}{f(x_h(s))} \geq \frac{x^\Delta(s)}{f(x(\sigma(s)))}, \quad \text{for } s \in [T, \infty)_{\mathbb{T}}.$$

From (2.29), we have

$$(F(x(s)))^\Delta \geq \frac{x^\Delta(s)}{f(x(\sigma(s)))}, \quad \text{for } s \in [T, \infty)_{\mathbb{T}}.$$

From (1.2), we have

$$F(x(s)) = \int_{x(T)}^{x(s)} \frac{d\theta}{f(\theta)} \leq L_1, \quad \text{for } s \in [T, \infty)_{\mathbb{T}}.$$

Hence, it follows that

$$\int_T^t \frac{x^\Delta(s)}{f(x(\sigma(s)))} \Delta s \leq F(x(t)) - F(x(T)) \leq F(x(t)) \leq L_1.$$

using (2.28) for all $t \in [T, \infty)_{\mathbb{T}}$, we have

$$(2.30) \quad \int_T^t \frac{P(s)x^\Delta(s)}{f(x(\sigma(s)))} \Delta s = P(T)\Lambda(t) \leq P(T)L_1.$$

From (2.27) and (2.30), we get

$$w(t) \leq w(T) + P(T)L_1 - \int_T^t [\phi^\sigma(s)q(s) - e(s)] \Delta s,$$

for $t \in [T, \infty)_{\mathbb{T}}$. In view of condition (2.24), it follows from the last inequality that there exists a sufficiently large $T_1 \geq T$ such that

$$(2.31) \quad x^\Delta(t) < 0, \quad \text{for } t \in [T_1, \infty)_{\mathbb{T}}.$$

From (2.24), there exists $T_2 \geq T_1$ such that

$$(2.32) \quad \int_{T_2}^t [\phi^\sigma(s)q(s) - e(s)] \Delta s \geq 0, \quad \text{for all } t \geq T_2.$$

Multiplying both sides of Eq. (2.22) by $\phi^\sigma(t)$ and then integrating it from T_2 to t , we get

$$(2.33) \quad \int_{T_2}^t \phi^\sigma(s)(r(s)x^\Delta(s))^\Delta \Delta s + \int_{T_2}^t \phi^\sigma(s)p(s)x^\Delta(s) \Delta s + \int_{T_2}^t \phi^\sigma(s)q(s)f(x(\sigma(s))) \Delta s = \int_{T_2}^t \phi^\sigma(s)G(s, x^\sigma(s)) \Delta s.$$

Proceeding as in the proof of Theorem 2.1, one can easily get $\lim_{t \rightarrow \infty} x(t) = -\infty$, which is a contradiction. This completes the proof. \square

Corollary 2.3. *Assume that there exist $e(t)$ and a C^1_{rd} function ϕ such that (2.1), (2.2) and (2.24) hold, then every solution of Eq. (2.22) is oscillatory.*

Theorem 2.4. *Assume that f satisfies (1.2). Also, assume that there exist $e(t)$ and a C_{rd}^1 function ϕ such that (2.1) and (2.23) hold. If*

$$(2.34) \quad \int_{t_0}^{\infty} \frac{\Delta t}{\phi(t)r(t)} < \infty,$$

$$(2.35) \quad \int_{t_0}^{\infty} [\phi^\sigma(t)q(t) - e(t)]\Delta t = \infty,$$

and

$$(2.36) \quad \int_{t_0}^{\infty} \frac{1}{\phi(t)r(t)} \left[\int_{t_0}^t [\phi^\sigma(s)q(s) - e(s)]\Delta s \right] \Delta t = \infty,$$

then every solution of Eq. (2.22) is either oscillatory or tends to zero as t tends to infinity.

