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GORENSTEIN INJECTIVE, PROJECTIVE AND FLAT (PRE)COVERS

E. ENOCHS, S. ESTRADA AND A. IACOB

ABSTRACT. We prove that if the ring R is left noetherian and if the class $\mathcal{GI}$ of Gorenstein injective modules is closed under filtrations, then $\mathcal{GI}$ is precovering. We extend this result to the category of complexes. We also prove that when R is commutative noetherian and such that the character modules of Gorenstein injective modules are Gorenstein flat, the class of Gorenstein injective complexes is both covering and enveloping. This is the case when the ring is commutative noetherian with a dualizing complex. The second part of the paper deals with Gorenstein projective and flat complexes. We prove the existence of special Gorenstein projective precovers over commutative noetherian rings of finite Krull dimension.

1. INTRODUCTION

The starting point of Gorenstein homological algebra was in 1966–1967 when in a series of conferences Auslander introduced a class of finitely generated modules that have a complete resolution. Auslander used these modules to define the notion of the G-dimension of a finite module over a commutative noetherian local ring. In 1969, Auslander and Bridger extended the definition to two sided noetherian rings. Calling the modules of G-dimension zero Gorenstein projective modules, in 1995 Enochs and Jenda, defined in [**11**] the Gorenstein projective modules (whether finitely generated or not) and Gorenstein injective modules over arbitrary rings. Another extension of the G-dimension is based on Gorenstein flat modules. These modules were introduced by Enochs, Jenda and Torrecillas [**14**].

Gorenstein homological algebra is the relative version of homological algebra that uses Gorenstein injective, Gorenstein projective and Gorenstein flat resolutions instead of the classical injective, projective and flat resolutions. But while the existence of the classical resolutions over arbitrary rings is well known, things are a little different when it comes to Gorenstein homological algebra. The main

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open problems in this area concern the existence of the Gorenstein (injective, projective and flat) resolutions. This is the reason why the existence of the Gorenstein (injective, projective, flat) precovers and preenvelopes both for modules and for complexes of modules has been studied intensively in recent years.

In Section 2, we give a sufficient condition for the class $\mathcal{GI}$ of Gorenstein injective modules being a precovering class. We prove that if R is left noetherian and if the class of Gorenstein injective modules is closed under filtrations, then $\mathcal{GI}$ is precovering in R-Mod. The converse is also true when we assume the existence of special Gorenstein injective precovers. In particular, this is the case when the class $\mathcal{GI}$ is covering. We extend our results to the category of complexes. We prove that if the class of Gorenstein injective injective modules is closed under filtrations then the class of Gorenstein injective complexes is precovering in $\mathrm{Ch}(R)$.

We also extend some of the results of [**15**] to the category of complexes. We prove that when R is commutative noetherian and such that for every Gorenstein injective module M, its character module M^+ is Gorenstein flat, the class $\mathcal{GI}(C)$ of Gorenstein injective complexes is covering. We also prove that over such rings the class of Gorenstein injective complexes is enveloping. In particular, this is the case when R is commutative noetherian with a dualizing complex.

In Section 3 we consider the existence of Gorenstein flat and Gorenstein projective (pre)covers. Enochs and López-Ramos [**13**] proved the existence of Gorenstein flat covers over coherent rings. And Jørgensen [**24**] showed the existence of Gorenstein projective precovers over commutative noetherian rings with dualizing complexes. More recently, Murfet and Salarian [**26**] extended his result to commutative noetherian rings of finite Krull dimension. We extend some of the results on the existence of Gorenstein projective and Gorenstein flat (pre)covers to the category of complexes of R-modules over noetherian rings. We show the existence of Gorenstein flat covers over two sided noetherian rings. And we prove the existence of *special* Gorenstein projective precovers over commutative noetherian rings of finite Krull dimension. This apparently "slight" variation is crucial from a homotopical point of view since it allows us to define Gorenstein projective cofibrant replacements of modules in the category of unbounded complexes.

2. Gorenstein injective (pre)covers and envelopes

Throughout the paper, R denotes an associative ring with 1. By R-module, we mean left R-module. And by R-Mod we mean, the category of left R-modules.

We recall that by [**12**, Definition 10.1.1], a module G is Gorenstein injective if there is an exact complex $\ldots \to E_1 \to E_0 \to E_{-1} \to \ldots$ of injective modules which remains exact under application of the functors $\mathrm{Hom}(\mathrm{Inj}, -)$, where Inj stands for all injective modules, and such that $G = \mathrm{Ker}(E_0 \to E_{-1})$.

The notions of precover and cover, preenvelope and envelope with respect to a class of modules $\mathcal{C}$ were introduced by Enochs [**7**]. The definitions carry to more general categories than module categories. In this paper, we also consider covers and envelopes in $\mathrm{Ch}(R)$, the category of chain complexes of left R-modules.