Proof. Assume that Eq. (2.22) has a nonoscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we assume that $x(t) > 0$ on $[T, \infty)_{\mathbb{T}}$ for some $T \in [t_0, \infty)_{\mathbb{T}}$. Proceeding as in the proof of Theorem 2.3 to get (2.31), i.e., there exists a sufficiently large $T_1 \geq T$ such that

$$x^\Delta(t) < 0, \quad t \in [T_1, \infty)_{\mathbb{T}},$$

and then

$$w^\Delta(t) \leq -[\phi^\sigma(t)q(t) - e(t)] + \frac{P(t)x^\Delta(t)}{f(x(\sigma(t)))} \leq -[\phi^\sigma(t)q(t) - e(t)].$$

Integrating the above inequality from T_1 to $t (\geq T_1)$, we get

$$\begin{aligned} \int_{T_1}^t [\phi^\sigma(s)q(s) - e(s)]\Delta s &\leq - \int_{T_1}^t \left(\frac{\phi(s)r(s)x^\Delta(s)}{f(x(s))} \right)^\Delta \Delta s, \\ &\leq \frac{\phi(T_1)r(T_1)x^\Delta(T_1)}{f(x(t))} - \frac{\phi(t)r(t)x^\Delta(t)}{f(x(t))}, \\ &\leq -\phi(t)r(t) \frac{x^\Delta(t)}{f(x(t))}, \quad \text{for } t \in [T_1, \infty)_{\mathbb{T}}. \end{aligned}$$

Therefore,

$$\frac{1}{\phi(t)r(t)} \int_{T_1}^t [\phi^\sigma(s)q(s) - e(s)]\Delta s \leq -\frac{x^\Delta(t)}{f(x(t))}.$$

Integrating from T_1 to $t (t \geq T_1)$, we get

$$(2.37) \quad \int_{T_1}^t \frac{1}{\phi(s)r(s)} \left[\int_{T_1}^s [\phi^\sigma(u)q(u) - e(u)]\Delta u \right] \Delta s \leq - \int_{T_1}^t \frac{x^\Delta(s)}{f(x(s))} \Delta s.$$

Defining

$$F(x(s)) := \int_{x(T)}^{x(s)} \frac{d\theta}{f(\theta)},$$

then

$$(2.38) \quad \begin{aligned} (F(x(s)))^\Delta &= \int_0^1 F'(x_h(s)) dh x^\Delta(s) = \int_0^1 \frac{1}{f(x_h(s))} dh x^\Delta(s), \\ &\leq \int_0^1 \frac{1}{f(x(s))} dh x^\Delta(s) = \frac{x^\Delta(s)}{f(x(s))}. \end{aligned}$$

Since $x(t) > 0$ and $x^\Delta(t) < 0$, then, we have $\lim_{t \rightarrow \infty} x(t) = N \geq 0$. If we assume that $N > 0$, then

$$-F(x(t)) = \int_{x(t)}^{x(T_1)} \frac{ds}{f(s)} \leq \int_N^{x(T_1)} \frac{ds}{f(s)} < \infty.$$

Consequently,

$$\begin{aligned} \int_{T_1}^t \frac{1}{\phi(s)r(s)} \left[\int_{T_1}^s [\phi^\sigma(u)q(u) - e(u)] \Delta u \right] \Delta s &\leq - \int_{T_1}^t (F(x(s)))^\Delta \Delta s \\ &= -F(x(t)) < \infty. \end{aligned}$$

If $t \rightarrow \infty$, we get a contradiction to (2.36). This completes the proof. \square

Corollary 2.4. *Assume that there exist $e(t)$ and a C_{rd}^1 function ϕ such that (2.1), (2.2), (2.34), (2.35) and (2.36) hold, then every solution of Eq. (2.22) is either oscillatory or tends to zero.*

In the following, we present oscillation criteria for Eq. (2.22) when f satisfies the sublinear condition (1.3).

Theorem 2.5. *Assume that f satisfies (1.3). Also, assume that there exist $e(t)$ and a C_{rd}^1 function ϕ such that (2.1), (2.2), (2.34), (2.35) and (2.36) hold, then every solution of Eq. (2.22) is oscillatory.*

Proof. Assume that Eq. (2.22) has a nonoscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we assume that $x(t) > 0$ on $[T, \infty)_{\mathbb{T}}$ for some $T \in [t_0, \infty)_{\mathbb{T}}$. Proceeding as in the proof of Theorem 2.4 to get (2.31), i.e., there exists a sufficiently large $T_1 \geq T$ such that

$$x^\Delta(t) < 0, \quad t \in [T_1, \infty)_{\mathbb{T}}.$$

From (2.37), we get

$$\int_{T_1}^t \frac{1}{\phi(s)r(s)} \left[\int_{T_1}^s [\phi^\sigma(u)q(u) - e(u)] \Delta u \right] \Delta s \leq - \int_{T_1}^t \frac{x^\Delta(s)}{f(x(s))} \Delta s.$$

Defining

$$F(x(s)) := \int_{x(T)}^{x(s)} \frac{d\theta}{f(\theta)},$$

we get

$$\begin{aligned} (F(x(s)))^\Delta &= \int_0^1 F'(x_h(s)) dh x^\Delta(s) = \int_0^1 \frac{1}{f(x_h(s))} dh x^\Delta(s), \\ &\leq \int_0^1 \frac{1}{f(x(s))} dh x^\Delta(s) = \frac{x^\Delta(s)}{f(x(s))}. \end{aligned}$$

Therefore, we have

$$\int_{T_1}^t \frac{1}{\phi(s)r(s)} \left[\int_{T_1}^s [\phi^\sigma(u)q(u) - e(u)] \Delta u \right] \Delta s \leq - \int_{T_1}^t (F(x(s)))^\Delta \Delta s = -F(x(t)),$$

Since $x(t) > 0$ and $x^\Delta(t) < 0$, then, we have $\lim_{t \rightarrow \infty} x(t) \geq 0$, and

$$-F(x(t)) = \int_{x(t)}^{x(T_1)} \frac{ds}{f(s)} \leq \int_0^{x(T_1)} \frac{ds}{f(s)} < \infty.$$

Thus

$$\int_{T_1}^t \frac{1}{\phi(s)r(s)} \left[\int_{T_1}^s [\phi^\sigma(u)q(u) - e(u)] \Delta u \right] \Delta s < \infty.$$

If $t \rightarrow \infty$, we get a contradiction to (2.36). This completes the proof. □

Example 2.1. Consider the second order nonlinear delay dynamic equation

$$\begin{aligned} (2.39) \quad &(-t^{\gamma-1}(x^\Delta(t))^\gamma)^\Delta - \frac{t^{\gamma-1}}{\sigma(t)} (x^\Delta(t))^\gamma + \left(1 - \frac{t^\gamma}{\sigma(t)}\right) x^\sigma(t)(1 + (x^\sigma(t))^2) \\ &= x^\sigma(t)(1 + (x^\sigma(t))^2), \end{aligned}$$

where γ is the quotient of odd positive integers and f satisfies $xf(x) > 0$ for $x \neq 0$.

Here, $r(t) = -t^{\gamma-1}$, $p(t) = \frac{-t^{\gamma-1}}{\sigma(t)}$, $q(t) = \left(1 - \frac{t^\gamma}{\sigma(t)}\right)$, $e(t) = -\sigma(t)$, $f(x) = x(1 + x^2)$ and $g(x) = x^\gamma$. To apply Corollary 2.1, take $\phi(t) = -t$. It is easy to see that the assumption (2.1) holds. Therefore, we have

$$\phi(t)r(t) = t^\gamma > 0, \quad \phi^\Delta(t)r(t) = \phi^\sigma(t)p(t) = t^{\gamma-1}.$$

Also,

$$\int_{t_0}^\infty g^{-1}\left(\frac{1}{\phi(t)r(t)}\right) \Delta t = \int_{t_0}^\infty g^{-1}\left(\frac{1}{t^\gamma}\right) \Delta t = \int_{t_0}^\infty \left(\frac{1}{t^\gamma}\right)^{\frac{1}{\gamma}} \Delta t = \int_{t_0}^\infty \frac{1}{t} \Delta t = \infty,$$

and

$$\int_{t_0}^\infty [\phi^\sigma(t)q(t) - e(t)] \Delta t = \int_{t_0}^\infty t^\gamma \Delta t = \infty.$$

Hence, every solution of Eq. (2.39) is oscillatory.