Definition 1. ([**18**, Definition 1.2.3]) If $\mathcal{A}$ is an abelian category and $\mathcal{F}$ a class of objects of $\mathcal{A}$, then an $\mathcal{F}$-precover of an object X of $\mathcal{A}$ is a morphism $\phi\colon F \to X$, where $F \in \mathcal{F}$ and such that $\mathrm{Hom}(G, F) \to \mathrm{Hom}(G, X)$ is surjective for all $G \in \mathcal{F}$. If furthermore any $f\colon F \to F$ with $\phi \circ f = \phi$ is an automorphism of F, then ϕ is said to be an $\mathcal{F}$-cover of X (clearly an $\mathcal{F}$-cover of X is unique if it exists). If every M in $\mathcal{A}$ has an $\mathcal{F}$-(pre)cover, the class $\mathcal{F}$ is said to be (pre)covering.

The dual notions are that of an $\mathcal{F}$-preenvelope $\beta\colon X \to F$ and of an $\mathcal{F}$-envelope. And so we have the notion of (pre)enveloping class.

When the class $\mathcal{F}$ is known, the notions of $\mathcal{F}$-(pre)covers and $\mathcal{F}$-(pre)envelopes take the name of $\mathcal{F}$. Thus, in case $\mathcal{F}$ is the class of Gorenstein injective modules, an $\mathcal{F}$-(pre)cover is called a Gorenstein injective (pre)cover.

For a class $\mathcal{F}$ of objects of $\mathcal{A}$, $\mathcal{F}^{\perp}$ denotes the class of objects C of $\mathcal{A}$ such that $\mathrm{Ext}^1(F, C) = 0$ for all $F \in \mathcal{F}$. Similarly, ${}^{\perp}\mathcal{F}$ denotes the class of objects D such that $\mathrm{Ext}^1(D, F) = 0$ for all $F \in \mathcal{F}$.

A pair of classes $(\mathcal{F}, \mathcal{C})$ of objects of $\mathcal{A}$ is said to be a cotorsion pair on $\mathcal{A}$ if $\mathcal{F}^{\perp} = \mathcal{C}$ and ${}^{\perp}\mathcal{C} = \mathcal{F}$.

Definition 2. A cotorsion pair $(\mathcal{F}, \mathcal{C})$ on $\mathcal{A}$ is said to be perfect if every object X of $\mathcal{A}$ has an $\mathcal{F}$-cover and a $\mathcal{C}$-envelope.

An $\mathcal{F}$-precover $\phi\colon F \to X$ is said to be a special $\mathcal{F}$-precover if $\mathrm{Ker}(\phi) \in \mathcal{F}^{\perp}$. An $\mathcal{F}$-preenvelope $\beta\colon X \to F$ is special if $\mathrm{Coker}(\beta) \in {}^{\perp}\mathcal{F}$.

By [**18**, Proposition 1.2.3], any $\mathcal{F}$ cover is a special $\mathcal{F}$ precover, and any $\mathcal{F}$ envelope is a special $\mathcal{F}$ preenvelope.

A cotorsion pair $(\mathcal{F}, \mathcal{C})$ on $\mathcal{A}$ is said to be complete if every object has a special epic $\mathcal{F}$-precover and a special monic $\mathcal{C}$-preenvelope.

Definition 3. ([**18**, Definition 1.2.10]) Let $\mathcal{A}$ be an abelian category with enough projectives and injectives. A cotorsion pair $(\mathcal{F}, \mathcal{C})$ in $\mathcal{A}$ is called hereditary if one of the following equivalent statements holds:

1. $\mathcal{F}$ is resolving, that is, $\mathcal{F}$ is closed under taking kernels of epimorphisms.
2. $\mathcal{C}$ is coresolving, that is, $\mathcal{C}$ is closed under taking cokernels of monomorphisms.
3. $\mathrm{Ext}^i(F, C) = 0$ for any $F \in \mathcal{F}$, $C \in \mathcal{C}$ and $i \geq 1$.

Over Gorenstein rings the existence of Gorenstein injective covers is known (see for instance, [**12**, Theorem 11.1.3]). We give sufficient conditions for the existence of Gorenstein injective precovers and covers over noetherian rings.

We recall the following definitions.

Definition 4. A direct system of modules $(X_\alpha | \alpha \leq \lambda)$ is said to be continuous if $X_0 = 0$ and if for each limit ordinal $\beta \leq \lambda$, we have $X_\beta = \varinjlim X_\alpha$ with the limit over the $\alpha < \beta$. The direct system $(X_\alpha | \alpha \leq \lambda)$ is said to be a system of monomorphisms if all the morphisms in the system are monomorphisms.

If $(X_\alpha | \alpha \leq \lambda)$ is a continuous direct system of R-modules then for this to be a system of monomorphisms, it suffices that $X_\alpha \to X_{\alpha+1}$ is monomorphism whenever $\alpha + 1 \leq \lambda$.

Definition 5. Let $\mathcal{L}$ be a class of R-modules. An R-module X of A is said to be a direct transfinite extension of objects of $\mathcal{L}$ if $X = \varinjlim X_\alpha$ for a continuous direct system $(X_\alpha|\alpha \leq \lambda)$ of monomorphisms such that $\mathrm{Coker}(X_\alpha \to X_{\alpha+1})$ is in $\mathcal{L}$ whenever $\alpha + 1 \leq \lambda$.