Example 2.2. Consider the equation ($\mathbb{T} = \mathbb{N}$)

$$\begin{aligned} (2.40) \quad &\Delta\left(\frac{-t}{(\sigma(t))^\gamma} (\Delta x(t))^\gamma\right) + \frac{1}{(\sigma(t))^\gamma} \Delta(x(t))^\gamma - 2t\sigma(t)x^\sigma(t) \\ &= t\sigma(t)x^\sigma(t)(1 + (x^\sigma(t))^2) \end{aligned}$$

where γ is the quotient of odd positive integers, $f^\Delta(t) = \Delta f(t) = f(t + 1) - f(t)$, $\sigma(t) = t + 1$.

Here, $r(t) = \frac{-t}{(\sigma(t))^\gamma}$, $p(t) = \frac{1}{(\sigma(t))^\gamma}$, $q(t) = -2t\sigma(t)$, $e(t) = -t$, $f(x) = x$ and $g(x) = x^\gamma$. To apply Corollary 2.1, take $\phi(t) = \frac{-1}{t}$. It is easy to see that the assumption (2.1) holds. Therefore, we have

$$\phi(t)r(t) = \frac{1}{(t + 1)^\gamma} > 0, \quad \phi^\Delta(t)r(t) = \phi^\sigma(t)p(t) = \frac{-1}{(t + 1)^{\gamma+1}}.$$

Also,

$$\begin{aligned} \int_{t_0}^\infty g^{-1}\left(\frac{1}{\phi(t)r(t)}\right)\Delta t &= \int_{t_0}^\infty g^{-1}(t + 1)^\gamma \Delta t \\ &= \int_{t_0}^\infty ((t + 1)^\gamma)^{\frac{1}{\gamma}} \Delta t = \sum_{t=n_0}^\infty (t + 1)\Delta t = \infty, \end{aligned}$$

and

$$\int_{t_0}^\infty [\phi^\sigma(t)q(t) - e(t)]\Delta t = 3 \sum_{t=n_0}^\infty t = \infty.$$

Hence, every solution of Eq. (2.40) is oscillatory.

Remark 2.2. The results of [6, 16, 18] can not be applied to Eq. (2.39) and Eq. (2.40) when $r(t)$ is not positive, but according to Corollary 2.1, this equation is oscillatory.

Example 2.3. Consider the equation ($\mathbb{T} = \mathbb{N}$)

$$\begin{aligned} \Delta((-1)^n \Delta x(t)) + (-1)^n \left[1 - \left(1 + \frac{1}{t}\right)^\alpha\right] \Delta x(t) + (-1)^n \left(\frac{\sigma(t)}{t}\right)^\alpha x^\sigma(t) \\ (2.41) \quad = \frac{(\sigma(t))^{\alpha-1} x^\sigma(t)}{t}, \end{aligned}$$

where $f^\Delta(t) = \Delta f(t) = f(t + 1) - f(t)$, $\sigma(t) = t + 1$, f satisfies $xf(x) > 0$ for $x \neq 0$. Here, $r(t) = (-1)^n$, $p(t) = (-1)^n [1 - (1 + \frac{1}{t})^\alpha]$, $q(t) = (-1)^n (\frac{\sigma(t)}{t})^\alpha$, $0 < \alpha \leq 1$, $f(x) = x$ and $e(t) = \frac{(-1)^n}{t\sigma(t)}$. To apply Corollary 2.3, take $\phi(t) = \frac{(-1)^n}{t^\alpha}$. It is easy to see that the assumption (2.1) holds. Therefore, we have

$$\phi(t)r(t) = \frac{1}{t^\alpha} > 0,$$

$$P(t) = 0 \quad \text{i.e. } (\phi^\Delta(t)r(t) = \phi^\sigma(t)p(t)) \quad \text{and} \quad \Delta P(t) = 0.$$

Also,

$$\int_{t_0}^\infty \frac{\Delta t}{\phi(t)r(t)} = \sum_{t=n_0}^\infty t^\alpha = \infty,$$

and

$$\int_{t_0}^\infty [\phi^\sigma(t)q(t) - e(t)]\Delta t = \sum_{t=n_0}^\infty \left[\frac{1}{t^\alpha} - \frac{(-1)^n}{t(t + 1)}\right] = \infty.$$

Hence, every solution of Eq. (2.41) is oscillatory.