Definition 6. By a filtration of a module M we mean that for an ordinal number λ, we have a continuous well-ordered chain $(M_\alpha|\alpha \leq \lambda)$ of submodules of M with $M_0 = 0$ and with $M_\lambda = M$. We say that λ is the length of the filtration. If $\mathcal{C}$ is any class of modules, this filtration is said to be a $\mathcal{C}$-filtration if for every $\alpha + 1 \leq \lambda$ we have that $M_{\alpha+1}/M_\alpha$ is isomorphic to some $C \in \mathcal{C}$.

The class of all $\mathcal{C}$-filtered modules is denoted $\mathrm{Filt}(\mathcal{C})$.

Roughly speaking, $\mathrm{Filt}(\mathcal{C})$ is the class of all transfinite extensions of modules in $\mathcal{C}$. It is known ([**8**, Theorem 5.5] and [**27**], Theorem in the Introduction) that if $\mathcal{C}$ is a *set* of modules, then $\mathrm{Filt}(\mathcal{C})$ is precovering.

Our first result is a sufficient condition for the existence of Gorenstein injective precovers. It is known ([**16**]) that when R is left noetherian, the class of Gorenstein injective left R-modules is a Kaplansky class.

Since we use this property in our proof, we recall the following definition.

Definition 7. ([**16**, Definition 2.1]) Let R be a ring and say $\mathcal{F}$ be a class of R-modules. Then $\mathcal{F}$ is said to be a Kaplansky class if there exists a cardinal κ such that for every $M \in \mathcal{F}$ and for each $x \in M$, there exists a submodule F of M such that $x \in F \subseteq M$, $F \in \mathcal{F}$, $M/F \in \mathcal{F}$ and $\mathrm{Card}(F) \leq \kappa$.

Lemma 1. *Let R be a left noetherian ring. There exists an infinite cardinal κ such that the class $\mathcal{GI}$ of Gorenstein injective left R-modules is a κ-Kaplansky class.*

Proof. This is [**16**, Proposition 2.6]. □

Proposition 1. *Let R be a left noetherian ring, κ an infinite cardinal as in Lemma 1, and let $\mathcal{X}$ denote a set of representatives of isomorphism classes of Gorenstein injective modules M such that $|M| \leq \kappa$. The following assertions are equivalent:*

(1) *$\mathcal{GI}$ is closed under $\mathcal{X}$–filtrations.*
(2) *$\mathcal{GI} = \mathrm{Filt}(\mathcal{X})$.*

Proof. (2) $\Rightarrow$ (1) Clear.

(1) $\Rightarrow$ (2) By (1), it is clear that $\mathrm{Filt}(\mathcal{X}) \subseteq \mathcal{GI}$. Conversely, let $G \neq 0$ be a Gorenstein injective module and let $\{g_\alpha, \alpha < \lambda\}$ be a generating set for G. Let $G_0 = 0$ and let $G_1 \in \mathcal{GI}$ such that $g_1 \in G_1$ and $G/G_1 \in \mathcal{GI}$ – this is possible as $\mathcal{GI}$ is κ-Kaplansky. Assume that G_α is defined such that $G_\alpha \in \mathcal{GI}$, $\sum_{\beta<\alpha} Rg_\beta \subset G_\alpha$, and such that G/G_α is Gorenstein injective.

Consider another ordinal, say γ. If γ is a successor ordinal, say $\gamma = \alpha + 1$, then by the Kaplansky property, there is $G_{\alpha+1}$ such that $\overline{g_\alpha} \in G_{\alpha+1}/G_\alpha \subset G/G_\alpha$ and such that G/G_α, $G_{\alpha+1}/G_\alpha$ are Gorenstein injective. Then $\sum_{\beta\leq\alpha} Rg_\beta \subset G_{\alpha+1}$. Further, the exact sequence $0 \to G_\alpha \to G_{\alpha+1} \to G_{\alpha+1}/G_\alpha \to 0$ with both G_α

and $G_{\alpha+1}/G_\alpha$ in $\mathcal{GI}$ gives that $G_{\alpha+1} \in \mathcal{GI}$ (since the class $\mathcal{GI}$ is closed under extensions).

If $\gamma \leq \lambda$ is a limit ordinal, then we set $G_\gamma = \bigcup_{\alpha<\gamma} G_\alpha$. From (1), we have that $G_\gamma \in \mathcal{GI}$. Now, since the sequence $0 \to G_\gamma \to G \to G/G_\gamma \to 0$ is exact with G_γ and G Gorenstein injectives, and $\mathcal{GI}$ is closed under cokernels of monomorphisms ([**22**, Theorem 2.6]), it follows that G/G_γ is Gorenstein injective, so we can continue the induction. The process clearly terminates. Thus $G \in \mathrm{Filt}(\mathcal{X})$. □

Theorem 1. *Under the assumptions of Proposition 1, the class of Gorenstein injective modules is precovering.*

Proof. By [**8**, Theorem 5.5], or by [**27**], Theorem in the Introduction, $\mathrm{Filt}(\mathcal{X})$ is precovering. □

If moreover every R-module has a special $\mathcal{GI}$-precover, then the converse is also true, that is, R is left noetherian and $\mathcal{GI}$ is closed under $\mathcal{X}$-filtrations. The first claim follows from [**4**, Proposition 3.15] because the class $\mathcal{GI}$ being precovering requires R to be left noetherian. The second statement follows from the next proposition.

Proposition 2. *Let R be a left noetherian ring. If every R-module has a special Gorenstein injective precover then the class $\mathcal{GI}$ of Gorenstein injective modules is closed under direct transfinite extensions.*

Proof. Let $(G_\alpha|\alpha \leq \lambda)$ be a direct system of monomorphisms, with each $G_\alpha \in \mathcal{GI}$, and let $G = \varinjlim G_\alpha$. Since for each α, we have $G_\alpha \in {}^\perp(\mathcal{GI}^\perp)$, it follows that $G = \varinjlim G_\alpha \in {}^\perp(\mathcal{GI}^\perp)$ (by [**5**, Theorem 1.2]).
For each α, consider $\oplus_{E\in X} E^{(\mathrm{Hom}(E,G_\alpha))} \to G_\alpha$, where the map is the evaluation map and X is a representative set of indecomposable injective modules E. This is an injective precover of G_α, and since G_α is Gorenstein injective, $\oplus_{E\in X} E^{(\mathrm{Hom}(E,G_\alpha))} \to G_\alpha$ is surjective. Also this way of constructing a precover is functorial. The map $G_\alpha \to G_\beta$ gives rise to a map $E_\alpha \to E_\beta$. Since $E_\alpha \to G_\alpha$ was constructed in a functorial manner, we have that when $\alpha \leq \beta \leq \gamma$ the map $E_\alpha \to E_\gamma$ is the composition of the two maps $E_\alpha \to E_\beta$ and $E_\beta \to E_\gamma$.

Then we have an exact sequence $E \to G \to 0$ with $E = \varinjlim E_\alpha$ an injective module. It follows that G has a surjective injective cover and therefore, a surjective special Gorenstein injective precover. So there is an exact sequence

$$0 \to A \to \overline{G} \to G \to 0$$

with $A \in \mathcal{GI}^\perp$ and $\overline{G}$ Gorenstein injective. But by the above, we have that $\mathrm{Ext}^1(G, A) = 0$. So G is a direct summand of $\overline{G} \in \mathcal{GI}$. □

Corollary 1. *Let R be a left noetherian ring. If the class $\mathcal{GI}$ of Gorenstein injective modules, is covering, then $\mathcal{GI}$ is closed under transfinite extensions.*

Proposition 3. *When the ring R is left noetherian and the class of Gorenstein injective modules is closed under direct limits, the class $\mathcal{GI}$ is covering.*

Proof. Since R is left noetherian, the class of Gorenstein injective modules is Kaplansky. Since $\mathcal{GI}$ is also closed under direct limits, it is precovering (by Theorem 1). A precovering class that is also closed under direct limits is covering ([**12**, Corollary, 5.2.7]) $\square$

Corollary 2. ([**12**, Theorem 11.1.3]) *Over a Gorenstein ring, the class of Gorenstein injective modules is covering.*

Corollary 3. ([**23**, Theorem 3.3]) *If R is commutative noetherian with a dualizing complex, then the class of Gorenstein injective modules is covering.*

Proof. By [**3**, Theorem 6.9], $\mathcal{GI}$ is closed under direct limits. By Proposition 3, it is covering. $\square$

We extend our results to the category of complexes of R-modules over a two sided noetherian ring R. We use the notation $\mathcal{GI}(C)$ for the class of Gorenstein injective complexes.

It is known that when R is a left noetherian ring, a complex of left R-modules is Gorenstein injective if and only if each component is a Gorenstein injective R-module ([**25**], Theorem 8). Using this result we prove the following proposition.

Proposition 4. *Let R be a left noetherian ring. If the class of Gorenstein injective R-modules is closed under filtrations, then the class of Gorenstein injective complexes is precovering in* $\mathrm{Ch}(R)$.

Proof. Again, let κ be an infinite regular cardinal as in Lemma 1, and let $\mathcal{X}$ denote a set of representatives of isomorphism classes of Gorenstein injective modules M such that $|M| \leq \kappa$. Then $\mathcal{GI} = \mathrm{Filt}(\mathcal{X})$. Since $\mathcal{GI}$ is closed under filtrations, it follows that each complex of Gorenstein injective modules is filtered by bounded below complexes with components in $\mathcal{X}$ ([**27**], Proposition 4.3). In particular, the class of complexes of Gorenstein injective modules is deconstructible. By [**25**, Theorem 8], this is the class of Gorenstein injective complexes. By [**27**, Theorem page 195], the class of Gorenstein injective complexes is precovering. $\square$

Proposition 5. *Let R be left noetherian. If the class of Gorenstein injective modules is closed under direct limits, then the class of Gorenstein injective complexes is covering in* $\mathrm{Ch}(R)$.