Remark 2.3. The results of [3, 15, 16, 18] can not be applied to Eq. (2.41) when $r(t) = (-1)^n$, but according to Corollary 2.3, this equation is oscillatory for $\mathbb{T} = \mathbb{N}$.

Example 2.4. Consider the equation ($\mathbb{T} = \mathbb{R}$)

$$(2.42) \quad (t^{\gamma-1}x'(t))' + \frac{\lambda}{t^2}(x^\sigma(t))^5 = \frac{(x^\sigma(t))^5}{t\sigma(t)((x^\sigma(t))^2 + t + 1)}$$

where $0 < \gamma \leq 1$, f satisfies $xf(x) > 0$ for $x \neq 0$ and the superlinear condition (1.2).

Here, $r(t) = t^{\gamma-1}$, $p(t) = 0$, $q(t) = \frac{\lambda}{t^2}$, $f(x) = x^5$ and $e(t) = \frac{1}{t}$. To apply Theorem 2.3, take $\phi(t) = t$. It is easy to see that the assumption (2.1) holds. Therefore, we have

$$\begin{aligned} \phi(t)r(t) &= t^\gamma > 0, \\ P(t) &= t^{\gamma-1} > 0 \quad \text{and} \quad P'(t) = (\gamma - 1)t^{\gamma-2} \leq 0, \quad 0 < \gamma \leq 1. \end{aligned}$$

Also,

$$\int_{t_0}^\infty \frac{\Delta t}{\phi(t)r(t)} = \int_{t_0}^\infty \frac{1}{t^\gamma} dt = \infty, \quad 0 < \gamma \leq 1.$$

and

$$\int_{t_0}^\infty [\phi^\sigma(t)q(t) - e(t)]\Delta t = \int_{t_0}^\infty (\lambda - 1)\frac{1}{t} dt = \infty.$$

Hence, every solution of Eq. (2.42) is oscillatory if $\lambda > 1$.

Example 2.5. Consider the equation ($\mathbb{T} = 2\overline{\mathbb{Z}}$)

$$(2.43) \quad \Delta_2(-\Delta_2x(t)) - \sigma(t)(x^\sigma(t))^3 = t(x^\sigma(t))^3 \quad t \geq t_0 := 2,$$

where $f^\Delta(t) = \Delta_2f(t) = [f(2t) - f(t)]/(2t - t)$, $\sigma(t) = 2t$, f satisfies $xf(x) > 0$ for $x \neq 0$ and the superlinear condition (1.2).

Here, $r(t) = -1$, $p(t) = 0$, $q(t) = -\sigma(t)$, $f(x) = x^3$ and $e(t) = -t\sigma(t)$. To apply Theorem 2.3, take $\phi(t) = -t$. It is easy to see that the assumption (2.1) holds. Therefore, we have

$$\begin{aligned} \phi(t)r(t) &= t > 0, \\ P(t) &= 1 > 0 \quad \text{and} \quad \Delta_2P(t) = 0. \end{aligned}$$

Also,

$$\int_2^\infty \frac{\Delta_2t}{\phi(t)r(t)} = \int_2^\infty \frac{1}{t} \Delta_2t = \infty,$$

and

$$\int_2^\infty [\phi^\sigma(t)q(t) - e(t)]\Delta_2t = 6 \int_2^\infty t^2 \Delta_2t = \infty.$$

Hence, every solution of Eq. (2.43) is oscillatory.

Remark 2.4. The results of [3, 10, 11, 12, 13, 15, 19] can not be applied to Eq. (2.43) when $r(t) = -1 < 0$, but according to Theorem 2.3, this equation is oscillatory for $\mathbb{T} = 2\overline{\mathbb{Z}}$.

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