Proof. By Proposition 3, the class of Gorenstein injective modules is covering. Also, the class of Gorenstein injective modules is closed under extensions, direct products, and by our assumptions, closed under direct limits. By [**10**, Theorem 3.13], the class of complexes of Gorenstein injective modules is covering. By [**25**, Theorem 8], these are the Gorenstein injective complexes. $\square$

We give a sufficient condition for the class of Gorenstein injective complexes being covering. Since it involves Gorenstein flat modules, we recall the following

Definition 8. ([**12**, Definition 10.3.1]) A module N is Gorenstein flat if there is an exact and $\mathrm{Inj}\otimes$-exact sequence $\ldots \to F_1 \to F_0 \to F_{-1} \to F_{-2} \to \ldots$ of flat modules such that $N = \mathrm{Ker}(F_0 \to F_{-1})$.

In [**15**, Theorem 1], we proved the following result: when the ring R is commutative noetherian and with the property that the character modules of the Gorenstein injective modules are Gorenstein flat, the class of Gorenstein injective modules is closed under direct limits, and so it is covering in R-Mod.

By Proposition 5 and [**15**, Theorem 1], we have the following theorem.

Theorem 2. *Let R be a commutative noetherian ring. Assume that the character modules of Gorenstein injective modules are Gorenstein flat. Then the class of Gorenstein injective complexes is covering.*

Example 1. If the ring R is commutative noetherian with a dualizing complex, then the class of Gorenstin injective complexes is covering.

Gorenstein injective envelopes of complexes

In [**15**], we proved that the class of Gorenstein injective modules is enveloping over a commutative noetherian ring with the property that the character modules of Gorenstein injective modules are Gorenstein flat. In particular, this shows the existence of Gorenstein injective envelopes over commutative noetherian rings with dualizing complexes.

We extend this result to the category of complexes. By $\mathcal{GI}(C)$, we denote the class of Gorenstein injective complexes.

First we prove that if the ring R is noetherian then the class of Gorenstein injective complexes is enveloping if and only if its left orthogonal class ${}^{\perp}\mathcal{GI}(C)$ is covering.

We start with the following result.

Proposition 6. *Let R be a left noetherian ring. Then $({}^{\perp}\mathcal{GI}(C), \mathcal{GI}(C))$ is a complete hereditary cotorsion pair in the category $\mathrm{Ch}(R)$ of complexes of R-modules.*

Proof. By [**20**], $({}^{\perp}\mathcal{GI}(C), \mathcal{GI}(C))$ is a complete cotorsion pair whenever R is any left noetherian ring.

Since the class of Gorenstein injective modules is coresolving in R-Mod, it follows (by [**25**, Theorem 8]) that $\mathcal{GI}(C)$ is coresolving in $\mathrm{Ch}(R)$. By [**17**, Lemma 2.10], the cotorsion pair $({}^{\perp}\mathcal{GI}(C), \mathcal{GI}(C))$ is hereditary. □

The following result is proved for modules in [**13**, Theorem 1.4]. The argument carries to the category of complexes

Theorem 3. ([**13**, Theorem 1.4]) *Let $(\mathcal{L}, \mathcal{C})$ be a hereditary cotorsion pair in $\mathrm{Ch}(R)$. Then the following statements are equivalent:*

(1) *$(\mathcal{L}, \mathcal{C})$ is perfect.*
(2) *Every complex of R-modules has a $\mathcal{C}$ envelope and every $C \in \mathcal{C}$ has an $\mathcal{L}$-cover.*
(3) *Every complex of R-modules has an $\mathcal{L}$-cover and every $L \in \mathcal{L}$ has a $\mathcal{C}$-envelope.*

Using Theorem 3 and the argument of [**15**, Proposition 2], we obtain the following theorem.

Theorem 4. *Let R be a left noetherian ring. The following statements are equivalent:*

(1) *The cotorsion pair $({}^{\perp}\mathcal{GI}(C), \mathcal{GI}(C))$ is perfect.*
(2) *The class $\mathcal{GI}(C)$ is enveloping.*
(3) *The class ${}^{\perp}\mathcal{GI}(C)$ is covering.*

For the following result we recall some definitions from [**18**].

Given two complexes C and D, let $\underline{\mathrm{Hom}}(C,D) = Z(\mathcal{H}\mathrm{om}(C,D))$. $\underline{\mathrm{Hom}}(C,D))$ can be made into a complex with $\underline{\mathrm{Hom}}(C,D)_m$ the abelian group of morphisms from C to $D[m]$ and with a boundary operator given by the following: if $f \in \underline{\mathrm{Hom}}(C,D)_m$ then $\delta_m(f)\colon C \to D[m+1]$ with $\delta_m(f)_n = (-1)^m \delta_D f^n$ for any $n \in \mathbb{Z}$.

The right derived functors of $\underline{\mathrm{Hom}}(C,D)$ are denoted $\underline{\mathrm{Ext}}^i(C,D)$.

We also recall that if C is a complex of right R-modules and D is a complex of left R-modules then the usual tensor product complex of C and D is the complex of Z-modules $C \otimes^{\cdot} D$ with $(C \otimes^{\cdot} D)_n = \oplus_{t \in Z}(C_t \otimes_R D_{n-t})$ and differentials

$$\delta(x \otimes y) = \delta_t^C(x) \otimes y + (-1)^t x \otimes \delta_{n-t}^D(y)$$

for $x \in C_t$ and $y \in D_{n-t}$.

In [**18**], García Rozas introduced another tensor product by the following: if C is again a complex of right R-modules and D is a complex of left R-modules, then $C \otimes D$ is defined to be $\frac{C \otimes^{\cdot} D}{B(C \otimes^{\cdot} D)}$. Then with the maps

$$\frac{(C \otimes^{\cdot} D)_n}{B_n(C \otimes^{\cdot} D)} \to \frac{(C \otimes^{\cdot} D)_{n-1}}{B_{n-1}(C \otimes^{\cdot} D)}$$

$x \otimes y \to \delta_C(x) \otimes y$, where $x \otimes y$ is used to denote the coset in $\frac{C \otimes^{\cdot} D}{B(C \otimes^{\cdot} D)}$, we get a complex. The right derived functors of the tensor product $- \otimes -$ are denoted by $\mathrm{Tor}_i(-,-)$.

By $\mathcal{GF}(C)$, we will denote the class of Gorenstein flat complexes. It is known that over a two-sided noetherian ring these are the complexes of Gorenstein flat modules ([**10**]).

Proposition 7. *Let R be a commutative noetherian ring with the property that the character modules of Gorenstein injective modules are Gorenstein flat. Then a complex K is in ${}^{\perp}\mathcal{GI}(C)$ if and only if $K^+ \in \mathcal{GF}(C)^{\perp}$.*

Proof. Given the canonical isomorphism for $\underline{\mathrm{Ext}}$ and Tor ([**18**, Proposition 2.4.1 and Lemma 5.4.2]) the proof of [**15**, Lemma 5] applies. $\square$

Now we can prove the following proposition.

Proposition 8. *Let R be commutative noetherian and such that the character modules of Gorenstein injective modules are Gorenstein flat. Then the class ${}^{\perp}\mathcal{GI}(C)$ is closed under pure quotients.*

Proof. Let $0 \to A \to B \to C \to 0$ be a pure exact sequence of complexes with $B \in^{\perp} \mathcal{GI}(C)$. Then the sequence $0 \to C^+ \to B^+ \to A^+ \to 0$ is split exact. So $B^+ \simeq A^+ \oplus C^+$. By Proposition 7, the complex B^+ is in $\mathcal{GF}(C)^{\perp}$. It follows that both A^+ and C^+ are in $\mathcal{GF}(C)^\perp$. By Proposition 7 again, A and C are both in ${}^\perp\mathcal{GI}(C)$. □

Theorem 5. *Let R be a commutative noetherian ring such that the character modules of Gorenstein injective modules are Gorenstein flat. Then the class of Gorenstein injective complexes is enveloping in* $\mathrm{Ch}(R)$.

Proof. By Theorem 4, it suffices to prove that the class ${}^\perp\mathcal{GI}(C)$ is covering. By Proposition 6, the class ${}^\perp\mathcal{GI}(C)$ is precovering, so it is closed under direct sums. Since the direct limit of an inductive family is a pure quotient of the direct sum, by Proposition 8, every direct limit of complexes in ${}^\perp\mathcal{GI}(C)$ is still in ${}^\perp\mathcal{GI}(C)$. It follows that ${}^\perp\mathcal{GI}(C)$ is covering in $\mathrm{Ch}(R)$. □

Corollary 4. *If R is a commutative noetherian ring with a dualizing complex, then every complex of R-modules has a Gorenstein injective envelope.*

3. Gorenstein flat and Gorenstein projective precovers for complexes

We recall the following definition.

Definition 9. ([**12**, Definition 10.2.1]) A module M is Gorenstein projective if there is an exact and $\mathrm{Hom}(-, \mathrm{Proj})$ exact complex $\ldots \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \ldots$ of projective modules such that $M = \mathrm{Ker}(P_0 \to P_{-1})$.

The Gorenstein projective complexes are defined in a similar manner, but working with resolutions of complexes.

We recall that for two complexes X and Y, $\mathrm{Hom}(X, Y)$ denotes the group of morphisms of complexes from X to Y.

We also recall that a complex P is projective if the functor $\mathrm{Hom}(P, -)$ is exact. Equivalently, P is projective if and only if P is exact and for each $n \in Z$, $\mathrm{Ker}(P_n \to P_{n-1})$ is a projective module. For example, if M is a projective module, then the complex

$$\ldots \to 0 \to M \xrightarrow{\mathrm{Id}} M \to 0 \to \ldots$$

is projective. In fact, any projective complex is uniquely up to isomorphism the direct sum of such complexes (one such complex for each $n \in Z$).

By [**18**], a complex D is called Gorenstein projective if there exists an exact sequence of complexes

$$\ldots \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \ldots$$

such that

1) for each $i \in Z$, P_i is a projective complex,
2) $\mathrm{Ker}(P_0 \to P_{-1}) = D$,
3) the sequence remains exact when $\mathrm{Hom}(-, P)$ is applied to it for any projective complex P.

In this section we prove the existence of Gorenstein flat covers of complexes over two sided noetherian rings and the existence of *special* Gorenstein projective precovers of complexes over commutative noetherian rings of finite Krull dimension. This result generalizes and improves [**26**, Theorem A1] in two directions: on one side it is established for the category $\mathrm{Ch}(R)$ of unbounded complexes, and on the other hand, we prove that our Gorenstein precover is *special*. This property has been shown to be crucial in defining the cofibrant and fibrant replacements in (abelian) model category structures on $\mathrm{Ch}(R)$ (see [**21**]). We would like to stress that our methods are necessarily different from those of [**26**].

We begin by proving the existence of Gorenstein flat precovers and covers over two sided noetherian rings. The following result improves [**10**, Theorem 4.3], where the existence of Gorenstein flat covers is proved under the additional hypothesis that $\mathcal{GF}$ is closed under products.

Proposition 9. *Let R be a two sided noetherian ring. The class of Gorenstein flat complexes is covering in* $\mathrm{Ch}(R)$.

Proof. By [**16**, Proposition 2.10], the class of Gorenstein flat modules is Kaplansky and closed under direct limits. Then by [**27**], this class is deconstructible. By [**27**, Proposition 4.3], the class of complexes of Gorenstein flat modules is deconstructible, so it is precovering. But over a two sided noetherian ring a complex is Gorenstein flat if and only if it is a complex of Gorenstein flat modules ([**10**, Lemmas 4.1 and 4.2]). So the class of Gorenstein flat complexes is precovering. This class of complexes is also closed under direct limits, so it is covering. □

Next, we consider the question of the existence of Gorenstein projective precovers for complexes.

For modules, Enochs and Jenda showed that when R is a Gorenstein ring, the class of Gorenstein projective modules is precovering. Then Jørgensen showed the existence of Gorenstein projective precovers over commutative noetherian rings with dualizing complexes. Recently, Murfet and Salarian extended his result to commutative noetherian rings of finite Krull dimension.

Their goal in [**26**] was to introduce a triangulated category of totally acyclic complexes of flat modules which plays the role of $K_{\mathrm{tac}}(\mathrm{Proj}\, R)$ for any noetherian ring R (in fact they work in a more general setting, that of complexes of flat sheaves over noetherian schemes).

To accomplish this they started with a construction developed by Neeman who defined $N(\mathrm{Flat})$ as the Verdier quotient $\frac{K(\mathrm{Flat})}{K_{\mathrm{pac}}(\mathrm{Flat})}$ with $K(\mathrm{Flat})$ the homotopy category of complexes of flat modules and $K_{\mathrm{pac}}(\mathrm{Flat})$ the full subcategory of pure acyclic complexes in $K(\mathrm{Flat})$ (it is known that a complex of flat modules is pure acyclic if and only if it is a flat complex in the sense of García Rozas' definition [**21**]). Then they considered the full subcategory of $N(\mathrm{Flat})$, $N_{\mathrm{tac}}(\mathrm{Flat})$, of N-totally acyclic complexes of flat modules (i.e., exact and $\mathrm{Inj} \otimes$-exact complexes of flat modules). Their results in [**26**] indicate that this is the "correct" triangulated category one can use in order to generalize aspects of Gorenstein homological algebra to schemes.

We show that when R is noetherian the class of N-totally acyclic complexes of flat modules is precovering.

In the following we use $\widetilde{\text{Gor Flat}}$ to denote, the class of exact complexes F with $Z_n(F) \in \mathcal{GF}$ for each n. Since the class of Gorenstein flat modules is Kaplansky and also closed under direct limits, extensions and retracts, the class $\widetilde{\text{Gor Flat}}$ is covering in $\text{Ch}(R)$ (by [**21**, Theorem 4.12], or see also [**16**, Corollary 2.11] and [**9**, Corollary 3.1]). By [**21**], its right orthogonal class $\widetilde{\text{Gor Flat}}^{\perp}$ consists of the complexes X with each $X_n \in \mathcal{GF}^{\perp}$ and such that for any $G \in \widetilde{\text{Gor Flat}}$, every $u \in \text{Hom}(G, X)$ is homotopic to zero.

Proposition 10. *Let R be a noetherian ring. Then the class of N-totally acyclic complexes of flat modules is precovering in* $\text{Ch}(R)$.

Proof. Let P be a complex of flat R-modules. Since the class of $\widetilde{\text{Gor Flat}}$ complexes is covering, there is an exact sequence

$$0 \to K \to F \to P \to 0$$

with $F \in \widetilde{\text{Gor Flat}}$ and with $K \in \widetilde{\text{Gor Flat}}^{\perp}$. In particular, each module K_n is in $\mathcal{GF}^{\perp}$.

For each n we have an exact sequence

$$0 \to K_n \to F_n \to P_n \to 0$$

Since P_n is flat and $K_n \in \mathcal{GF}^{\perp}$, the sequence is split exact. So K_n is a direct summand of F_n, so it is Gorenstein flat. But then $K_n \in \mathcal{GF} \bigcap \mathcal{GF}^{\perp}$ gives that K_n is flat for each n. Therefore, F_n is flat for each n. So the complex F is N-totally acyclic. Also, for each N-totally acyclic complex D, we have that D is in $\widetilde{\text{Gor Flat}}$, so $\text{Ext}^1(D, K) = 0$.

Let X be any complex of R-modules. Since the class of exact complexes of flat modules, $\text{dw}(\text{Flat}) \bigcap \mathcal{E}$, is precovering ([**2**, Example 2]), there is an exact sequence

$$0 \to H \to P \to X \to 0$$

with P an exact complex of flat modules and with H in $(\text{dw}(\text{Flat}) \bigcap \mathcal{E})^{\perp}$.

By the above there is an exact sequence

$$0 \to K \to F \to P \to 0$$

with F an N-totally acyclic complex of flat modules and $K \in N_{\text{tac}}(\text{Flat})^{\perp}$.

We form the commutative diagram

$$\begin{tikzcd}
 & K \arrow[r, equal] \arrow[d] & K \arrow[d] & & \\
0 \arrow[r] & M \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H \arrow[r] & P \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

So we have an exact sequence

$$0 \to M \to F \to X \to 0$$

with an N-totally acyclic complex F of flat modules. Both K and H are in $N_{\text{tac}}(\text{Flat})^{\perp}$, so M also satisfies $\text{Ext}^1(D, M) = 0$ for any N-totally acyclic complex D. □

We recall that over a commutative noetherian ring of finite Krull dimension d, every Gorenstein flat module M has finite Gorenstein projective dimension and $G.p.d_R(M) \leq d$.

Now, we prove the existence of *special* Gorenstein projective precovers in $\text{Ch}(R)$ over a commutative noetherian ring R of finite Krull dimension.

The proof uses the fact that over such a ring R, a complex is Gorenstein projective if and only if it is a complex of Gorenstein projective R-modules ([**10**, Theorem 5.1]).

Proposition 11. *If R is commutative noetherian of finite Krull dimension, then every complex X of R-modules has a special Gorenstein projective precover.*

Proof. Let $\dim R = d$.

We show first that every Gorenstein flat complex G has a special Gorenstein projective precover. Let

$$0 \to \overline{G} \to P_{d-1} \to \ldots \to P_0 \to G \to 0$$

be a partial projective resolution of G. Then for each j, we have an exact sequence of modules

$$0 \to \overline{G}_j \to P_{d-1,j} \to \ldots \to P_{0,j} \to G_j \to 0$$

Since $Gpd\ G_j \leq d$ it follows that each $\overline{G}_j$ is Gorenstein projective. Thus $\overline{G}$ is a Gorenstein projective complex (by [**10**, Theorem 5.1]). So $\overline{G}$ has an exact and $\text{Hom}(-, \text{Proj})$ exact complex of projective complexes

$$0 \to \overline{G} \to T_{d-1} \to \ldots \to T_0 \to \ldots$$

Let $T = \text{Ker}(T_{-1} \to T_{-2})$. Then T is a Gorenstein projective complex, and we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \overline{G} \arrow[r] \arrow[d, equal] & T_{d-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & T_1 \arrow[r] \arrow[d] & T_0 \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overline{G} \arrow[r] & P_{d-1} \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & G \arrow[r] & 0
\end{tikzcd}$$

Therefore, we have an exact sequence

$$0 \to T_{d-1} \to P_{d-1} \oplus T_{d-2} \to \ldots \to P_1 \oplus T_0 \to P_0 \oplus T \xrightarrow{\delta} G \to 0$$

Let $V = \operatorname{Ker} \delta$. Then V has finite projective dimension, so $\operatorname{Ext}^1(W, V) = 0$ for any Gorenstein projective complex W.

We have an exact sequence $0 \to V \to P_0 \oplus T \to G \to 0$ with $P_0 \oplus T$ Gorenstein projective and with V of finite projective dimension. Thus $P_0 \oplus T \to G$ is a special Gorenstein projective precover.

Now, we prove that every complex X has a special Gorenstein projective precover.

Let X be any complex of R-modules. By Proposition 9, there exists an exact sequence

$$0 \to Y \to G \to X \to 0$$

with Gorenstein flat G and with $\operatorname{Ext}^1(U, Y) = 0$ for any Gorenstein flat complex U.

By the above, there is an exact sequence

$$0 \to L \to P \to G \to 0$$

with P Gorenstein projective and with the complex L of finite projective dimension.

Form the pullback diagram

$$\begin{tikzcd}
 & L \arrow[r, equal] \arrow[d] & L \arrow[d] & & \\
0 \arrow[r] & M \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r] & G \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

Since $L \in \operatorname{Gor} \operatorname{Proj}^{\perp}$ and $Y \in \operatorname{Gor} \operatorname{Flat}^{\perp}$ and the sequence $0 \to L \to M \to Y \to 0$ is exact, it follows that $M \in \operatorname{Gor} \operatorname{Proj}^{\perp}$.

So $0 \to M \to P \to X \to 0$ is exact with P a Gorenstein projective and with $M \in \operatorname{Gor} \operatorname{Proj}^{\perp}$. □

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E. Enochs, Department of Mathematics, University of Kentucky, Lexington, KY 40506-0027, USA,
e-mail: `enochs@ms.uky.edu`

S. Estrada, Department of Applied Mathematics, University of Murcia, 30100 Murcia, Spain,
e-mail: `sestrada@um.es`

A. Iacob, Department of Mathematical Sciences, Georgia Southern University 0203 Georgia Ave., Statesboro, GA 30460-8093 Georgia Southern University, USA,
e-mail: `aiacob@georgiasouthern.edu`